

# COMPLEXITY THEORY

## Lecture 3: Undecidability

Markus Krötzsch

Knowledge-Based Systems

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# Decidability and Computability

**Review:** A language is

- **recognisable** (or **semi-decidable**, or **recursively enumerable**) if it is the language of all words recognised by some Turing machine
- **decidable** (or **recursive**) if it is the language of a Turing machine that always halts, even on inputs that are not accepted
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- **undecidable** if it is not decidable

Instead of acceptance of words, we can also consider other computations:

**Definition 3.1:** A TM  $\mathcal{M}$  **computes** a partial function  $f_{\mathcal{M}} : \Sigma^* \rightarrow \Sigma^*$  as follows. We have  $f_{\mathcal{M}}(w) = v$  for a word  $w \in \Sigma^*$  if  $\mathcal{M}$  halts on input  $w$  with a tape that contains only the word  $v \in \Sigma^*$  (followed by blanks).

In this case, the function  $f_{\mathcal{M}}$  is called **computable**.

Total, computable functions are called **recursive**.

Functions may therefore be computable or uncomputable.

# Undecidability is Real

A fundamental insight of computer science and mathematics is that there are undecidable languages:

**Theorem 3.2:** There are undecidable languages over every alphabet  $\Sigma$ .

**Proof:** See exercise. □

Analogously, there are uncomputable functions.

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We do not know if the language  $L_\pi$  is decidable, but it might be (e.g., if every finite sequence of digits occurred in  $\pi$ , which, however, is not known to be true today).

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$L_{\pi 7}$  is decidable:

- Option 1:  $\pi$  contains sequences of arbitrary many 7. Then  $L_{\pi 7}$  is decided by a TM that accepts all words of the form  $7^n$ .
- Option 2:  $\pi$  contains sequences of 7s only up to a certain maximal length  $\ell$ . Then  $L_{\pi 7}$  is decided by a TM that accepts all words of the form  $7^n$  with  $n \leq \ell$ .

In each possible case, we have a practical algorithm – we just don't know which one is correct.

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**Definition 3.5:** We define  $S(n)$  as the largest number of steps that any DTM with  $n$  states and tape alphabet  $\Gamma = \{\mathbf{x}, \sqcup\}$  executes on the empty tape, before it eventually halts.

**Observation:**  $S$  is well defined.

- The number of TMs with at most  $n$  states is finite
- Among the relevant  $n$ -state TMs there must be a largest number of steps before halting (TMs that do not halt are ignored)

# Busy Beaver

A small variation of the step counter function leads to the Busy-Beaver Problem:



Tibor Radó, BB inventor

**Definition 3.6:** The **Busy-Beaver function**  $\Sigma : \mathbb{N} \rightarrow \mathbb{N}$  is a total function, where  $\Sigma(n)$  is the maximal number of  $\mathbf{x}$  that a DTM with at most  $n$  states and tape alphabet  $\Gamma = \{\mathbf{x}, \sqcup\}$  can write when starting on the empty tape and before it eventually halts.

**Note:** The exact value of  $\Sigma(n)$  depends on details of the TM definition.

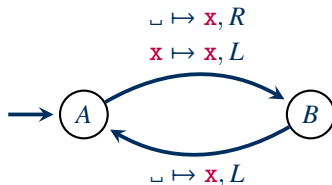
Most works in this area assume a two-sided infinite tape that can be extended to the left and to the right if necessary. We (often) assume a partial transition function in the following; such a TM halts if no transition is defined.



# Example

The Busy-Beaver number  $\Sigma(2)$  is 4 when using a two-way infinite tape.

The following TM (with partial transition function) implements this behaviour:



We obtain:  $A \sqcup \vdash \mathbf{x} B \sqcup \vdash A \mathbf{x} \mathbf{x} \vdash B \sqcup \mathbf{x} \mathbf{x} \vdash A \sqcup \mathbf{x} \mathbf{x} \mathbf{x} \vdash \mathbf{x} B \mathbf{x} \mathbf{x} \mathbf{x}$

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- Let  $k$  be the total number of states in  $\mathcal{M}_\Sigma$ ,  $\mathcal{M}_{+1}$ , and  $\mathcal{M}_{\times 2}$ . There is a TM  $\mathcal{I}_k$  with  $k + 1$  states that writes the word  $\mathbf{x}^k$  to the empty tape.



