Towards a Navigation Paradigm for Triadic Concepts

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Abstract. The simple formalization and the intuitive graphical representation are main reasons for the growing popularity of Formal Concept Analysis (FCA). FCA gives the user the possibility to explore the structure of data and understand correlations and implications in the data set. Recently, triadic FCA (3FCA) has become increasingly popular, but exploring triadic conceptual landscapes is not easy, especially because of the less immediate structure of the space of triadic concepts. Even more, available graphical representations of trilattices are barely intelligible and hard to obtain even for small data sets. Driven by practical requirements, we propose a new navigation paradigm for triadic conceptual landscapes based on a neighborhood notion arising from appropriately defined dyadic concept lattices. Understanding the corresponding reachability relation gives also new theoretical insights about the behavior of triadic concepts and the corresponding triadic data sets.

1 Introduction

With the advent of the information society and the rise of data science, understanding big collections of information and knowledge and representing them in intuitive ways is more important than ever. Formal concept analysis is well-known for its capabilities addressing knowledge processing and knowledge representation as well as offering reasoning support for understanding the structure of large collections of data.

For dyadic FCA – the original version of FCA based on a binary incidence relation – this has proven to be the case through the graphical representation of the concept lattice that offers a intuitive visualization and hence understanding of the data. From this graphical representation, one can read any relation between objects and attributes, but also implications holding in the data. For cases where the concept lattice gets too big to be represented in a readable way, "local" navigational paradigms have been proposed, where only one concept and its direct neighbor concepts are visualized and the user can explore the concept lattice by successively moving to neighboring concepts [2, 7].

Dyadic FCA was extended in [11] by Rudolf Wille and Fritz Lehmann to the triadic case, featuring a ternary instead of a binary incidence relation. The use of FCA increased over the last years, still there was little focus on applications of triadic FCA (3FCA), mainly because of its higher complexity and unavailability of a graphical representation for trilattices which quickly become impossible to draw even for small data sets. Although there are a lot of data collections that map perfectly to a triadic representation, for instance collaborative tagging scenarios or folksonomies ([10]), there is no good support for helping humans understand the structure of the triconcepts in a tricontext. Despite the fact that 3FCA is just an extension of FCA, the graphical representation for the dyadic case does not have an intuitive extension to the triadic case ([1, 6, 8]). Wille and Lehmann proposed a way to graphically represent a triadic context by using a triadic diagram in [11], inspired by the concept lattice from the dyadic case. However, the geometric representation obtained does not give much insight into the structure of the tricontext and cannot be easily read and understood. Furthermore, even a small set of triadic data can generate a large amount of triconcepts.

For the reasons mentioned above, we intend to present in this article a method to locally display a smaller part of the space of triconcepts, instead of displaying all of them at once. Our goal is to find an intuitive navigation strategy that allows for moving from one such local view to other, adjacent ones. Furthermore, we will formally analyze the properties of this strategy and ultimately suggest algorithms for producing the structures necessary for browsing the space of triconcepts using developed and theoretically well-understood methods.

Exploiting the fact that triconcepts are built three-dimensional, the navigation strategy we propose makes use of the elegance and the expressive power of dyadic concept lattices. Navigation starts local, with a triconcept. Herefrom, we fix what we call a *perspective*, i.e., one of the three dimensions (extent, intent or modus) and then collect all so-called *directly reachable triconcepts*. For each perspective, the triconcepts directly reachable via this perspective can be arranged in a dyadic concept lattice, hence navigating among them benefits from all advantages concept lattices are offering. After selecting a directly reachable triconcept, one may change the perspective and move towards another set of reachable triconcepts, exploring again another concept lattice. Despite of its apparent growth of computational complexity, this approach allows to cope with large sets of triconcepts. Moreover, the local navigation strategy discussed in this paper gives rise to a list of theoretical questions: reachability of all triconcepts, the existence and the number of reachability clusters, their structure and a method to navigate from one to another. Understanding these clusters proves to be not trivial and gives interesting insights about the inherent conceptual structure of triadic data.

2 Preliminaries

This section introduces the basic notions of triadic formal concept analysis. For further information about the dyadic case or more specific results about 3FCA we refer the interested reader to the standard literature [3, 4, 11, 12].

Definition 1. *A* triadic context *(also:* tricontext*) is a quadruple* (K_1, K_2, K_3, Y)*, where* K_1, K_2 *and* K_3 *are sets and* $Y \subseteq K_1 \times K_2 \times K_3$ *is a ternary relation be-*

tween them. The elements of K_1, K_2, K_3 are called (formal) objects, attributes and conditions, respectively. An element $(g, m, b) \in Y$ is read object g has attribute m under condition b.

The following definition shows how dyadic contexts can be obtained from a triadic one in a natural way.

Definition 2 (Derived contexts). *Every triadic context (K_1, K_2, K_3, Y) gives rise to the following dyadic contexts:*
$\mathbb{K}^{(1)} := (K_1, K_2 \times K_3, Y^{(1)})$ *with* $gY^{(1)}(m, b) :\Leftrightarrow (g, m, b) \in Y$,
$\mathbb{K}^{(2)} := (K_2, K_1 \times K_3, Y^{(2)})$ *with* $mY^{(2)}(g, b) :\Leftrightarrow (g, m, b) \in Y$, *and*
$\mathbb{K}^{(3)} := (K_3, K_1 \times K_2, Y^{(3)})$ *with* $bY^{(3)}(g, m) :\Leftrightarrow (g, m, b) \in Y$.
For $\{i, j, k\} = \{1, 2, 3\}$ and $A_k \subseteq K_k$, we define $\mathbb{K}^{(ij)}_{A_k} := (K_i, K_j, Y^{(ij)}_{A_k})$, where $(a_i, a_j) \in Y^{(ij)}_{A_k}$ if and only if $(a_i, a_j, a_k) \in Y$ for all $a_k \in A_k$.

Intuitively, the contexts $\mathbb{K}^{(i)}$ represent "flattened" versions of the triadic context, obtained by putting the "slices" of (K_1, K_2, K_3, Y) side by side. Moreover, $\mathbb{K}^{(ij)}_{A_k}$ corresponds to the intersection of all those slices that correspond to elements of A_k.

In triadic FCA, there are two extensions for the dyadic derivation operators.

Definition 3 ((i)-derivation operators). *For $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$ and for $X \subseteq K_i$ and $Z \subseteq K_j \times K_k$ the (i)-derivation operators are defined by:*
$X \mapsto X^{(i)} := \{(a_j, a_k) \in K_j \times K_k \mid (a_i, a_j, a_k) \in Y \text{ for all } a_i \in X\}$.
$Z \mapsto Z^{(i)} := \{a_i \in K_i \mid (a_i, a_j, a_k) \in Y \text{ for all } (a_j, a_k) \in Z\}$.

Obviously, these derivation operators correspond to the derivation operators of the dyadic contexts $\mathbb{K}^{(i)}, i \in \{1, 2, 3\}$.

