TECHNISCHE UNIVERSITÄT DRESDEN

THE MORE THE WORST-CASE-MERRIER

A GENERALIZED CONDORCET JURY THEOREM FOR BELIEF FUSION

Jonas Karge and Sebastian Rudolph
Computational Logic Group

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Introduction

Belief Fusion

Belief Fusion as opposed to Belief Revision:

- **Belief Revision**: combination of two pieces of information with preference given to one of them
- **Belief Fusion**: combining several pieces of information without strict preferences

Two alternative goals (Everaere et al. 2010):

(1) fair fusion procedure (synthesis view)

(2) obtain correct piece of information (epistemic view)

We focus on the second goal: epistemic view aka **truth-tracking**.

Possible Application – Smart Dust

Smart Dust: micro-electro mechanical system consisting of (possibly thousands) of “motes” carrying sensors that can gather information

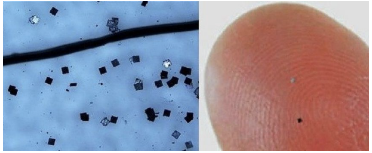

Figure: A Smart Dust Mote[1]

Applications: general engineering, health, environmental monitoring...

[1] https://medium.com/@bhargavravinuthala/a-brief-introduction-to-smart-dust-technology-by-bhargav-1d498e7c60fe

Smart Dust – Environmental Monitoring Scenario

Hypothetical Smart Dust system for detecting **geological activity**

Manufacturer's **guarantees** regarding **reliability** of provided motes:

- certain percentage **malfunctioning** (production errors / deployment risk);
- probability of functional mote **correctly spotting** patterns that precede earthquakes or landslides;
- motes have **heterogeneously** distributed levels of competence about which only statistical guarantees can be given (can also depend on location in area).

$\Rightarrow$ data delivered by motes to be aggregated

Idea: apply voting methods (potential predictions = set of alternatives to vote on)

The Condorcet Jury Theorem (CJT)

Marie Jean Antoine Nicolas Caritat Marquis de Condorcet

Theorem: For odd-numbered **homogenous** groups of **independent** and **reliable** agents in a **dichotomic** voting setting, the probability that majority voting identifies the correct alternative

- increases monotonically with the number of agents and (non-asymptotic part)
- converges to 1 as the number of agents goes to infinity. (asymptotic part)

Formal Framework

Voting

Define **approval voting** and obtain simpler voting mechanisms as special cases.

Given: finite set of n agents $\mathcal{A} = \{a_1, \ldots, a_n\}$
finite set of m choices $\mathcal{W} = \{\omega_1, \ldots, \omega_m\}$

- **approval voting (instance)**: relation $V \subseteq \mathcal{A} \times \mathcal{W}$
  $(a_i, \omega_j) \in V$ means agent a_i approves choice ω_j
- given $\omega \in \mathcal{W}$, obtain **score** $\#_V\, \omega$ as overall number of votes that ω receives, i.e.,

$$\#_V\, \omega = |\{a_i \in \mathcal{A}_n \mid (a_i, \omega) \in V\}|$$

- ω **wins approval vote** V if it receives strictly more votes than any other choice:

$$\#_V\, \omega > \max_{\omega' \in \mathcal{W} \setminus \{\omega\}} \#_V\, \omega'$$

The Probabilistic Framework

Make **probabilistic assumptions** explicit that underlie the CJT.

Random process chooses ω_* (the actual world state) and generates V, governed by **joint probability distribution** $\mathbb{P}$ over Bernoulli (i.e., $\{0, 1\}$-valued) random variables

$$\begin{array}{ccc} X_*^{\omega_1}, & \dots, & X_*^{\omega_m}, \\ X_1^{\omega_1}, & \dots, & X_1^{\omega_m}, \\ \vdots & \vdots\vdots\vdots & \vdots \\ X_n^{\omega_1}, & \dots, & X_n^{\omega_m}. \end{array}$$

- $X_*^{\omega_j}$ is 1 if ω_j is the actual world state (i.e., $\omega_j = \omega_*$), and 0 otherwise,
- $X_i^{\omega_j}$ is 1 if a_i voted for ω_j (that is, $(a_i, \omega_j) \in V$) and 0 otherwise.

