DEDUCTION SYSTEMS

Tableau Procedures I

Sebastian Rudolph

Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary

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- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  $\rightsquigarrow$ idea: constructive decision procedure that tries to build models
- analog: truth tables in propositional logic

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$$(p \vee q) \rightarrow (\neg p \vee \neg q)$$

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$$(\neg p \wedge \neg q) \vee (\neg p \vee \neg q)$$

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$$(p \vee q) \rightarrow (\neg p \vee \neg q)$$

negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:

$$\begin{aligned} \neg(p \vee q) \vee (\neg p \vee \neg q) \\ (\neg p \wedge \neg q) \vee (\neg p \vee \neg q) \\ (\neg p \wedge \neg q) \vee \neg p \vee \neg q \end{aligned}$$

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Simple Tableau

$$(\neg p \wedge \neg q) \vee \neg p \vee \neg q$$

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$$\neg p \wedge \neg q \qquad\qquad \neg p \qquad\qquad \neg q$$

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches

Simple Tableau

$$(\neg p \land \neg q) \lor \neg p \lor \neg q$$

$$\neg p \land \neg q \qquad\qquad \neg p \qquad\qquad \neg q$$

$$\neg p$$

$$\neg q$$

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches

Simple Tableau

$$(\neg p \wedge \neg q) \vee \neg p \vee \neg q$$

$\neg p \wedge \neg q$ $\qquad$ $\neg p$ $\qquad$ $\neg q$

$\neg p$

$\neg q$

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches
- compare: truth table

$I(p)$	$I(q)$	$I(\neg p)$	$I(\neg q)$	$I(p \vee q)$	$I(\neg p \vee \neg q)$	$I((p \vee q) \rightarrow (\neg p \vee \neg q))$
t	t	f	f	t	f	f
t	f	f	t	t	t	t
f	t	t	f	t	t	t
f	f	t	t	f	t	t

Simple Tableau with Contradiction

$(\neg p \vee q) \wedge p \wedge \neg q$

Simple Tableau with Contradiction

$$(\neg p \lor q) \land p \land \neg q$$

$$\neg p \lor q$$

$$p$$

$$\neg q$$

Simple Tableau with Contradiction

$$(\neg p \lor q) \land p \land \neg q$$

$$\neg p \lor q$$

$$p$$

$$\neg q$$

$$\neg p \qquad\qquad q$$

Simple Tableau with Contradiction

$$
\begin{tikzcd}[row sep=small]
& (\neg p \vee q) \wedge p \wedge \neg q \arrow[d, no head, phantom] & \\
& \neg p \vee q \arrow[d, no head, phantom] & \\
& p \arrow[d, no head, phantom] & \\
& \neg q \arrow[dl, no head] \arrow[dr, no head] & \\
\neg p & & q
\end{tikzcd}
$$

Simple Tableau with Contradiction

$$
\begin{array}{ccc}
\multicolumn{3}{c}{(\neg p \lor q) \land p \land \neg q} \\
\multicolumn{3}{c}{\neg p \lor q} \\
\multicolumn{3}{c}{p} \\
\multicolumn{3}{c}{\neg q} \\
\neg p & & q \\
\bot & &
\end{array}
$$

- if a branch contains an atomic contradiction (clash), we call this branch closed

Simple Tableau with Contradiction

$$(\neg p \vee q) \wedge p \wedge \neg q$$

$$\neg p \vee q$$

$$p$$

$$\neg q$$

$$\neg p \qquad\qquad q$$

$$\bot \qquad\qquad \bot$$

- if a branch contains an atomic contradiction (clash), we call this branch closed
- a tableau is closed, if all its branches are

Simple Tableau with Contradiction

$$
\begin{array}{ccc}
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\multicolumn{3}{c}{\neg p \vee q} \\
\multicolumn{3}{c}{p} \\
\multicolumn{3}{c}{\neg q} \\
\swarrow & & \searrow \\
\neg p & & q \\
\bot & & \bot
\end{array}
$$

