

FOUNDATIONS OF COMPLEXITY THEORY

Lecture 13: Space Hierarchy and Gaps

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TU Dresden, December 14, 2020

Review

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A Hierarchy for Space

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Review: Time Hierarchy Theorems

Time Hierarchy Theorem 12.12 If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is time-constructible, and $g \cdot \log g \in o(f)$, then

 $\mathsf{DTime}_*(g) \subsetneq \mathsf{DTime}_*(f)$

Nondeterministic Time Hierarchy Theorem 12.14 If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is time-constructible, and $g(n + 1) \in o(f(n))$, then

 $\mathsf{NTime}_*(g) \subsetneq \mathsf{NTime}_*(f)$

In particular, we find that $P \neq ExpTime$ and $NP \neq NExpTime$:



Space Hierarchy

For space, we can always assume a single working tape:

- Tape reduction leads to a constant-factor increase in space
- Constant factors can be eliminated by space compression

Therefore, $DSpace_{i}(f) = DSpace_{1}(f)$.

Space turns out to be easier to separate - we get:

Space Hierarchy Theorem 13.1: If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is spaceconstructible, and $g \in o(f)$, then

 $\mathsf{DSpace}(g) \subsetneq \mathsf{DSpace}(f)$

Challenge: TMs can run forever even within bounded space.

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Proving the Space Hierarchy Theorem (1)

Proof (continued): It remains to show that \mathcal{D} implements diagonalisation:

$L(\mathcal{D}) \in \mathsf{DSpace}(f)$:

- f is space-constructible, so both the marking of tape symbols and the initialisation of the counter are possible in DSpace(f)
- The simulation is performed so that the marked O(f)-space is not left

There is *w* such that $\langle \mathcal{M}, w \rangle \in \mathbf{L}(\mathcal{D})$ iff $\langle \mathcal{M}, w \rangle \notin \mathbf{L}(\mathcal{M})$:

- As for time, we argue that some w is long enough to ensure that f is sufficiently larger than g, so \mathcal{D} 's simulation can finish.
- The countdown measures $2^{f(n)}$ steps. The number of possible distinct configurations of \mathcal{M} on w is $|Q| \cdot n \cdot g(n) \cdot |\Gamma|^{g(n)} \in 2^{O(g(n) + \log n)}$, and due to $f(n) \ge \log n$ and $g \in o(f)$, this number is smaller than $2^{f(n)}$ for large enough *n*.
- If \mathcal{M} has d tape symbols, then \mathcal{D} can encode each in $\log d$ space, and due to \mathcal{M} 's space bound \mathcal{D} 's simulation needs at most $\log d \cdot g(n) \in o(f(n))$ cells.

Therefore, there is w for which \mathcal{D} simulates \mathcal{M} long enough to obtain (and flip) its output, or to detect that it is not terminating (and to accept, flipping again). Foundations of Complexity Theory slide 7 of 19

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Proving the Space Hierarchy Theorem (1)

Space Hierarchy Theorem 13.1: If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is spaceconstructible, and $g \in o(f)$, then

$\mathsf{DSpace}(g) \subsetneq \mathsf{DSpace}(f)$

Proof: Again, we construct a diagonalisation machine \mathcal{D} . We define a multi-tape TM \mathcal{D} for inputs of the form $\langle \mathcal{M}, w \rangle$ (other cases do not matter), assuming that $|\langle \mathcal{M}, w \rangle| = n$

- Compute f(n) in unary to mark the available space on the working tape
- Initialise a separate countdown tape with the largest binary number that can be written in f(n) space
- Simulate \mathcal{M} on $\langle \mathcal{M}, w \rangle$, making sure that only previously marked tape cells are used
- Time-bound the simulation using the content of the countdown tape by decrementing the counter in each simulated step
- If *M* rejects (in this space bound) or if the time bound is reached without *M* halting, then accept; otherwise, if M accepts or uses unmarked space, reject

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Space Hierarchies

Like for time, we get some useful corollaries:

Corollary 13.2: PSpace ⊆ ExpSpace

Proof: As for time, but easier.

