TECHNISCHE UNIVERSITÄT DRESDEN

# FOUNDATIONS OF COMPLEXITY THEORY

**Lecture 2: Turing Machines and Languages**

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TU Dresden, October 27, 2020

## A Model for Computation

### Clear

To understand computational problems we need to have a formal understanding of what an **algorithm** is.

**Example 2.1 (Hilbert's Tenth Problem):**
"Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers."
(→ Wikipedia)

### Question

How can we model the notion of an algorithm?

### Answer

With Turing machines.



## Turing Machines

Let us fix a blank symbol ␣.

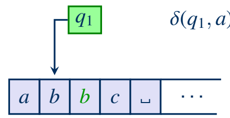
**Definition 2.2:** A (deterministic) Turing Machine $\mathcal{M} = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ consists of

- a finite set $Q$ of states,
- an input alphabet $\Sigma$ not containing ␣,
- a tape alphabet $\Gamma$ such that $\Gamma \supseteq \Sigma \cup \{␣\}$.
- a transition function $\delta \colon Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$
- an initial state $q_0 \in Q$,
- an accepting state $q_{\text{accept}} \in Q$, and
- an rejecting state $q_{\text{reject}} \in Q$ such that $q_{\text{accept}} \neq q_{\text{reject}}$.



## Turing Machines

**Example 2.3:**




$\delta(q_1, a) = (q_2, b, \mathsf{L})$

- The tape is bounded on the left, but unbounded on the right; the content of the tape is a finite word over $\Gamma$, followed by an infinite sequence of ␣.
- The head of the machine is at exactly one position of the tape
- The head can read only one symbol at a time
- The head moves and writes according to the transition function $\delta$; the current state also changes accordingly
- The head will stay put when attempting to cross the left tape end

## Configurations

Observation: to describe the current step of a computation of a TM it is enough to know

- the content of the tape,
- the current state, and
- the position of the head

**Definition 2.4:** A configuration of a TM $\mathcal{M}$ is a word $uqv$ such that

- $q \in Q$,
- $uv \in \Gamma^*$

Some special configurations:

- The **start configuration** for some input word $w \in \Sigma^*$ is the configuration $q_0 w$
- A configuration $uqv$ is **accepting** if $q = q_{\text{accept}}$.
- A configuration $uqv$ is **rejecting** if $q = q_{\text{reject}}$.



## Computation

We write

- $C \vdash_{\mathcal{M}} C'$ only if $C'$ can be reached from $C$ by one computation step of $\mathcal{M}$;
- $C \vdash^*_{\mathcal{M}} C'$ only if $C'$ can be reached from $C$ in a finite number of computation steps of $\mathcal{M}$.

We say that $\mathcal{M}$ **halts** on input $w$ if and only if there is a finite sequence of configurations

$$C_0 \vdash_{\mathcal{M}} C_1 \vdash_{\mathcal{M}} \cdots \vdash_{\mathcal{M}} C_\ell$$

such that $C_0$ is the start configuration of $\mathcal{M}$ on input $w$ and $C_\ell$ is an accepting or rejecting configuration. Otherwise $\mathcal{M}$ **loops** on input $w$.

We say that $\mathcal{M}$ **accepts** the input $w$ only if $\mathcal{M}$ halts on input $w$ with an accepting configuration.



## Recognisability and Decidability

**Definition 2.5:** Let $\mathcal{M}$ be a Turing machine with input alphabet $\Sigma$. The language accepted by $\mathcal{M}$ is the set

$$\mathbf{L}(\mathcal{M}) := \{ w \in \Sigma^* \mid \mathcal{M} \text{ accepts } w \}.$$

A language $\mathbf{L} \subseteq \Sigma^*$ is called Turing-recognisable (recursively enumerable) if and only if there exists a Turing machine $\mathcal{M}$ with input alphabet $\Sigma$ such that $\mathbf{L} = \mathbf{L}(\mathcal{M})$. In this case we say that $\mathcal{M}$ recognises $\mathbf{L}$.

A language $\mathbf{L} \subseteq \Sigma^*$ is called Turing-decidable (decidable, recursive) if and only if there exists a Turing machine $\mathcal{M}$ such that $\mathbf{L} = \mathbf{L}(\mathcal{M})$ and $\mathcal{M}$ halts on every input. In this case we say that $\mathcal{M}$ decides $\mathbf{L}$.