Definition 4 ((i, j, X_k)-derivation operators). *For $\{i, j, k\} = \{1, 2, 3\}$ and $X_i \subseteq K_i, X_j \subseteq K_j, X_k \subseteq K_k$, the (i, j, X_k)-derivation operators are defined by*
$X_i \mapsto X_i^{(i,j,X_k)} := \{a_j \in K_j \mid (a_i, a_j, a_k) \in Y \text{ for all } (a_i, a_k) \in X_i \times X_k\}$
$X_j \mapsto X_j^{(i,j,X_k)} := \{a_i \in K_i \mid (a_i, a_j, a_k) \in Y \text{ for all } (a_j, a_k) \in X_j \times X_k\}$.

The (i, j, X_k)-derivation operators correspond to those of the dyadic contexts $(K_i, K_j, Y^{(ij)}_{X_k})$.

Similar to the notion of formal concepts in dyadic FCA, triadic concepts can be defined ([11]). A triadic concept is a maximal box of incidences (Proposition 1) and can be generated using derivation operators (Proposition 2).

Definition 5. *A* triadic concept *(short:* triconcept*) of $\mathbb{K} := (K_1, K_2, K_3, Y)$ is a triple (A_1, A_2, A_3) with $A_i \subseteq K_i$ for $i \in \{1, 2, 3\}$ and $A_i = (A_j \times A_k)^{(i)}$ for every $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$. The sets A_1, A_2, and A_3 are called* extent, intent, *and* modus *of the triadic concept, respectively. We let $\mathfrak{T}(\mathbb{K})$ denote the set of all triadic concepts of $\mathbb{K}$.*

Proposition 1. *The triconcepts of a triadic context (K_1, K_2, K_3, Y) are exactly the maximal triples $(A_1, A_2, A_3) \in \mathfrak{P}(K_1) \times \mathfrak{P}(K_2) \times \mathfrak{P}(K_3)$ with $A_1 \times A_2 \times A_3 \subseteq Y$, with respect to the component-wise set inclusion.*

Proposition 2. *For $X_i \subseteq K_i$ and $X_k \subseteq K_k$ with $\{i,j,k\} = \{1,2,3\}$, let $A_j := X_i^{(i,j,X_k)}, A_i := A_j^{(i,j,X_k)}$ and $A_k := (A_i \times A_j)^{(k)}$ (if $i < j$) or $A_k := (A_j \times A_i)^{(k)}$ (if $j < i$). Then (A_1, A_2, A_3) is the triadic concept $\mathfrak{b}_{ik}(X_i, X_k)$ with the property that it has the smallest k-th component among all triadic concepts (B_1, B_2, B_3) with the largest j-th component satisfying $X_i \subseteq B_i$ and $X_k \subseteq B_k$. In particular, $\mathfrak{b}_{ik}(A_i, A_k) = (A_1, A_2, A_3)$ for each triadic concept (A_1, A_2, A_3) of $\mathbb{K}$.*

3 Motivating example

In this section we present a small example, aiming to explain how the local navigation paradigm works in a set of triconcepts. The related theoretical aspects will be introduced in the following sections. For this, we consider the hostel tricontext from [5], whose trilattice is represented in Figure 1. The objects of the

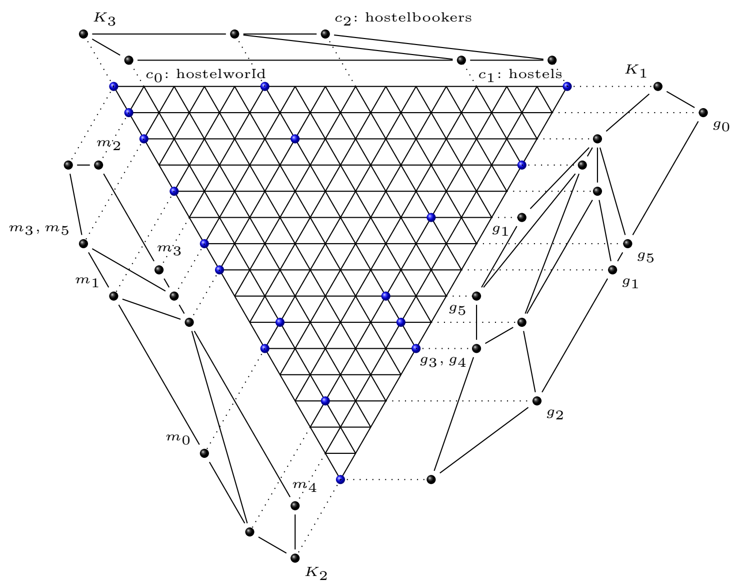

Fig. 1: Trilattice of the tricontext "Hostels".

triadic data set are hostels, the attributes services provided by the hostels, while the conditions are web portals where the hostels can be rated. The graphical representation as a 3-net displays all triconcepts and the equivalence classes to which they belong in a triadic diagram. The extent, intent and modus of a triconcept can be read by using the order diagrams displayed on the side of the trilattice. Global navigation in a 3-net becomes difficult for a (slightly) larger set of triconcepts and this is the case in many 3FCA applications. What graphical representation should be employed in the cases where a representation as a 3-net is not possible ([12])? The complexity of the trillatice structure and that of the order diagrams of the extents, intents and modi set makes a global navigation approach quite difficult.

To cope with the complexity of larger data sets, we propose a local navigation paradigm which starts from a triconcept (A_1, A_2, A_3) and the selection of one of its components (extent, intent or modus), which we then call *perspective*. We build the projected context $\mathbb{K}^{(ij)}_{A_k}$ along perspective k and compute its concept lattice. It can be proved that every dyadic concept of this projected context corresponds to exactly one triconcept in the original trilattice. These triconcepts are called *directly reachable* and navigation among them is performed in the underlying dyadic concept lattice.

To start local navigation, choose $T := (\{g_3, g_4, g_5\}, \{m_0, m_1, m_2, m_3, m_5\}, \{c_1, c_2\})$ and consider perspective 3 (i.e., modus). By projecting along $\{c_1, c_2\}$, we obtain the concept lattice displayed in Figure 2. Triconcept T corresponds to the leftmost dyadic concept in Figure 2. Moreover, all dyadic concepts correspond to some triconcepts, having either the same modus or a larger one. The navigation can be continued herefrom by choosing one of the directly reachable triconcepts from T and a perspective, i.e., one of the concepts of $\mathbb{K}^{(12)}_{\{c_1,c_2\}}$ and then navigating within the new concept lattice. For example, the rightmost concept of this lattice corresponds to the triconcept $(\{g_2, g_3, g_4\}, \{m_2, m_3, m_4\}, \{c_1, c_2\})$. By choosing perspective 1 (i.e., extent), the triconcepts reachable herefrom are represented in Figure 3.