Corollary 13.3: NL ⊊ PSpace

Proof: Savitch tells us that $NL \subseteq DSpace(\log^2 n)$. We can apply the Space Hierarchy Theorem since $\log^2 n \in o(n)$.

Corollary 13.4: For all real numbers 0 < a < b, we have DSpace $(n^a) \subsetneq$ DSpace (n^b) .

In other words: The hierarchy of distinct space classes is very fine-grained.

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The Gap TheoremThe following theorem snows why (for time):Special Gap Theorem 13.5: There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that
DTime $(f(n)) = DTime(2^{f(n)})$.This has been shown independently by Boris Trakhtenbrot (1964) and Allan Borodin
(1972).Reminder: For this we continue to use the strict definition of DTime(f) where no
constant factors are included (no hidden O(f)). This simplifies proofs; the factors
are easy to add back.

Why Constructibility?

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Proving the Gap Theorem

Special Gap Theorem 13.5: There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathsf{DTime}(f(n)) = \mathsf{DTime}(2^{f(n)})$.

Proof idea: We divide time into exponentially long intervals of the form:

 $[0,n], [n+1,2^n], [2^n+1,2^{2^n}], [2^{2^n}+1,2^{2^{2^n}}], \cdots$

(for some appropriate starting value *n*)

We are looking for gaps of time where no TM halts, since:

- for every finite set of TMs,
- and every finite set of inputs to these TMs,
- there is some interval of the above form $[m + 1, 2^m]$

such none of the TMs halts in between m + 1 and 2^m steps on any of the inputs.

The task of f is to find the start m of such a gap for a suitable set of TMs and words

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Gaps in Time

We consider an (effectively computable) enumeration of all Turing machines:

$\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \dots$

Definition 13.6: For arbitrary numbers $i, a, b \in \mathbb{N}$ with $a \leq b$, we say that $\operatorname{Gap}_i(a, b)$ is true if:

- Given any TM \mathcal{M}_j with $0 \le j \le i$,
- and any input string *w* for \mathcal{M}_j of length |w| = i,
- \mathcal{M}_j on input w will halt in less than a steps, in more than b steps, or not at all.

Lemma 13.7: Given $i, a, b \ge 0$ with $a \le b$, it is decidable if $\text{Gap}_i(a, b)$ holds.

Proof: We just need to ensure that none of the finitely many TMs M_0, \ldots, M_i will halt after *a* to *b* steps on any of the finitely many inputs of length *i*. This can be checked by simulating TM runs for at most *b* steps.

Find the Gap

We can now define the value f(n) of f for some $n \ge 0$:

Let in(n) denote the number of runs of TMs $\mathcal{M}_0, \ldots, \mathcal{M}_n$ on words of length n, i.e.,

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in(n) = |\Sigma_0|^n + \dots + |\Sigma_n|^n where \Sigma_i is the input alphabet of \mathcal{M}_i
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We recursively define a series of numbers k_0, k_1, k_2, \ldots by setting $k_0 = 2n$ and $k_{i+1} = 2^{k_i}$ for $i \ge 0$, and we consider the following list of intervals:

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[k_0 + 1, k_1], [k_1 + 1, k_2], \cdots, [k_{in(n)} + 1, k_{in(n)+1}]
[2n+1, 2^{2n}], [2^{2n}+1, 2^{2^{2n}}], \cdots, [2^{2^{n}+1}, 2^{2^{2^{n}}}]
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Let f(n) be the least number k_i with $0 \le i \le in(n)$ such that $\text{Gap}_n(k_i + 1, k_{i+1})$ is true.

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Properties of *f*

We first establish some basic properties of our definition of *f*:

Claim: The function *f* is well-defined.