## Example

**Claim 2.6:** The language $\mathbf{L} := \{ \mathtt{a}^{2^n} \mid n \geq 0 \}$ is decidable.

**Proof:** A Turing machine $\mathcal{M}$ that decides $\mathbf{L}$ is

$\mathcal{M}$ := On input $w$, where $w$ is a string

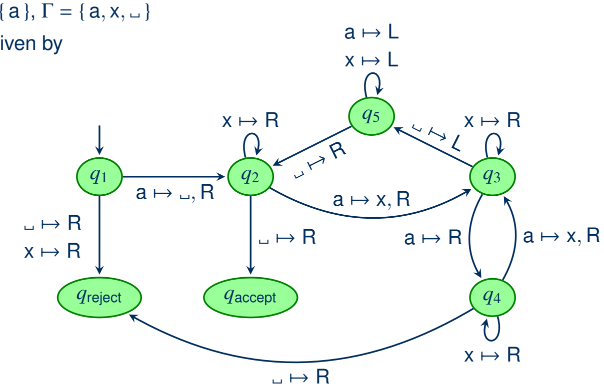
- Go from left to right over the tape and cross off every other a
- If in the first step the tape contained a single a, accept
- If in the first step the number of a's on the tape was odd, reject
- Return the head the beginning of the tape
- Go to the first step

## Example (cont'd)

Formally, $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_1, q_{\text{accept}}, q_{\text{reject}})$, where

- $Q = \{ q_1, q_2, q_3, q_4, q_5, q_{\text{accept}}, q_{\text{reject}} \}$
- $\Sigma = \{ \mathsf{a} \}$, $\Gamma = \{ \mathsf{a}, \mathsf{x}, \sqcup \}$

and $\delta$ is given by






## Problems as Languages

### Observation

- Languages can be used to model computational problems.
- For this, a suitable **encoding** is necessary
- TMs must be able to decode the encoding

**Example 2.7 (Graph-Connectedness):** The question whether a graph is connected or not can be seen as the **word problem** of the following language

$$\text{GCONN} := \{ \langle G \rangle \mid G \text{ is a connected graph} \},$$

where $\langle G \rangle$ is (for example) the adjacency matrix encoded in binary.

**Notation 2.8:** The encoding of objects $O_1, \ldots, O_n$ we denote by $\langle O_1, \ldots, O_n \rangle$.



## The Church-Turing Thesis

It turns out that Turing-machines are **equivalent** to a number of formalisations of the intuitive notion of an **algorithm**

- $\lambda$-calculus
- while-programs
- $\mu$-recursive functions
- Random-Access Machines
- ...

Because of this it is believed that Turing-machines completely capture the intuitive notion of an algorithm. $\rightsquigarrow$ **Church-Turing Thesis**:

> "A function on the natural numbers is intuitively computable if and only if it can be computed by a Turing machine."

(→ Wikipedia: Church-Turing Thesis)



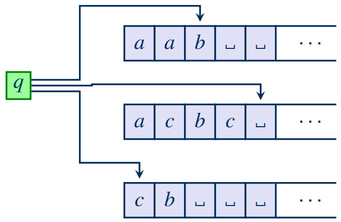
## Variations of Turing-Machines

It has also been shown that deterministic, single-tape Turing machines are equivalent to a wide range of other forms of Turing machines:

- Multi-tape Turing machines
- Nondeterministic Turing machines
- Turing machines with doubly-infinite tape
- Multi-head Turing machines
- Two-dimensional Turing machines
- Write-once Turing machines
- Two-stack machines
- Two-counter machines
- ...

## Multi-Tape Turing Machines

*k*-**tape Turing machines** are a variant of Turing machines that have $k$ tapes.