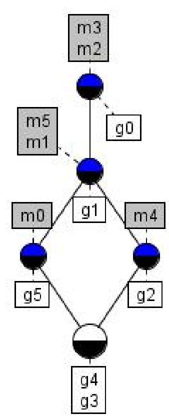

Fig. 2: Directly reachable triconcepts from T using perspective 3. The extent and intent of the triconcepts can be read from the concept lattice, while the modus is computed using the corresponding derivation operator $(\cdot)^3$ in the tricontext.

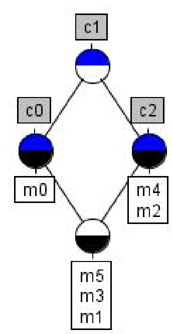

Fig. 3: Reachable triconcepts from T using perspective 3 and then 1. Only intent and modus are displayed in the concept lattice, the extent is computed using the corresponding derivation operator $(\cdot)^1$ in the tricontext.

This example shows how triconcepts can be clustered according to their reachability and how we can navigate from one triconcept to another. We might ask whether all concepts might be reachable by this approach or not, what are the maximal strongly connected components of the reachability relation, i.e., the reachability clusters, what are the properties of the set of reachability clusters and how can we set up a local navigation paradigm herefrom. By changing perspectives, all concepts in this example prove to be reachable (though not directly reachable). We will prove later on that this will not always be the case.

Motivated by this short example, we introduce in the following sections the theoretical aspects and considerations of the proposed navigation paradigm.

4 Reachable triconcepts

This section aims to define the exploration paradigm exemplified in Section 3 and to discuss some theoretical issues. The following propositions are direct consequences of Proposition 2. For every triconcept, by projecting along one of the dimensions, we obtain a formal dyadic context, where the projection of the triconcept is a dyadic concept of the corresponding concept lattice (Proposition 3). Moreover, every dyadic concept herefrom generates a triconcept (Proposition 4).

Proposition 3. *Let* $(A, B, C) \in \mathfrak{T}(\mathbb{K})$ *be a triadic concept. Then* $(A, B) \in \mathfrak{B}(\mathbb{K}_C^{(12)})$.

Proposition 4. *Let* $(A, B, C) \in \mathfrak{T}(\mathbb{K})$ *be a triadic concept. Let* $(A_1, A_2) \in \mathfrak{B}(\mathbb{K}_C^{(12)})$. *Then* $(A_1, A_2, (A_1 \times A_2)^{(3)}) \in \mathfrak{T}(\mathbb{K})$.

By the above propositions, we conclude that given a triconcept (A, B, C), fixing either its extent, or its intent or modus, gives rise to a (dyadic) concept lattice, every concept of which can be deterministically turned into a triconcept by computing the missing component using an appropriate triadic derivation operator (for instance $(\cdot)^{(3)}$). Based on this, we are now able to define a reachability relation between triconcepts.

Definition 6. *For* (A_1, A_2, A_3) *and* (B_1, B_2, B_3) *triadic concepts, we say that* (B_1, B_2, B_3) *is* directly reachable *from* (A_1, A_2, A_3) using perspective (1) *and we write* $(A_1, A_2, A_3) \prec_1 (B_1, B_2, B_3)$ *if and only if* $(B_2, B_3) \in \mathfrak{B}(\mathbb{K}_{A_1}^{(23)})$. *Analogously, we can define direct reachability using perspectives* (2) *and* (3).

We say that (B_1, B_2, B_3) *is* directly reachable *from* (A_1, A_2, A_3) *if it is directly reachable using at least one of the three perspectives, that is, formally* $(A_1, A_2, A_3) \prec (B_1, B_2, B_3) :\Leftrightarrow [(A_1, A_2, A_3) \prec_1 (B_1, B_2, B_3)] \vee [(A_1, A_2, A_3) \prec_2 (B_1, B_2, B_3)] \vee [(A_1, A_2, A_3) \prec_3 (B_1, B_2, B_3)]$.

By Proposition 3, two triconcepts having the same extent, or the same intent, or modus are always mutually directly reachable. Hence, in a trilattice diagram, all triconcepts aligned on the same line (i.e., being equivalent with respect to one of the three preorders) are mutually directly reachable:

Proposition 5. *Let* $(A_1, A_2, A_3), (B_1, B_2, B_3)$ *be two triconcepts. If* $A_i = B_i$ *for an* $i \in \{1, 2, 3\}$ *then* $(A_1, A_2, A_3) \prec_i (B_1, B_2, B_3)$ *and* $(B_1, B_2, B_3) \prec_i (A_1, A_2, A_3)$.

Definition 7. *We define the* reachability relation *between two triconcepts as being the transitive closure of the direct reachability relation* $\prec$. *We denote this relation by* $\lhd$.

Definition 8. *The equivalence class of a triconcept* (A_1, A_2, A_3) *with respect to the preorder* $\lhd$ *on* $\mathfrak{T}(\mathbb{K})$ *will be called a* reachability cluster *and denoted by* $[(A_1, A_2, A_3)]$.

Intuitively, the reachability cluster of (A_1, A_2, A_3) contains all triconcepts which are mutually reachable from (A_1, A_2, A_3) When considering $\prec$ as directed edge relation of a graph, reachability clusters correspond to the strongly connected components of that graph.

The following results are providing a better understanding of the reachability clusters and their structure. We prove that there exist triconcepts which are always reachable (Proposition 6). Moreover, the induced order on the set of reachability clusters always has a greatest element.

Proposition 6. *The trivial triconcepts* $\theta_1 := (K_1, K_2, (K_1 \times K_2)^{(3)})$, $\theta_2 := (K_1, K_3, (K_1 \times K_3)^{(2)})$ *and* $\theta_3 := (K_2, K_3, (K_2 \times K_3)^{(1)})$ *are always reachable. Moreover, they are always directly reachable.*

Proof. Let us assume, without restricting generality that $(K_1 \times K_2)^{(3)} = (K_1 \times K_3)^{(2)} = (K_2 \times K_3)^{(1)} = \emptyset$. Let $(A, B, C) \in \mathfrak{T}(\mathbb{K})$. Using perspective (3), we have that $(A, B) \in \mathfrak{B}(\mathbb{K}_C^{(12)})$. The greatest and the lowest elements of $\mathbb{K}_C^{(12)}$ are $(K_1, \emptyset)$ and $(\emptyset, K_2)$, respectively. Hence $(A, B, C) \lhd \theta_2$ and $(A, B, C) \lhd \theta_3$. By choosing another perspective, θ_1 is directly reached from (A, B, C).

In particular, if $(A, B, C) = \theta_1$, then the trivial triconcepts θ_2 and θ_3 are reachable by perspective (1) . □

Corollary 1. *The ordered set* $(\mathfrak{T}(\mathbb{K})/\sim, \leq)$ *has always a greatest element, the reachability cluster of the trivial concepts. We denote this cluster by* ∇.

Proposition 7. *If* (A, B, C) *is a triconcept with either* $A = K_1$, *or* $B = K_2$, *or* $C = K_3$, *then* $(A, B, C) \in \nabla$.