Proof: For finding f(n), we consider in(n) + 1 intervals. Since there are only in(n) runs of TMs $\mathcal{M}_0, \ldots, \mathcal{M}_n$, at least one interval remains a "gap" where no TM run halts.

Claim: The function *f* is computable.

Proof: We can compute in(n) and k_i for any i, and we can decide $Gap_i(k_i + 1, k_{i+1})$.

Papadimitriou: "notice the fantastically fast growth, as well as the decidedly unnatural definition of this function."

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Finishing the Proof

We can now complete the proof of the theorem:

Claim: $DTime(f(n)) = DTime(2^{f(n)}).$

Consider any $\mathbf{L} \in \text{DTime}(2^{f(n)})$. Then there is an $2^{f(n)}$ -time bounded TM \mathcal{M}_i with $\mathbf{L} = \mathbf{L}(\mathcal{M}_i)$.

For any input *w* with $|w| \ge j$:

- The definition of f(|w|) took the run of \mathcal{M}_i on w into account
- \mathcal{M}_i on w halts after less than f(|w|) steps, or not until after $2^{f(|w|)}$ steps (maybe never)
- Since \mathcal{M}_i runs in time DTime($2^{f(n)}$), it must halt in DTime(f(n)) on w

For the finitely many inputs *w* with |w| < j:

- We can augment the state space of \mathcal{M}_i to run a finite automaton to decide these cases
- This will work in DTime(*f*(*n*))

Therefore we have $\mathbf{L} \in \mathsf{DTime}(f(n))$.

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Discussion: The case |w| < i

Borodin says: It is meaningful to state complexity results if they hold for "almost every" input (i.e., for all but a finite number)

Papadimitriou says: These words can be handled since we can check the length and then recognise the word in less than 2*i* steps

Really?

- If we do these < 2i steps before running \mathcal{M}_i , the modified TM runs in $\mathsf{DTime}(f(n) + 2i)$
- This does not show $L \in DTime(f(n))$

A more detailed argument:

- Make the intervals larger: $[k_i + 1, 2^{k_i+2n} + 2n]$, that is $k_{i+1} = 2^{k_i+2n} + 2n$.
- Select f(n) to be $k_i + 2n + 1$ if the least gap starts at $k_i + 1$.

The same pigeon hole argument as before ensures that an empty interval is found.

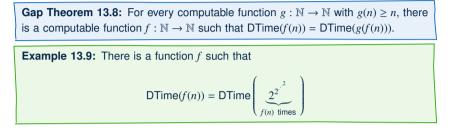
But now the f(n) time bounded machine \mathcal{M}_i from the proof will be sure to stop after f(n) - 2n - 1 steps, so a shift of $2i \le 2n$ to account for the finitely many cases will not make it use more than f(n) steps either

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Discussion: Generalising the Gap Theorem

- Our proof uses the function $n \mapsto 2^n$ to define intervals
- Any other computable function could be used without affecting the argument

This leads to a generalised Gap Theorem:



Moreover, the Gap Theorem can also be shown for space (and for other resources) in a similar fashion (space is a bit easier since the case of short words |w| < j is easy to handle in very little space)

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Discussion: Significance of the Gap Theorem

What have we learned?

- More time (or space) does not always increase computational power
- · However, this only works for extremely fast-growing, very unnatural functions

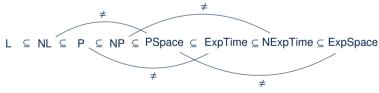
"Fortunately, the gap phenomenon cannot happen for time bounds *t* that anyone would ever be interested in"¹

Main insight: better stick to constructible functions

¹ Allender, Loui, Reagan: Complexity	Theory. In Computing Handbook, 3	3rd ed., CRC Press, 2014
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Summary and Outlook

Hierarchy theorems tell us that more time/space leads to more power:



However, they don't help us in comparing different resources and machine types (P vs. NP, or PSpace vs. ExpTime)

With non-constructible functions as time/space bounds, arbitrary (constructible or not) boosts in resources do not lead to more power