## Multi-Tape Turing Machines

**Definition 2.9:** Let $k \in \mathbb{N}$. Then a (deterministic) $k$-**tape Turing machine** is a tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where

- $Q$, $\Sigma$, $\Gamma$, $q_0$, $q_{\text{accept}}$, $q_{\text{reject}}$ are as for TMs
- $\delta$ is a transition function for $k$ tapes, i.e.,

$$\delta\colon Q \times \Gamma^k \to Q \times \Gamma^k \times \{\mathsf{L}, \mathsf{R}, \mathsf{N}\}^k$$

**Running $\mathcal{M}$ on input** $w \in \Sigma^*$ means to start $\mathcal{M}$ with the content of the first tape being $w$ and all other tapes blank.

The notions of a **configuration** and of the **language accepted by** $\mathcal{M}$ are defined analogously to the single-tape case.

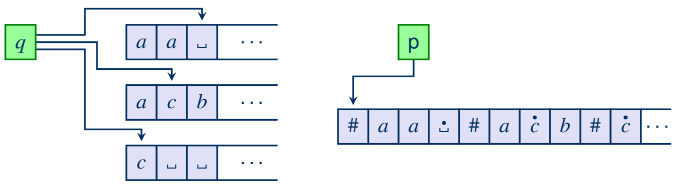


## Multi-Tape Turing Machines

**Theorem 2.10:** Every multi-tape Turing machine has an equivalent single-tape Turing machine.

**Proof:** Let $\mathcal{M}$ be a $k$-tape Turing machine. Simulate $\mathcal{M}$ with a single-tape TM $S$ by

- keeping the content of all $k$ tapes on a single tape, separated by #
- marking the positions of the individual heads using special symbols






## Multi-Tape Turing Machines

$S :=$ On input $w = w_1 \dots w_n$

- Format the tape to contain the word

$$\#\dot{w}_1 w_2 \dots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \dots \#$$

- Scan the tape from the first # to the $(k+1)$-th # to determine the symbols below the markers.
- Update all tapes according to $\mathcal{M}$'s transition function with a second pass over the tape; if any head of $\mathcal{M}$ moves to some previously unread portion of its tape, insert a blank symbol at the corresponding position and shift the right tape contents by one cell
- Repeat until the accepting or rejecting state is reached.

□

## Nondeterministic Turing Machines

### Goal

Allow transitions to be **nondeterministic**.

### Approach

Change transition function from

$$\delta\colon Q \times \Gamma \to Q \times \Gamma \times \{\,\mathsf{L}, \mathsf{R}\,\}$$

to

$$\delta\colon Q \times \Gamma \to 2^{Q \times \Gamma \times \{\,\mathsf{L}, \mathsf{R}\,\}}.$$

The notions of **accepting** and **rejecting computations** are defined accordingly.
Note: there may be more than one or no computation of a nondeterministic TM on a given input.
A nondeterministic TM $\mathcal{M}$ **accepts** an input $w$ if and only if **there exists** some accepting computation of $\mathcal{M}$ on input $w$.



## Nondeterministic Turing Machines

**Theorem 2.11:** Every nondeterministic TM has an equivalent deterministic TM.

**Proof:** Let $N$ be a nondeterministic TM. We construct a deterministic TM $D$ that is equivalent to $N$, i.e., $\mathbf{L}(N) = \mathbf{L}(D)$.

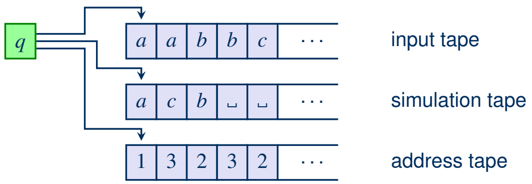
### Idea

- $D$ deterministically traverses in breath-first order the tree of configuration of $N$, where each branch represents a different possibility for $N$ to continue.
- For this, successively try out all possible choices of transitions allowed by $N$.



## Nondeterministic Turing Machines

Sketch of $D$:




Without loss of generality, we assume that the maximal number of choices in $\delta$ is at most 2, i.e.,

$$\max(\{\, |\delta(q, x)| \mid q \in Q, x \in \Gamma \,\}) \leq 2.$$



## Nondeterministic Turing Machines

$D$ works as follows:

(1) Start: input tape contains input $w$, simulation and address tape empty

(2) Initialise the address tape with 0.

(3) Copy $w$ to the simulation tape.

(4) Simulate one finite computation of $N$ on $w$ on the simulation tape.
  - Interpret the address tape as a list of choices to make during this computation.
  - If an accepting configuration is reached at the end of the simulation, accept.