Proof. Every trivial concept is reachable from (A, B, C). Let us assume that $A = K_1$. Take now $\theta_1 := (K_1, K_2, \emptyset)$ and choose perspective (1). We obtain the context $\mathbb{K}_{K_1}^{(23)} := (K_2, K_3, Y_{K_1}^{(23)})$. We want to prove that $(B, C) \in \mathfrak{B}(\mathbb{K}_{K_1}^{(23)})$.

We know that $B = (K_1 \times C)^{(2)} = \{m \in K_2 \mid \forall g \in K_1, \forall b \in C. (g, m, b) \in Y\}$. Also, by definition, $C^{(2,3,K_1)} = \{m \in K_2 \mid \forall g \in K_1, \forall b \in C. (g, m, b) \in Y\}$, hence $B = C^{(2,3,K_1)}$. Analogously, $C = B^{(2,3,K_1)}$. □

Remark 1. (1) If (A_1, B_1, C_1) and (A_2, B_2, C_2) are triconcepts with $A_1 = K_1$ or $B_1 = K_2$ or $C_1 = K_3$ and $(A_1, B_1, C_1) \lhd (A_2, B_2, C_2)$, then $(A_2, B_2, C_2) \in \nabla$.

(2) If $(A_1, B_1, C_1) \in \nabla$ and $(A_1, B_1, C_1) \triangleleft (A_2, B_2, C_2)$ then $(A_2, B_2, C_2) \in \nabla$. The converse does not hold true, i.e., more than one reachability cluster is possible. Take for example $K_1 := \{g_1, g_2\}, K_2 := \{m_1, m_2\}$, and $K_3 := \{b_1, b_2\}$ with $Y := \{(g_1, m_1, b_1)\}$. In this context there are exactly two reachability clusters, $\nabla = \{\theta_1, \theta_2, \theta_3\}$ and $\{(g_1, m_1, b_1)\}$.

Example 1. In general, triconcepts might be structured in more than one cluster, as the following examples show. A more profound discussion about the depth and width of the ordered set of reachability clusters will be given in Section 5.

(1) A tricontext with more than two clusters:

b1	m1	m2	m3
g1	×		
g2			
g3			

b2	m1	m2	m3
g1	×		
g2		×	
g3			

b3	m1	m2	m3
g1	×		
g2		×	
g3			×

The triconcepts are partitioned in clusters the following way:
$C_1 = \{(\{g_3\}, \{m_3\}, \{b_3\})\}, C_2 = \{(\{g_2\}, \{m_2\}, \{b_2, b_3\})\}, C_3 = \{(\{g_1\}, \{m_1\}, \{b_1, b_2, b_3\}), (\{g_1, g_2, g_3\}, \{m_1, m_2, m_3\}, \emptyset), (\{g_1, g_2, g_3\}, \emptyset, \{b_1, b_2, b_3\}), (\emptyset, \{m_1, m_2, m_3\}, \{b_1, b_2, b_3\})\}$, and $C_1 \leq C_2 \leq C_3$. Thereby, the triconcepts $(\{g_3\}, \{m_3\}, \{b_3\})$ and $(\{g_2\}, \{m_2\}, \{b_2, b_3\})$ have disjoint extents and intents, but $(\{g_3\}, \{m_3\}, \{b_3\}) \prec_3 (\{g_2\}, \{m_2\}, \{b_2, b_3\})$.

(2) A tricontext with exactly two clusters

b1	m1	m2
g1	×	
g2		×

b2	m1	m2
g1		
g2		×

b3	m1	m2
g1		
g2	×	

The triconcepts are partitioned in clusters the following way:
$C_1 = \{(\{g_1\}, \{m_1\}, \{b_1\}), (\{g_2\}, \{m_2\}, \{b_1, b_2\}), (\{g_2\}, \{m_1\}, \{b_3\})\}$, $C_2 = \{(\{g_1, g_2\}, \{m_1, m_2\}, \emptyset), (\{g_1, g_2\}, \emptyset, \{b_1, b_2, b_3\}), (\emptyset, \{m_1, m_2\}, \{b_1, b_2, b_3\})\}$, and $C_1 \leq C_2$.

(3) A tricontext with a single cluster

b1	m1	m2	m3
g1	×		
g2			
g3			

b2	m1	m2	m3
g1		×	
g2			
g3		×	

b3	m1	m2	m3
g1	×	×	
g2			
g3			

The triconcepts are the following:
$C = \{(\{g_1\}, \{m_1\}, \{b_1, b_3\}), (\{g_1\}, \{m_2\}, \{b_2, b_3\}), (\{g_1\}, \{m_1, m_2\}, \{b_3\}), (\{g_1, g_3\}, \{m_2\}, \{b_2\}), (\{g_1, g_2, g_3\}, \{m_1, m_2, m_3\}, \emptyset), (\{g_1, g_2, g_3\}, \emptyset, \{b_1, b_2, b_3\}), (\emptyset, \{m_1, m_2, m_3\}, \{b_1, b_2, b_3\})\}$.

5 Reachability in Composed Tricontexts

There is a way of composing several tricontexts such that the reachability clusters of the composed tricontext coincide with the union of the reachability clusters of the constituents, except for the greatest cluster. We will exploit this correspondency later.

Definition 9. *Given tricontexts* $\mathbb{K}_1 := (K_1^1, K_2^1, K_3^1, Y^1), \ldots, \mathbb{K}_n := (K_1^n, K_2^n, K_3^n, Y^n)$*, with* K_j^i *and* K_k^i *being disjoint for all* $j \neq k$ *and all* $i \in \{1,2,3\}$*, their* composition $\mathbb{K}_1 \uplus \ldots \uplus \mathbb{K}_n$ *is the tricontext* $\mathbb{K} := (K_1, K_2, K_3, Y)$ *with* $K_i := \bigcup_{k=1}^n K_i^k$ *and* $Y := \bigcup_{k=1}^n Y^k$*.*

Proposition 8. *Let* $(K_1, K_2, K_3, Y) = \mathbb{K}_1 \uplus \ldots \uplus \mathbb{K}_n$ *with* $n \geq 2$ *and all* K_j^i *being non-empty. Then* (A_1, A_2, A_3) *is a triconcept of* (K_1, K_2, K_3, Y) *iff*

- A_1, A_2, A_3 *are all non-empty and* (A_1, A_2, A_3) *is a triconcept of some* $\mathbb{K}_j$ *or*
- (A_1, A_2, A_3) *is one of* $(\emptyset, K_2, K_3)$ *or* $(K_1, \emptyset, K_3)$ *or* $(K_1, K_2, \emptyset)$*.*

Proof. "If": First, consider a triconcept (A_1, A_2, A_3) of some $\mathbb{K}_j$ with A_1, A_2, and A_3 nonempty. Now suppose (A_1, A_2, A_3) were not a triconcept of $\mathbb{K}$, i.e., at least one of A_1, A_2, A_3 can be enlarged. W.l.o.g., assume some $a \in K_1 \setminus A_1$ with $(A_1 \cup \{a\}) \times A_2 \times A_3 \subseteq Y$. Now, for $a_2 \in A_2$ and $a_3 \in A_3$, we have $(a, a_2, a_3) \in Y$, implying $a \in K_j$ and thus $(A_1 \cup \{a\}) \times A_2 \times A_3 \in Y_j$, contradicting that (A_1, A_2, A_3) is a triconcept of $\mathbb{K}_j$.