- if a branch contains an atomic contradiction (clash), we call this branch closed
- a tableau is closed, if all its branches are
- a complete tableau without open branches shows the formula's unsatisfiability

Constructing a Model from the Tableau

$$
\begin{array}{ccc}
 & (\neg p \land \neg q) \lor \neg p \lor \neg q & \\
\neg p \land \neg q & \neg p & \neg q \\
\neg p & & \\
\neg q & &
\end{array}
$$

- given an open branch, we can construct a model

Constructing a Model from the Tableau

$$(\neg p \wedge \neg q) \vee \neg p \vee \neg q$$

$\neg p \wedge \neg q$ | $\neg p$ | $\neg q$

$\neg p$

$\neg q$

- given an open branch, we can construct a model
- let I(p)=false and let I(q)=false

Constructing a Model from the Tableau

$$(\neg p \land \neg q) \lor \neg p \lor \neg q$$

$\neg p \land \neg q$ $\qquad$ $\neg p$ $\qquad$ $\neg q$

$\neg p$

$\neg q$

- given an open branch, we can construct a model
- let I(p)=false and let I(q)=false
- let I(p)=false (I(q) is irrelevant since not in the branch, default assignment false)

Constructing a Model from the Tableau

$$(\neg p \wedge \neg q) \vee \neg p \vee \neg q$$

$\neg p \wedge \neg q$ | $\neg p$ | $\neg q$

$\neg p$

$\neg q$

- given an open branch, we can construct a model
- let I(p)=false and let I(q)=false
- let I(p)=false (I(q) is irrelevant since not in the branch, default assignment false)
- let I(q)=false (I(p) is irrelevant since not in the branch, default assignment false)

Propositional Tableau

- not always exponentially many combinations have to be checked (as opposed to truth table method)
- branches can be built one after the other $\rightsquigarrow$ only polynomial space needed
- if we care about satisfiability we can stop after constructing the first complete open branch

Construction with only one Branch in Memory

$$(\neg p \vee q) \wedge p \wedge q$$

Construction with only one Branch in Memory

$$(\neg p \vee q) \wedge p \wedge q$$

$$\neg p^{1a} \vee q^{1b}$$

$$p$$

$$q$$

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked

Construction with only one Branch in Memory

$$(\neg p \lor q) \land p \land q$$

$$\neg p^{1a} \lor q^{1b}$$

$$p$$

$$q$$

$$\neg p^{1a}$$

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked

Construction with only one Branch in Memory

$$(\neg p \vee q) \wedge p \wedge q$$

$$\neg p^{1a} \vee q^{1b}$$

$$p$$

$$q$$

$$\neg p^{1a}$$

$$\perp^{1a}$$

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
- when encountering a contradiction caused by a choice, remove marked formulae and try next choice

Construction with only one Branch in Memory

$$(\neg p \lor q) \land p \land q$$

$$\neg p^{1a} \lor q^{1b}$$

$$p$$

$$q$$

$$\cancel{\neg p^{1a}}$$

$$\cancel{\bot^{1a}}$$

$$q^{1b}$$

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
- when encountering a contradiction caused by a choice, remove marked formulae and try next choice

From Propositional Tableau to Tableau for DLs

How can the tableaux be extended for checking satisfiability of $\mathcal{ALC}$ concepts?
NB: initially, we assume no underlying knowledge base, thus unsatisfiability means that the concept is contradictory "by itself".

- tableau represents an element of the domain (plus its "environment")

From Propositional Tableau to Tableau for DLs

How can the tableaux be extended for checking satisfiability of $\mathcal{ALC}$ concepts?
NB: initially, we assume no underlying knowledge base, thus unsatisfiability means that the concept is contradictory "by itself".