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- When executing  $\mathcal{I}_k$ ,  $\mathcal{M}_{\times 2}$ ,  $\mathcal{M}_\Sigma$ , and  $\mathcal{M}_{+1}$  after another, the result is a TM with  $\leq 2k$  states that writes  $\Sigma(2k) + 1$  times  $\mathbf{x}$  before halting.

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- Hence  $\Sigma(2k) \geq \Sigma(2k) + 1$  – contradiction. □

# Proof Notes

**Note 1:** The proof involves an interesting idea of using TMs as “sub-routines” in other TMs. We will use this again later on.

**Note 2:** If a TM can compute  $f : \mathbb{N} \rightarrow \mathbb{N}$  in the usual binary encoding, it is not hard to get a TM for  $x^n \mapsto x^{f(n)}$  by just using unary encoding instead.

**Note 3:** Transforming an arbitrary TM into one that uses only symbols  $\{x, \sqcup\}$  on its tape is slightly more involved, but doable.

**Note 4:** To execute TMs after one another, we can assume w.l.o.g. that they terminate in a unique state that has no possible transitions. Then one can combine TMs by identifying this unique final state with the starting state of the next TM, which decreases the total number of states by merging states.

**Note 5:** Busy Beaver is increasing with its input, i.e.,  $\Sigma(m) \leq \Sigma(2k)$  for any  $m < 2k$ , so the proof works even if the composed machine has less than  $2k$  states.

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For  $n = 10$ , one has found a lower bound of the form  $\Sigma(10) > 3^{3^{3^{\dots^3}}}$ , where the complete expression has more than  $7.6 \times 10^{12}$  occurrences of the number 3.

The value for  $\Sigma(5)$  is obtained by a TM that runs for 47, 176, 870 steps, as proven in July 2024 [Reference].

# Universality

# The Universal Machine

A first important observation of Turing was that TMs are powerful enough to simulate other TMs:

Step 1: Encode Turing Machines  $\mathcal{M}$  as words  $\langle \mathcal{M} \rangle$

Step 2: Construct a **universal Turing Machine**  $\mathcal{U}$ , which gets  $\langle \mathcal{M} \rangle$  as input and then simulates  $\mathcal{M}$

# Step 1: encoding Turing Machines

Any reasonable encoding of a TM  $\mathcal{M} = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$  is usable, e.g., the following (for DTMs):

- We use an alphabet  $\{0, 1, \#\}$
- States are enumerated in any order (beginning with  $q_0$ ), and encoded in binary:  
 $Q = \{q_0, \dots, q_n\} \rightsquigarrow \langle Q \rangle = \text{bin}(0)\#\dots\#\text{bin}(n)$
- We also encode  $\Gamma$  and the directions  $\{R, L\}$  in binary
- A transition  $\delta(q_i, \sigma_n) = \langle q_j, \sigma_m, D \rangle$  is encoded as 5-tuple:  
 $\text{enc}(q_i, \sigma_n) = \text{bin}(i)\#\text{bin}(n)\#\text{bin}(j)\#\text{bin}(m)\#\text{bin}(D)$
- The transition function is encoded as a list of all these tuples, separated with  $\#$ :  
 $\langle \delta \rangle = (\text{enc}(q_i, \sigma_n)\#)_{q_i \in Q, \sigma_i \in \Gamma}$
- Combining everything, we set  $\langle \mathcal{M} \rangle = \langle Q \rangle\#\#\langle \Sigma \rangle\#\#\langle \Gamma \rangle\#\#\langle \delta \rangle\#\#\langle q_{\text{accept}} \rangle\#\#\langle q_{\text{reject}} \rangle$

We can also encode arbitrary words to match this encoding:

- For a word  $w = a_1 \dots a_\ell$  we define  $\langle w \rangle = \text{bin}(a_1)\#\dots\#\text{bin}(a_\ell)$

## Step 2: The Universal Turing Machine

We define the universal TM  $\mathcal{U}$  as multi-tape TM:

Tape 1: Input tape of  $\mathcal{U}$ : contains  $\langle \mathcal{M} \rangle \#\#\langle w \rangle$