Second, $(A_1, A_2, A_3) = (\emptyset, K_2, K_3)$ is maximal unless for some a holds $\{a\} \times K_2 \times K_3 \subseteq Y$. Yet this contradicts the construction of Y. The cases of $(K_1, \emptyset, K_3)$ and $(K_1, K_2, \emptyset)$ follow by symmetry.

"Only if": for any triconcept (A_1, A_2, A_3) of (K_1, K_2, K_3, Y) with nonempty A_1, A_2, A_3, we find an $(a_1, a_2, a_3) \in A_1 \times A_2 \times A_3$. By construction, for every such (a_1, a_2, a_3) must exist some j with $a_1 \in K_1^j$ and $a_2 \in K_2^j$ and $a_3 \in K_3^j$. Consequently, $A_1 \subseteq K_1^j$ and $A_2 \subseteq K_2^j$ and $A_3 \subseteq K_3^j$. Moreover, maximality of (A_1, A_2, A_3) in (K_1, K_2, K_3, Y) implies maximality in $\mathbb{K}_j$.

Finally if one of the components of (A_1, A_2, A_3) is empty, the other two must be maximal by definition. □

Proposition 9. *Let* $\mathbb{K} = (K_1, K_2, K_3, Y) = \mathbb{K}_1 \uplus \ldots \uplus \mathbb{K}_n$ *with* $n \geq 2$ *and all* K_j^i *being non-empty. Then* (B_1, B_2, B_3) *is directly reachable from* (A_1, A_2, A_3) *in* $\mathbb{K}$ *iff*

- *they are triconcepts of the same* $\mathbb{K}_j$ *and* (B_1, B_2, B_3) *is directly reachable from* (A_1, A_2, A_3) *in* $\mathbb{K}_j$ *or*
- *one of* B_1, B_2, B_3 *is empty.*

Proof. "If": First assume (A_1, A_2, A_3) is directly reachable from (B_1, B_2, B_3) and both are are triconcepts of the same $\mathbb{K}_j$. W.l.o.g. let (1) be the corresponding perspective. Then $A_1 \subseteq B_1$. Moreover, none of A_1, A_2, A_3 is empty (otherwise (A_1, A_2, A_3) cannot be a triconcept of $\mathbb{K}_j$ due to Proposition 8). We find that $(B_2, B_3) \in \mathfrak{B}({\mathbb{K}_j}_{A_1}^{(23)})$. This implies $(B_2, B_3) \in \mathfrak{B}(\mathbb{K}_{A_1}^{(23)})$, thus (B_1, B_2, B_3) is directly reachable from (A_1, A_2, A_3) in $\mathbb{K}$.

Next, assume that one of B_1, B_2, B_3 is empty. W.l.o.g. assume $B_1 = \emptyset$. By Proposition 8, this entails $B_2 = K_2$ and $B_3 = K_3$. Then $(\emptyset, K_3) \in \mathfrak{B}(\mathbb{K}_{A_2}^{(13)})$

whenever $A_2 \neq \emptyset$ and $(\emptyset, K_2) \in \mathfrak{B}(\mathbb{K}_{A_3}^{(12)})$ whenever $A_3 \neq \emptyset$ (it is not possible that $A_2 = \emptyset = A_3$), therefore $(A_1, A_2, A_3) \prec (B_1, B_2, B_3)$ holds in $\mathbb{K}$.

"Only if": Assume $(A_1, A_2, A_3) \prec_i (B_1, B_2, B_3)$ in $\mathbb{K}$ and all of B_1, B_2, B_3 are nonempty. W.l.o.g. assume $i = 1$, i.e., $(B_2, B_3) \in \mathfrak{B}(\mathbb{K}_{A_1}^{(23)})$. Proposition 8 implies that (B_1, B_2, B_3) must be a triconcept of some $\mathbb{K}_j$. Then, due to $\emptyset \neq A_1 \subseteq B_1 \subseteq K_1^j$ we find that (A_1, A_2, A_3) cannot be a trivial triconcept, thus it is a triconcept of $\mathbb{K}_j$. Then $(B_2, B_3) \in \mathfrak{B}(\mathbb{K}_{A_1}^{(23)})$ implies $(B_2, B_3) \in \mathfrak{B}({\mathbb{K}_j}_{A_1}^{(23)})$ thus $(A_1, A_2, A_3) \prec_1 (B_1, B_2, B_3)$ holds in $\mathbb{K}_j$. □

Corollary 2. *Let $\mathbb{K} = (K_1, K_2, K_3, Y) = \mathbb{K}_1 \uplus \ldots \uplus \mathbb{K}_n$ with $n \geq 2$ and all K_j^i being non-empty. Then (B_1, B_2, B_3) is reachable from (A_1, A_2, A_3) in $\mathbb{K}$ iff*

- *they are triconcepts of the same $\mathbb{K}_j$ and (B_1, B_2, B_3) is reachable from (A_1, A_2, A_3) in $\mathbb{K}_j$ or*
- *one of B_1, B_2, B_3 is empty.*

Proof. This is a straightforward consequence of the previous proposition and the fact that all trivial triconcepts (those having one empty component) are together in the maximal cluster. □

Using the above results, we ask if there is any correlation between the cardinality of the three sets of a tricontext and the number of the reachability clusters we obtain. The first observation was that we can find qubic tricontexts (i.e., $|K_1| = |K_2| = |K_3| = n$), where the number of clusters equals $n + 1$.

Proposition 10. *Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a tricontext of size $n \times n \times n$ with $K_1 = \{k_i^1 \mid 1 \leq i \leq n\}$, $K_2 = \{k_i^2 \mid 1 \leq i \leq n\}$, $K_3 = \{k_i^3 \mid 1 \leq i \leq n\}$. Let the relation Y be the spatial main diagonal of the tricontext, meaning that a triple $(k_i^1, k_j^2, k_l^3) \in Y$ iff $i = j = k$. Then there are $n+1$ clusters, n minimal clusters and the maximal cluster.*

Proof. Considering Proposition 9, the conclusion is immediate, since $\mathbb{K} = (\{k_1^1\}, \{k_1^2\}, \{k_1^3\}, \{(k_1^1, k_1^2, k_1^3)\}) \uplus \ldots \uplus (\{k_n^1\}, \{k_n^2\}, \{k_n^3\}, \{(k_n^1, k_n^2, k_n^3)\})$ □

Based on this example, we assumed that the number of clusters is bounded by the minimal dimension of the tricontext plus one. This assumption proved to be false due to the following example.

Example 2. Consider the following $4 \times 6 \times 6$ tricontext $\mathbb{K}_{466}$.