- tableau represents an element of the domain (plus its "environment")
- tableau branch: finite set of propositions of the form $C(a)$, $r(a, b)$
- for existential quantifiers, new domain elements are introduced
- universal quantifiers propagate formulae (=concept expressions) to neighboring elements

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Propositional Logic – Some Logical Equivalences

- We aim at negations being present only in front of atomic concepts

$$\varphi \wedge \psi \equiv \psi \wedge \varphi \qquad \varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$$

$$\varphi \vee \psi \equiv \psi \vee \varphi \qquad \varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

$$\varphi \wedge (\psi \wedge \omega) \equiv (\varphi \wedge \psi) \wedge \omega \qquad \neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$$

$$\varphi \vee (\psi \vee \omega) \equiv (\varphi \vee \psi) \vee \omega \qquad \neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$$

$$\varphi \wedge \varphi \equiv \varphi \qquad \neg\neg\varphi \equiv \varphi$$

$$\varphi \vee \varphi \equiv \varphi$$

$$\varphi \wedge (\psi \vee \varphi) \equiv \varphi \qquad \varphi \vee (\psi \wedge \omega) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \omega)$$

$$\varphi \vee (\psi \wedge \varphi) \equiv \varphi \qquad \varphi \wedge (\psi \vee \omega) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \omega)$$

Further Logical Equivalences

$$
\begin{aligned}
\neg(C \sqcap D) &\rightsquigarrow \neg C \sqcup \neg D \\
\neg(D \sqcup D) &\rightsquigarrow \neg C \sqcap \neg D \\
\neg\neg C &\rightsquigarrow C
\end{aligned}
$$

$$
\begin{aligned}
\neg(\forall r.C) &\rightsquigarrow \exists r.(\neg C) \\
\neg(\exists r.C) &\rightsquigarrow \forall r.(\neg C) \\
\neg(\leqslant n\, s.C) &\rightsquigarrow \geqslant n+1\, s.C \\
\neg(\geqslant n\, s.C) &\rightsquigarrow \leqslant n-1\, s.C, \quad n \geq 1 \\
\neg(\geqslant 0\, s.C) &\rightsquigarrow \bot
\end{aligned}
$$

- apply these rules iteratively until none can be applied any more

$\rightsquigarrow$ equivalent concept in negation normal form

NNF Transformation

recursive definition of an NNF transformation:

if C atomic:

$$NNF(C) := C \qquad NNF(\neg C) := \neg C$$

otherwise:

$$NNF(\neg\neg C) := NNF(C)$$

$$NNF(C \sqcap D) := NNF(C) \sqcap NNF(D) \qquad NNF(\neg(C \sqcap D)) := NNF(\neg C) \sqcup NNF(\neg D)$$

$$NNF(C \sqcup D) := NNF(C) \sqcup NNF(D) \qquad NNF(\neg(C \sqcup D)) := NNF(\neg C) \sqcap NNF(\neg D)$$

$$NNF(\forall r.C) := \forall r.(NNF(C)) \qquad NNF(\neg(\forall r.C)) := \exists r.(NNF(\neg C))$$

$$NNF(\exists r.C) := \exists r.(NNF(C)) \qquad NNF(\neg(\exists r.C)) := \forall r.(NNF(\neg C))$$

$$NNF(\leqslant n\, s.C) := \leqslant n\, s.(NNF(C)) \qquad NNF(\neg(\leqslant n\, s.C)) := \geqslant n+1\, s.(NNF(C))$$

$$NNF(\geqslant n\, s.C) := \geqslant n\, s.(NNF(C)) \qquad NNF(\neg(\geqslant n\, s.C)) := \leqslant n-1\, s.(NNF(C)) \quad \text{if } n \geq 1$$

$$NNF(\geqslant 0\, s.C) := \top \qquad NNF(\neg(\geqslant 0\, s.C)) := \bot \quad \text{otherwise}$$

NNF Transformation – Example

$$
\begin{aligned}
& NNF(\neg(\neg C \sqcap (\neg D \sqcup E))) \\
= {} & NNF(\neg\neg C) \sqcup NNF(\neg(\neg D \sqcup E)) \\
= {} & NNF(C) \sqcup NNF(\neg(\neg D \sqcup E)) \\
= {} & C \sqcup NNF(\neg(\neg D \sqcup E)) \\
= {} & C \sqcup (NNF(\neg\neg D) \sqcap NNF(\neg E)) \\
= {} & C \sqcup (NNF(D) \sqcap NNF(\neg E)) \\
= {} & C \sqcup (D \sqcap NNF(\neg E)) \\
= {} & C \sqcup (D \sqcap \neg E)
\end{aligned}
$$