Tape 2: Working tape of  $\mathcal{U}$

Tape 3: Stores the state of the simulated TM

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The working principle of  $\mathcal{U}$  is easily sketched:

- $\mathcal{U}$  validates the input, copies  $\langle w \rangle$  to Tape 4, moves the head on Tape 4 to the start and initialises Tape 3 with  $\text{bin}(0)$  (i.e.,  $\langle q_0 \rangle$ ).
- In each step  $\mathcal{U}$  reads an (encoded) symbol from the head position on Tape 4, and searches for the simulated state (Tape 3) a matching transition in  $\langle \mathcal{M} \rangle$  on Tape 1  
(w.l.o.g. assume that the final states of the encoded TM have no transitions):
  - Transition found: update state on Tape 3; replace the encoded symbol on Tape 4 by the new symbol; move the head on Tape 4 accordingly
  - Transition not found: if the state on Tape 3 is  $q_{\text{accept}}$ , then go to the final accepting state; else go to the final rejecting state

# The Theory of Software

**Theorem 3.8:** There is a **universal Turing Machine**  $\mathcal{U}$ , that, when given an input  $\langle \mathcal{M} \rangle \#\# \langle w \rangle$ , simulates the behaviour of a DTM  $\mathcal{M}$  on  $w$ :

- If  $\mathcal{M}$  halts on  $w$ , then  $\mathcal{U}$  halts on  $\langle \mathcal{M} \rangle \#\# \langle w \rangle$  with the same result
- If  $\mathcal{M}$  does not halt on  $w$ , then  $\mathcal{U}$  does not halt on  $\langle \mathcal{M} \rangle \#\# \langle w \rangle$  either

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Practical consequences:

- Universal computers are possible
- We don’t have to buy a new computer for every application
- Software exists

# Undecidable Problems and Reductions

# The Halting Problem

A classical undecidable problem:

**Definition 3.9:** The **Halting Problem** consists in the following question:

Given a TM  $\mathcal{M}$  and a word  $w$ ,  
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We can formulate the Halting Problem as a word problem by encoding  $\mathcal{M}$  and  $w$ :

**Definition 3.10:** The **Halting Problem** is the word problem for the language

$$\mathbf{P}_{\text{Halt}} = \{\langle \mathcal{M} \rangle \#\#\langle w \rangle \mid \mathcal{M} \text{ halts on input } w\},$$

where  $\langle \mathcal{M} \rangle$  and  $\langle w \rangle$  are suitable encodings of  $\mathcal{M}$  and  $w$ , for which  $\#\#$  can be used as separator.

**Remark:** Wrongly encoded inputs are rejected.

# “Proof” by Intuition

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One can easily give an algorithm  $\mathcal{A}$  that verifies Goldbach’s conjecture systematically by testing it for every even number starting with 4:

- Success: Test the next even number
- Failure: Terminate with output “Goldbach was wrong!”

The question “Will  $\mathcal{A}$  halt?” therefore is equivalent to the question “Is Goldbach’s conjecture wrong?”

Many other important open problems could be solved in this way.

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Then one can construct a TM  $\mathcal{D}$  that does the following:

- (1) Check if the given input is a TM encoding  $\langle \mathcal{M} \rangle$
- (2) Simulate  $\mathcal{H}$  on input  $\langle \mathcal{M} \rangle \#\#\langle \mathcal{M} \rangle$ , that is, check if  $\mathcal{M}$  halts on  $\langle \mathcal{M} \rangle$
- (3) If yes, enter an infinite loop;  
if no, halt and accept

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Will  $\mathcal{D}$  accept the input  $\langle \mathcal{D} \rangle$ ?

$\mathcal{D}$  halts and accepts if and only if  $\mathcal{D}$  does not halt

Contradiction. □

# Proof by Reduction

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An algorithm:

- Input: natural number  $k$  (in binary)
- Iterate over all Turing machines  $\mathcal{M}$  that have  $k$  states and tape alphabet  $\{x, \sqcup\}$ :
  - Decide if  $\mathcal{M}$  halts on the empty input  $\varepsilon$   
(possible if the Halting problem is decidable)
  - If yes, then simulate  $\mathcal{M}$  on the empty input and, when  $\mathcal{M}$  has halted, count the number of  $x$  on the tape  
(possible, since there are universal TMs)
- Output: the maximal number of  $x$  written.