α	1	2	3	4	5	6
a	×					
b		×				
c			×			
d						
e						
f						

β	1	2	3	4	5	6
a	×					
b						
c						
d				×		
e					×	
f						

γ	1	2	3	4	5	6
a						
b		×				
c						
d				×		
e						
f						×

δ	1	2	3	4	5	6
a						
b						
c			×			
d						
e					×	
f						×

Besides the maximal cluster, we have six minimal ones which are all singletons consisting of the following triconcepts, respectively:
$C_1 := (\{a_1\},\{b_1\},\{c_1,c_2\}), C_2 := (\{a_2\},\{b_2\},\{c_1,c_3\}), C_3 := (\{a_3\},\{b_3\},\{c_1,c_4\}),$
$C_4 := (\{a_4\},\{b_4\},\{c_2,c_3\}), C_5 := (\{a_5\},\{b_5\},\{c_2,c_4\}), C_6 := (\{a_6\},\{b_6\},\{c_3,c_4\}).$

Another assumption, about the number of cluster assumed to be the maximal dimension of the tricontext plus one, could be disproven.

Example 3. Given the tricontext $\mathbb{K}_{466} = (G, M, B, Y)$ from Example 2, we define $\mathbb{K}_{646} := (B, G, M, \{(b, g, m)|(g, m, b) \in Y\})$ as well as
$\mathbb{K}_{664} := (M, B, G, \{(m, b, g)|(g, m, b) \in Y\})$, in words, we obtain $\mathbb{K}_{646}$ and $\mathbb{K}_{664}$ by rotating $\mathbb{K}_{466}$ twice. We now let $\mathbb{K}_{16^3} := \mathbb{K}_{466} \uplus \mathbb{K}_{646} \uplus \mathbb{K}_{664}$ be the $16 \times 16 \times 16$ context built by composing the three. Combining Example 2 with Corollary 2, we obtain that $\mathbb{K}_{16^3}$ has 19 clusters, viz. the maximal one and $6 + 6 + 6 = 18$ minimal ones.

Remark 2. The issue of whether the number of clusters or minimal clusters is bounded and what could be an estimation of that bound remains an open question.

6 Properties of reachability clusters

This section is devoted to the study of several properties of reachability clusters. We prove that reachability clusters can be found among some concepts of the context of reachable triconcepts, more exactly as object concepts.

Proposition 11. *Let* $(A_1, B_1, C_1), (A_2, B_2, C_2) \in \mathfrak{T}(\mathbb{K})$ *with* $(A_1, B_1, C_1) \prec_3 (A_2, B_2, C_2)$. *Then* $Y^{12}_{C_2} \subseteq Y^{12}_{C_1}$.

Proof. Let $(g, m) \in Y^{12}_{C_2}$. Then, for every $b \in C_2$, we have $(g, m, b) \in Y$. Since $C_1 \subseteq C_2$, we have that for every $b \in C_1$, $(g, m, b) \in Y$, hence $(g, m) \in Y^{12}_{C_1}$. □

Let $\mathbb{K} := (K_1, K_2, K_3, Y)$ be a triadic context. Let $\mathbb{K}_\triangleleft := (\mathfrak{T}(\mathbb{K}), \mathfrak{T}(\mathbb{K}), \triangleleft)$ be the formal context of reachable triconcepts. The concepts of $\mathbb{K}_\triangleleft$ are exactly the pairs (A, B) having the property that every triconcept from B is reachable from any of A and (A, B) is maximal with this property. If we take a look at the concepts of the symmetric kernel of $\triangleleft$, i.e., $\triangleleft \cap \triangleleft^{-1}$, we get exactly the reachability clusters of triconcepts without the ordering between them.

Proposition 12. *Let* $(A, B) \in \mathbb{K}_\triangleleft$ *be a concept and denote by* $C := A \cap B$. *If* $C \neq \emptyset$, *then* C *is a set of mutually reachable concepts, i.e.,* $C \times C$ *is a rectangle of crosses in* $\mathbb{K}_\triangleleft$.

Proof. From the definition, we have that $\forall T_1, T_2 \in C, T_1 \triangleleft T_2$ and $T_2 \triangleleft T_1$. It follows that all the triconcepts from C are part of the same cluster. □

Remark 3. If we denote with $\mathcal{C}$ the set of clusters of triconcepts from $\mathbb{K}$ and with $\mathcal{I} := \{A \cap B \mid (A, B) \in \mathfrak{B}(\mathbb{K}_\triangleleft), A \cap B \neq \emptyset\}$, i.e., the set of all concepts having non disjoint extent and intent, then the previous proposition states that $\mathcal{I} \subset \mathcal{C}$.

Example 4. One might expect that there exist a one-to-one correspondence between concepts in the context of reachable triconcepts and reachability clusters. This would mean that the structure of reachability clusters is a concept lattice. The following example shows that there exist concepts in $\mathbb{K}_\triangleleft$, having disjoint extent and intent.

α	1	2	3	4
a	×			
b				
c		×		
d		×		

β	1	2	3	4
a				
b				
c		×		
d		×		

γ	1	2	3	4
a			×	
b			×	
c				
d				

δ	1	2	3	4
a			×	
b			×	
c				
d				×

We have $(\{a\},\{1\},\{\alpha\}) \triangleleft (\{a,b\},\{3\},\{\gamma,\delta\})$, $(\{a\},\{1\},\{\alpha\}) \triangleleft (\{c,d\},\{2\},\{\alpha,\beta\})$, $(\{d\},\{4\},\{\delta\}) \triangleleft (\{a,b\},\{3\},\{\gamma,\delta\})$, $(\{d\},\{4\},\{\delta\}) \triangleleft (\{c,d\},\{2\},\{\alpha,\beta\})$, and the context $\mathbb{T}$ is given by:

	T_1	T_2	T_3	T_4	T_5	T_6	T_7
$T_1 = (\{a\},\{1\},\{\alpha\})$	×		×	×	×	×	×
$T_2 = (\{d\},\{4\},\{\delta\})$		×	×	×	×	×	×
$T_3 = (\{a,b\},\{3\},\{\gamma,\delta\})$			×		×	×	×
$T_4 = (\{c,d\},\{2\},\{\alpha,\beta\})$				×	×	×	×
$T_5 = (\{a,b,c,d\},\{1,2,3,4\},\emptyset)$					×	×	×
$T_6 = (\{a,b,c,d\},\emptyset,\{\alpha,\beta,\gamma,\gamma\})$					×	×	×
$T_7 = (\emptyset,\{1,2,3,4\},\{\alpha,\beta,\gamma,\gamma\})$					×	×	×

The concept $(\{T_1,T_2\},\{T_3,T_4,T_5,T_6,T_7\})$ has disjoint extent and intent.

In the following, we are going to characterize the reachability clusters as object concepts in the context of reachable triconcepts.