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Tableau for $\mathcal{ALC}$ Concepts

- tableau for a propositional formula α: one element, labeled with subformulae of α
- tableau for an $\mathcal{ALC}$ concept C: graph (more precisely: tree) where the nodes are labeled with subformulae of C
- root labeled with C
- represents model for C (if complete and clash-free)
- non-root nodes are enforced by existential quantifiers

Definition

Let C be an $\mathcal{ALC}$ concept, $\mathsf{SF}(C)$ the set of all subformulae of C and $\mathsf{Rol}(C)$ the set of all roles occurring in C. A tableau for C is a tree $G = \langle V, E, L \rangle$, with nodes V, edges $E \subseteq V \times V$ and a labeling function L with $L\colon V \to 2^{\mathsf{SF}(C)}$ and $L\colon V \times V \to 2^{\mathsf{Rol}(C)}$.

Properties of the $\mathcal{ALC}$ Tableau Algorithm

- the algorithm is specified as a set of rules
- every rule breaks down a complex concept into its parts
- rules applicable in any order
- the algorithm is non-deterministic (due to disjunction)
- check for atomic contradictions

tableau algorithm for checking satisfiability of $\mathcal{ALC}$ concepts

Input:	an $\mathcal{ALC}$ concept in NNF	
Output:	`true`	if there is a clash-free tableau where no more rules can be applied
	`false`	otherwise (tableau closed)

Tableau Rules for $\mathcal{ALC}$ Concepts

$\sqcap$-rule: For an arbitrary $v \in V$ mit $C \sqcap D \in L(v)$ and $\{C, D\} \not\subseteq L(v)$, let $L(v) := L(v) \cup \{C, D\}$.

$\sqcup$-rule: For an arbitrary $v \in V$ with $C \sqcup D \in L(v)$ and $\{C, D\} \cap L(v) = \emptyset$, choose $X \in \{C, D\}$ and let $L(v) := L(v) \cup \{X\}$.

$\exists$-rule: For an arbitrary $v \in V$ with $\exists r.C \in L(v)$ such that there is no r-successor v' of v with $C \in L(v')$, let $V = V \cup \{v'\}$, $E = E \cup \{\langle v, v' \rangle\}$, $L(v') := \{C\}$ and $L(v, v') := \{r\}$ for v' a new node.

$\forall$-rule: For arbitrary $v, v' \in V$, v' r-neighbor of v, $\forall r.C \in L(v)$ and $C \notin L(v')$, let $L(v') := L(v') \cup \{C\}$.

- a node v' is an r-neighbor of a node v if $\langle v, v' \rangle \in E$ and $r \in L(v, v')$

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- a node v' is an r-neighbor of a node v if $\langle v, v' \rangle \in E$ and $r \in L(v, v')$
- rule application order: “don’t care” non-determinism

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- a node v' is an r-neighbor of a node v if $\langle v, v' \rangle \in E$ and $r \in L(v, v')$
- rule application order: “don’t care” non-determinism
- choice of disjunction: “don’t know” non-determinism

Tableau Algorithmus Example

$$C = \exists r.(A \sqcup \exists r.B) \sqcap \exists r.\neg A \sqcap \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))$$

u

$$L(u) = \{C\}$$

Tableau Algorithmus Example

$$C = \exists r.(A \sqcup \exists r.B) \sqcap \exists r.\neg A \sqcap \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))$$

u

$$L(u) = \{C, \exists r.(A \sqcup \exists r.B), \exists r.\neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))\}$$

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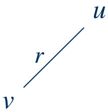

$$\begin{aligned} L(u) &= \{C, \exists r.(A \sqcup \exists r.B), \\ &\quad \exists r.\neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))\} \\ L(v) &= \{A \sqcup \exists r.B\} \end{aligned}$$

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$$\begin{tikzcd} & u \arrow[dl, dash, "r"'] \arrow[dr, dash, "r"] & \\ v & & w \end{tikzcd}$$

