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Complexity Theory

Exercise 1: Mathematica Foundations, and Decidability and Recognisability

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Exercise 1.1. Let M be a set. Show that there is no function

$$f\colon M \to 2^M$$

such that f is surjective.

Exercise 1.2. Show the following claims.

1. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
2. $|\mathbb{N}| = |\mathbb{Q}|$.
3. $|\mathbb{N}| \neq |\mathbb{R}|$.

Exercise 1.3. Show the following claims.

1. There exist non-regular languages.
2. There exist undecidable languages.
3. There exist non-Turing-recognizable languages.

Exercise 1.4. Let $G = \{V, E\}$ be a simple undirected graph such that $|V| \geq 2$. Show that G contains two or more nodes that have equal degree. That is, show that that there is a pair of nodes that occur in the same number of edges.

Exercise 1.5. Let $A = \{\, s \,\}$, where

$$s := \begin{cases} 0 & \text{if life will never be found on Mars,} \\ 1 & \text{if life will be found on Mars someday.} \end{cases}$$

Is A decidable? (For the purpose of this problem, assume that the question whether life will be found on Mars has an unambiguous "yes" or "no" answer.)

Exercise 1.6. Show that the class of Turing-decidable languages is closed under (1) union, (2) concatenation, (3) intersection, and (4) star.

$*$ **Exercise 1.7.** Show that the class of Turing-recognizable languages is closed under homomorphism.

Exercise 1.8. A *Turing machine with two-sided unbounded tape* is a single-tape Turing machine where the tape is unbounded on both sides. Argue that such machines can be simulated by ordinary Turing machines.

Exercise 1.9. Let $\mathsf{ALL}_{\mathsf{DFA}} = \{ \langle A \rangle \mid A \text{ is a DFA that accepts every word} \}$. Show that $\mathsf{ALL}_{\mathsf{DFA}}$ is decidable.

Exercise 1.10. Let $\mathsf{E}_{\mathsf{TM}} = \{ \langle M \rangle \mid M \text{ is a TM such that } \mathcal{L}(M) = \emptyset \}$. Show that $\overline{\mathsf{E}_{\mathsf{TM}}}$ is Turing-recognizable.

Exercise 1.11. Let C be a language. Prove that C is Turing-recognizable if and only if a decidable language D exists such that $C = \{ x \mid \exists y. \langle x, y \rangle \in D \}$.