Proposition 13. *Let C be a reachability cluster of triconcepts. Then there exists a concept in $(A,B) \in \mathbb{K}_\triangleleft$ with $C = A \cap B$.*

Proof. Consider $(C^{\triangleleft\triangleleft}, C^{\triangleleft})$. □

Proposition 14. *If $(A,B),(C,D) \in \mathfrak{B}(\mathbb{K}_\triangleleft)$ are two different concepts of the context $\mathbb{K}_\triangleleft$ of reachable triconcepts with $A \cap B \neq \emptyset$ and $C \cap D \neq \emptyset$, then $A \cap B \neq C \cap D$.*

Proof. Let $(A,B),(C,D) \in \mathfrak{B}(\mathbb{K}_\triangleleft)$ be two different concepts. We assume $A \cap B = C \cap D = M \neq \emptyset$. Since they are different concepts, we can conclude that they have different extents and intents, so $A \neq C$ and $B \neq D$. It follows that at least one of the extents and one of the intents is bigger than M.

If $A \neq M$, $B \neq M$, $C = M$ and $D = M$ (or the other way around) it contradicts the fact that $(C,D) \in \mathfrak{B}(\mathbb{K}_\triangleleft)$ because it is not maximal. We can conclude that at least the extent of one concept and the intent of the other concept are bigger than M. Hence, we can assume $A \neq M$ and $D \neq M$. Let $T_1 \in$

$A \setminus M, T_2 \in M, T_3 \in D \setminus M$. Since $T_2 \in M \subseteq B$ it follows $(T_1, T_2) \in I \Rightarrow T_1 \triangleleft T_2$. Since $T_2 \in M \subseteq C$ it follows $(T_2, T_3) \in I \Rightarrow T_2 \triangleleft T_3$. From the transitivity of the relation $\triangleleft$ we have $T_1 \triangleleft T_3$. Herefrom we conclude that for every $T \in A \setminus M$, we have $(T, T_3) \in I$, but since $M \subseteq C$ and $T_3 \in D$, we also have that for every $T \in M$, we have $(T, T_3) \in I$. It follows that T_3 should be in the intent of the concept (A, B), so $T_3 \in B \Rightarrow T_3 \in B \cap D = M$ and we reach a contradiction since we chose $T_3 \in D \setminus M$. Therefore, the two different concepts in $\mathfrak{B}(\mathbb{K}_\triangleleft)$ cannot have the same intersection of the extent and intent. □

Remark 4. The previous proposition proves that, by intersecting the extent and intent of the concepts in the context $\mathbb{K}_\triangleleft$ we cannot obtain the same cluster twice.

Proposition 15. *Let $(A, B), (C, D) \in \mathfrak{B}(\mathbb{K}_\triangleleft)$. Let $M := A \cap B \neq \emptyset$ and $N := C \cap D \neq \emptyset$. Then $M \cap N = \emptyset$.*

Proof. By the previous proposition, we know that $M \neq N$. Assume $M \cap N \neq \emptyset$. Let $a \in A \setminus M$, $b \in B \setminus M$, $x \in M \setminus (M \cap N)$, $y \in M \cap N$, and $z \in N \setminus (M \cap N)$ be arbitrary elements. Then we have $a \triangleleft y$ and $y \triangleleft z$ wherefrom follows that $a \triangleleft z$. Similarly, we have that from $x \triangleleft y$ and $y \triangleleft z$ follows $x \triangleleft z$. We also have that $y \triangleleft z$, hence for all $g \in A, g \triangleleft z$, i.e., $z \in A^\triangleleft = B$.

On the other hand, $z \triangleleft y \triangleleft x \triangleleft b$, hence $z \triangleleft b$ for all $b \in B$, i.e., $z \in B^\triangleleft = A$. We have that $z \in A \cap B = M$ and $z \in N$, thus $z \in M \cap N$. Contradiction! □

Proposition 16. *The sets defined in Remark 3, $\mathcal{C}$ and $\mathcal{I}$, are equal: $\mathcal{C} = \mathcal{I}$.*

Proof. The first part of the equivalence was proved in Proposition 12 which showed that $\mathcal{I} \subseteq \mathcal{C}$. For the converse inclusion, let $(A, B) \in \mathfrak{B}(\mathbb{K}_\triangleleft)$ be a concept and $M := A \cap B$. Assume M is not maximal and build $M' := M^{\triangleleft\triangleleft} \cap M^\triangleleft$. Then $M \subseteq M'$ which is a contradiction. □

Proposition 17. *Let $(A, B) \in \mathfrak{B}(\mathbb{K}_\triangleleft)$ with $A \cap B = \emptyset$. Then (A, B) is a concept of the contraordinal scale $(\mathfrak{T}(\mathbb{K}), \mathfrak{T}(\mathbb{K}), \not\trianglerighteq)$.*

Proof. Since $\triangleleft$ is a preorder, it makes sense to speak about the contraordinal scale induced by $\triangleleft$. The concepts of the contraordinal scale are exactly the pairs (A, B) with A oder ideal, B order filter, $A \cap B = \emptyset$, and $A \cup B = \mathfrak{T}(\mathbb{K})$. Let $(A, B) \in \mathfrak{B}(\mathbb{K}_\triangleleft)$ with $A \cap B = \emptyset$. Then for every $a \in A$ and every $b \in B$, we have $x \triangleleft y$ and $y \not\triangleleft x$, i.e., $x \not\trianglerighteq y$.

Let now $x \in A$ and $z \triangleleft x$. By transitivity, we get that for every $b \in B$, $z \triangleleft b$ and $z \in A$. Hence A is an order ideal. Dually, B is an order filter. We only have to prove that $B = \complement A$. Let $y \in \complement A$. Then for every $a \in A$, $y \not\triangleleft a$, which is equivalent to $a \not\trianglerighteq y$, i.e., $y \in B$. □

Concluding all the results obtained above, the following holds true:

Proposition 18. *Let $T \in \mathfrak{T}(\mathbb{K})$ be a triconcept. Then the cluster $[T]$ of T is generated by the object concept $\gamma(T)$ by $T^{\triangleleft\triangleleft} \cap T^\triangleleft = [T]$. Herefrom follows that the reachability clusters are generated by the object concepts of $(\mathfrak{T}(\mathbb{K}), \mathfrak{T}(\mathbb{K}), \triangleleft)$. If (A, B) is a proper concept which is not an object concept, then $A \cap B = \emptyset$.*

Remark 5. The above proposition states that reachability clusters are exactly the object concepts of the reachability context. This result gives a possibility to display all reachability clusters, along with a navigation support in a concept lattice, by highlighting the object concepts and deleting all the others, except the greatest concept.

7 Exploration strategy and algorithmics

Considering the theoretical aspects introduced in the previous paragraphs, we use reachability clusters to propose a strategy for navigating inside and between them. The purpose of this approach is to obtain a tool that can be used for navigation and visualization of a triadic context. Basically, starting from a triconcept, one can browse through all the others from the reachability cluster of that triconcept, navigate to another triconcept (not necessarily in the same cluster), moving back and forth among these triconcepts in order to explore as much as possible the triadic conceptual knowledge landscape.