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\end{aligned}$$

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L(v) &= \{A \sqcup \exists r.B, \neg A, \forall r.(\neg B \sqcup A), \cancel{A}\} \\
L(w) &= \{\neg A, \forall r.(\neg B \sqcup A)\}
\end{aligned}$$

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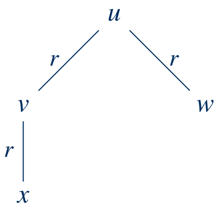

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Tableau Algorithmus Example

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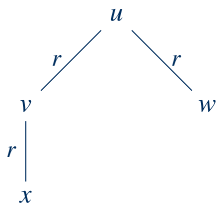

$$\begin{aligned}
L(u) &= \{C, \exists r.(A \sqcup \exists r.B), \\
&\quad \exists r.\neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))\} \\
L(v) &= \{A \sqcup \exists r.B, \neg A, \forall r.(\neg B \sqcup A), \cancel{A}, \exists r.B\} \\
L(w) &= \{\neg A, \forall r.(\neg B \sqcup A)\} \\
L(x) &= \{B\}
\end{aligned}$$

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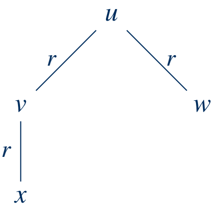

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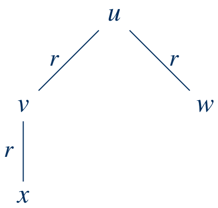

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$$\begin{tikzcd}
 & u \arrow[dl, dash, "r"'] \arrow[dr, dash, "r"] & \\
v \arrow[d, dash, "r"'] & & w \\
x & &
\end{tikzcd}$$

$$\begin{aligned}
L(u) &= \{C, \exists r.(A \sqcup \exists r.B), \\
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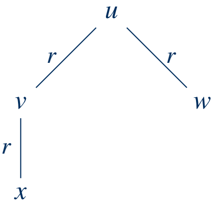

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L(x) &= \{B, \neg B \sqcup A, \cancel{\neg B}, A\}
\end{aligned}$$

Tableau Algorithm Example

the model $\mathcal{I}$ constructed by the algorithm is the following:

$$\begin{aligned} \Delta^{\mathcal{I}} &= \{u, v, w, x\} \\ A^{\mathcal{I}} &= \{x\} \\ B^{\mathcal{I}} &= \{x\} \\ r^{\mathcal{I}} &= \{\langle u, v\rangle, \langle u, w\rangle, \langle v, x\rangle\} \end{aligned}$$

Check that indeed $C^{\mathcal{I}} = \{u\}$, given the defined semantics of $\mathcal{ALC}$

Tableau Algorithm Properties

1. the model is finite: only finitely many elements in the domain
2. the model is tree-shaped: the tableau is a labeled tree

the algorithm will always construct finite trees

- from a clash-free tableau, we can construct a finite model
- if there is no clash-free tableau, there is no model

Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary

Tableau Properties

- the depth (number of nested quantifiers) decreases in every node
- every node is labeled only with subformulae of C
- C has only polynomially many subformulae
- if the ouput is `true` we can build a model out of the constructed tableau
- on the other hand, we can use a model of a satisfiable concept to construct a clash-free tableau for this concept

Tableau Algorithm for $\mathcal{ALC}$ Concepts

Theorem

1. the algorithm terminates for every input
2. if the output is true, then the input concept is satisfiable
3. if the input concept is satisfiable, then the output is true.

Tableau Algorithm for $\mathcal{ALC}$ Concepts

Theorem

1. the algorithm terminates for every input
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Corollary

Every $\mathcal{ALC}$ concept C has the following properties:

1. finite model property: If C has a model, then it has a finite one.
2. tree model property If C has a model, then it has a tree-shaped one.

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Summary

- we now have a constructive method for building model abstractions
- satisfiable $\mathcal{ALC}$ concepts always have a finite model that we can construct
- the algorithm is correct, complete and terminating
- serves as basis for practically implemented algorithms
- next: extension to knowledge bases