(5) Increment the content of the address tape by 1. Go to step 3.

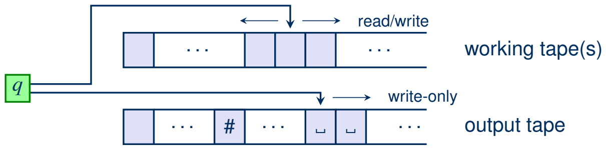
□

## Enumerators

**Definition 2.12:** A multi-tape Turing machine $\mathcal{M}$ is an enumerator if

- $\mathcal{M}$ has a designated write-only output-tape on which a symbol, once written, can never be changed and where the head can never move left;
- $\mathcal{M}$ has a marker symbol # separating words on the output tape.

We define the language generated by $\mathcal{M}$ to be the set $\mathbf{G}(\mathcal{M})$ of all words that eventually appear between two consecutive # on the output tape of $\mathcal{M}$ when started on the empty word as input.






## Enumerators

**Theorem 2.13:** A language **L** is Turing-recognisable if and only if there exists some enumerator $\mathcal{E}$ such that $\mathbf{G}(\mathcal{E}) = \mathbf{L}$.

**Proof:** Let $\mathcal{E}$ be an enumerator for **L**. Then the following TM accepts **L**:

$\mathcal{M}$ := On input $w$

- Simulate $\mathcal{E}$ on the empty input. Compare every string output by $\mathcal{E}$ with $w$
- If $w$ appears in the output of $\mathcal{E}$, accept



## Enumerators

Let $\mathbf{L} = \mathbf{L}(\mathcal{M})$ for some TM $\mathcal{M}$, and let $s_1, s_2, \ldots$ be an enumeration of $\Sigma^*$.
Then the following enumerator $\mathcal{E}$ enumerates **L**:

$\mathcal{E}$ := Ignore the input.

- Print the first # to initialise the output.
- Repeat for $i = 1, 2, 3, \ldots$
  - Run $\mathcal{M}$ for $i$ steps on each input $s_1, s_2, \ldots, s_i$
  - If any computation accepts, print the corresponding $s_j$ followed by #

□

**Theorem 2.14:** If **L** is Turing-recognisable, then there exists an enumerator for **L** that prints each word of **L** exactly once.



## Enumerators

**Theorem 2.15:** A language **L** is decidable if and only if there exists an enumerator for **L** that outputs exactly the words of **L** in some order of non-decreasing length.

**Proof:** Suppose **L** to be decidable, and let $\mathcal{M}$ be a TM that decides **L**.

- Define a TM $\mathcal{M}'$ that generates, on some scratch tape, all words over $\Sigma$ in some order of non-decreasing length. (Exercise!)
- An enumerator $\mathcal{E}$ works as follows:
  (1) Print the first # to initialise the output.
  (2) Run $\mathcal{M}'$ (enumerating words), followed by $\mathcal{M}$ (to check if the current word is accepted). If $\mathcal{M}$ accepts $w$, then print $w$ followed by #.

Then $\mathcal{E}$ enumerates exactly the words of **L** in some order of non-decreasing length.

## Enumerators

Now suppose **L** can be enumerated by some TM $\mathcal{E}$ in some order of non-decreasing length.

- If **L** is finite, then **L** is accepted by a finite automaton.
- If **L** is infinite, then we define a decider $\mathcal{M}$ for it as follows.

$\mathcal{M}$ := On input $w$

- Simulate $\mathcal{E}$ until it either outputs $w$ or some word longer than $w$
- If $\mathcal{E}$ outputs $w$, then accept, else reject.

**Observation**: since **L** is infinite, for each $w \in \Sigma^*$ the TM $\mathcal{E}$ will eventually generate $w$ or some word longer than $w$. Therefore, $\mathcal{M}$ always halts and thus decides **L**.

□



## Summary and Outlook

Turing Machines are a simple model of computation

Recognisable (semi-decidable) = recursively enumerable

Decidable = computable = recursive

Many variants of TMs exist – they normally recognise/decide the same languages

**What's next?**

- Actual complexity classes.
- Namely, the class of "efficiently" solvable problems: P