In order to be able to navigate through the data with the proposed paradigm the following steps are necessary:

(1) compute the triconcepts,
(2) compute the reachability relation between the triconcepts,
(3) compute the clusters of the tricontext,
(4) compute the partial order relation between the clusters.

The first step can be implemented by using Trias [9]. For the second step, we use the following procedure.

Listing 1.1: Procedure directlyReachable(T1, T2)

```
If T1.extent ⊆ T2.extent then
    B_e = extentProjectionContext(T1.extent)
    If (T2.intent)'_{B_e} =T2.modus and
    (T2.modus)'_{B_e}=T2.intent then
        Return true

If T1.intent ⊆ T2.intent then
    B_i = intentProjectionContext(T1.intent)
    If (T2.extent)'_{B_i} =T2.modus and
    (T2.modus)'_{B_i}=T2.extent then
        Return true

If T1.modus ⊆ T2.modus then
    B_m = modusProjectionContext(T1.modus)
    If (T2.extent)'_{B_m} =T2.intent and
    (T2.intent)'_{B_m}=T2.extent then
        Return true

Return false
```

The procedure checks whether the triconcept T_2 is directly reachable from the triconcept T_1. Thereby, $extentProjectionContext(T1.extent)$ represents the dyadic context obtained by projecting the triadic context on the extent dimension. This means that all the tricontexts selected have the extent equal or greater than $T1.extent$. The derivation $(T2.intent)'_{B_e}$ is considered to be a dyadic derivation in the obtained projection context B_e.

In order to obtain the reachability clusters, the most efficient method is to obtain the graph of the triconcepts with the direct reachability relation and compute the strongly connected components. This can be done by using existing algorithms for computing strongly connected components in directed graphs which have linear complexity. So for step 3, we consider the directed graph (since the direct reachability is not a symmetric relation) having the triconcept set as nodes and the edges given by the direct reachability relation. Then we obtain the clusters by computing all strongly connected components of the graph.

As proven earlier in the theoretical aspects of the navigation paradigm, the clusters correspond to nodes in a lattice, but not all the nodes in the lattice correspond to a cluster. Therefrom, the set of clusters is a partially ordered. The fact that they are object concepts in a particular concept lattice helps us navigate form one cluster to another. Also, this assures that we can reach any triconcept from the tricontext.

8 Conclusions and Further Work

We have proposed an approach to navigating in the space of triconcepts of a tricontext. To this end we defined three relatedness notions on the triconcepts based on extent, intent or modus. For each of these three perspectives, the triconcepts related to a given tricontext correspond to the concepts of a dyadic formal context, whence we can leverage the successful visualization approach of dyadic FCA by displaying, given a triconcept, all similar triconcepts in a lattice diagram. From such a diagram, a triconcept can be picked by a user, which will then be the starting point for the next visualization and navigation step.

We have investigated the reachability relation stemming from this navigation paradigm. As it turned out, for some tricontexts, not every triconcept can be reached from every other triconcept, although this seems to be the case in most practical scenarios. This gave rise to the notion of reachability clusters obtained as maximal sets of mutually reachable triconcepts. These clusters are in turn ordered by unidirectional reachability and form a partial order which always has a greatest element. Navigation can start either in one of the minimal clusters or the user can define its own constraints about included/excluded objects, attributes and/or conditions. By computing all triconcepts satisfying a given set of constraints, the user can choose them as navigation starting points. Not much more is known about the order of reachability clusters, some initial conjectures about upper bounds on their size or existence of suprema had to be refuted by counterexamples, which nevertheless provided some interesting structural insights and may pave the way to further investigations. As of yet, the only (and trivial) upper bound for the number of reachability clusters is the number of triconcepts, which may be exponential in the size of the tricontext. We, however, still conjecture that there is a polynomial bound.

Besides these open theoretical questions, future work on the topic has to include an implementation of the described navigation paradigm and user studies in order to confirm our hypothesis that this way of displaying and browsing the space of triconcepts is indeed accessible and intuitive for human users.

References

1. Cerf, L., Besson, J., Robardet, C., Boulicaut, J.: Closed patterns meet n-ary relations. TKDD 3(1) (2009)
2. Ferré, S., Ridoux, O.: Introduction to logical information systems. Inf. Process. Manage. 40(3), 383–419 (2004)
3. Ganter, B., Obiedkov, S.A.: Implications in triadic formal contexts. In: Wolff, K.E., Pfeiffer, H.D., Delugach, H.S. (eds.) Conceptual Structures at Work: 12th International Conference on Conceptual Structures, ICCS 2004, Huntsville, AL, USA, July 19-23, 2004. Proceedings. Lecture Notes in Computer Science, vol. 3127, pp. 186–195. Springer (2004)
4. Ganter, B., Wille, R.: Formal concept analysis - mathematical foundations. Springer (1999)
5. Glodeanu, C.V.: Tri-ordinal factor analysis. In: Cellier, P., Distel, F., Ganter, B. (eds.) Formal Concept Analysis, 11th International Conference, ICFCA 2013, Dresden, Germany, May 21-24, 2013. Proceedings. Lecture Notes in Computer Science, vol. 7880, pp. 125–140. Springer (2013)
6. Gnatyshak, D., Ignatov, D.I., Kuznetsov, S.O.: From triadic FCA to triclustering: Experimental comparison of some triclustering algorithms. In: Ojeda-Aciego, M., Outrata, J. (eds.) Proceedings of the Tenth International Conference on Concept Lattices and Their Applications, La Rochelle, France, October 15-18, 2013. CEUR Workshop Proceedings, vol. 1062, pp. 249–260. CEUR-WS.org (2013)
7. Godin, R., Missaoui, R., April, A.: Experimental comparison of navigation in a galois lattice with conventional information retrieval methods. International Journal of Man-Machine Studies 38(5), 747–767 (1993)
8. Ignatov, D.I., Kuznetsov, S.O., Poelmans, J., Zhukov, L.E.: Can triconcepts become triclusters? Int. J. General Systems 42(6), 572–593 (2013)
9. Jäschke, R., Hotho, A., Schmitz, C., Ganter, B., Stumme, G.: TRIAS - an algorithm for mining iceberg tri-lattices. In: Proceedings of the 6th IEEE International Conference on Data Mining (ICDM 2006). pp. 907–911. IEEE Computer Society (2006)
10. Jäschke, R., Hotho, A., Schmitz, C., Ganter, B., Stumme, G.: Discovering shared conceptualizations in folksonomies. Journal of Web Semantics 6(1), 38–53 (2008)
11. Lehmann, F., Wille, R.: A triadic approach to formal concept analysis. In: Ellis, G., Levinson, R., Rich, W., Sowa, J.F. (eds.) Proceedings of the Third International Conference on Conceptual Structures, ICCS '95. LNCS, vol. 954, pp. 32–43. Springer (1995)
12. Wille, R.: The basic theorem of triadic concept analysis. Order 12(2), 149–158 (1995)