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This algorithm would compute the Busy-Beaver function  $\Sigma : \mathbb{N} \rightarrow \mathbb{N}$ .

We have already shown that this is impossible – contradiction. □

# Turing Reductions

Our previous proof constructs an algorithm for one task (Busy Beaver) by calling subroutines for another task (the Halting Problem)

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**Informal Definition 3.13:** A problem **P** is **Turing reducible** to a problem **Q** (in Symbols:  $\mathbf{P} \leq_T \mathbf{Q}$ ), if **P** can be solved by a program that may call **Q** as a sub-program.

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**Example 3.14:** Our proof uses a reduction of the Busy-Beaver computation to the Halting problem. Note that the subroutine might be called exponentially many times here.

To make this more formal, we need oracles.

# Oracles

**Definition 3.15:** An **Oracle Turing Machine** (OTM) is a Turing machine  $\mathcal{M}$  with a special tape, called the oracle tape, and distinguished states  $q_?$ ,  $q_{\text{yes}}$ , and  $q_{\text{no}}$ . For a language  $\mathbf{O}$ , the **oracle machine**  $\mathcal{M}^{\mathbf{O}}$  can, in addition to the normal TM operations, do the following:

Whenever  $\mathcal{M}^{\mathbf{O}}$  reaches  $q_?$ , its next state is  $q_{\text{yes}}$  if the content of the oracle tape is in  $\mathbf{O}$ , and  $q_{\text{no}}$  otherwise.

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- The word problem for  $\mathbf{O}$  might be very hard or even undecidable
- Nevertheless, asking the oracle always takes just one step
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**Definition 3.16:** A problem  $\mathbf{P}$  is **Turing reducible** to a problem  $\mathbf{Q}$  (in symbols:  $\mathbf{P} \leq_T \mathbf{Q}$ ), if  $\mathbf{P}$  is decided by an OTM  $\mathcal{M}^{\mathbf{Q}}$  with oracle  $\mathbf{Q}$ .

# Undecidability via Turing Reductions

One can use Turing reductions to show undecidability:

**Theorem 3.17:** If  $\mathbf{P}$  is undecidable and  $\mathbf{P} \leq_T \mathbf{Q}$ , then  $\mathbf{Q}$  is undecidable.

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**Proof:** Via contrapositive: If  $\mathbf{P} \leq_T \mathbf{Q}$  and  $\mathbf{Q}$  is decidable, then we can implement the OTM that shows  $\mathbf{P} \leq_T \mathbf{Q}$  as a regular TM, which shows that  $\mathbf{P}$  is decidable.  $\square$

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Here is a small application:

**Theorem 3.18:** The language  $\mathbf{P}_{\text{Halt}} = \{\langle \mathcal{M} \rangle \#\#\langle w \rangle \mid \mathcal{M} \text{ does not halt on } w\}$  (the “Non-Halting Problem”) is undecidable.

**Proof sketch:** Decide Halting by using  $\mathbf{P}_{\text{Halt}}$  as an oracle and inverting the result. Check TM encoding first (wrong encodings are rejected by Halting and Non-Halting).  $\square$

# $\varepsilon$ -Halting

Special cases of the Halting Problem are usually not simpler:

**Definition 3.19:** The  $\varepsilon$ -Halting Problem consists in the following question:  
Given a TM  $\mathcal{M}$ ,  
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**Proof:** We define an oracle machine for deciding Halting:

- Input: A Turing machine  $\mathcal{M}$  and a word  $w$ .
- Construct a TM  $\mathcal{M}_w$  that runs in two phases:
  - (1) Delete the input tape and write the word  $w$  instead
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- Output: output of the  $\varepsilon$ -Halting Problem

This Turing-reduces Halting to  $\varepsilon$ -halting, so the latter is also undecidable. □



# Summary and Outlook

Busy Beaver is uncomputable

Halting is undecidable (for many reasons)

Oracles and Turing reductions formalise the notion of a “subroutine” and help us to transfer our insights from one problem to another

## What's next?

- Some more undecidability
- Recursion and self-referentiality
- Actual complexity classes