The Joint Probability Distribution – Assumptions

Definition: A joint distribution satisfies **agent approval independence** if for any $\omega, \omega_j \in \mathcal{W}$ and any sequence $v_1, \ldots, v_n$ of values from $\{0, 1\}$ the following holds:

$$\mathbb{P}\Big(\bigwedge_{i=1}^{n} X_i^{\omega_j} = v_i \mid [\omega_* = \omega]\Big) = \prod_{i=1}^{n} \mathbb{P}\Big(X_i^{\omega_j} = v_i \mid [\omega_* = \omega]\Big).$$

Definition: A joint probability distribution satisfies **Δp-group reliability** for some $\Delta p > 0$, if for every $\omega, \omega' \in \mathcal{W}$ with $\omega \neq \omega'$ the following holds:

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{P}\Big(X_i^{\omega} = 1 \,|\, [\omega_* = \omega]\Big) \geq \Delta p + \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}\Big(X_i^{\omega'} = 1 \,|\, [\omega_* = \omega]\Big).$$

A distribution satisfying both is called I&R (independent and reliable).

The Joint Probability Distribution – Further Properties

Definition: A joint distribution satisfies **homogeneity** if for any $\omega, \omega' \in \mathcal{W}$ and all $i, k \in \{1, \ldots, n\}$ the following holds:

$$\mathbb{P}\Big(X_i^{\omega} = 1 \mid [\omega_* = \omega']\Big) = \mathbb{P}\Big(X_k^{\omega} = 1 \mid [\omega_* = \omega']\Big).$$

Definition: A joint distribution satisfies **(vote) completeness** if for every $i \in \{1, \ldots, n\}$ the following holds:

$$\textstyle\sum_{j=1}^{m} X_i^{\omega_j} = 1.$$

Results

Prior Results

Definition: For a family $\mathcal{P}$ of joint probability distributions, the corresponding **worst-case success probability** $P^{\text{wcs}}_{m,n}$ for n agents and m choices is defined by

$$\min_{\substack{\mathbb{P}\in\mathcal{P} \\ \omega\in\mathcal{W}=\{\omega_1,\dots,\omega_m\}}} \mathbb{P}\Big(\bigwedge_{\omega_\dagger\in\mathcal{W}\setminus\{\omega\}} \textstyle\sum_{k=1}^n X_k^{\omega} > \sum_{k=1}^n X_k^{\omega_\dagger} \mid [\omega_* = \omega]\Big).$$

We can then summarize previous asymptotic results as follows:

- In any complete, homogeneous I&R setting holds $P^{\text{wcs}}_{2,n} \xrightarrow[n\to\infty]{} 1$ (Condorcet 1785).
- In any complete, homogeneous I&R setting holds $P^{\text{wcs}}_{m,n} \xrightarrow[n\to\infty]{} 1$ (List and Goodin 2001).
- In any homogeneous I&R setting holds $P^{\text{wcs}}_{m,n} \xrightarrow[n\to\infty]{} 1$ (Everaere, Konieczny, and Marquis 2010).
- In any complete I&R setting holds $P^{\text{wcs}}_{2,n} \xrightarrow[n\to\infty]{} 1$ (Owen, Grofman, and Feld 1989).

Main Result

Theorem: In any I&R setting with fixed $m \geq 2$ and $\Delta p > 0$ holds $P^{\text{wcs}}_{m,n} \xrightarrow[n \to \infty]{} 1$.

Note: assumptions relaxed – no homogeneity, no dichotomy, no vote completeness.

Proof Idea.

- Let $\omega_\dagger \in \mathcal{W} \setminus \{\omega_*\}$ denote an arbitrary but fixed "competitor" of ω_* in the approval vote.
- Apply **Chebyshev's inequality** to obtain lower bound for the probability of ω_* winning against $\omega_\dagger$.
- Obtain the probability for ω_* winning the approval vote against **all** competing $\omega_\dagger \in \mathcal{W} \setminus \{\omega_*\}$ simultaneously.

Estimates for Required Number of Agents

Theorem allows to derive bound on number n of agents required for success with probability of at least $P_{\min}$, given given reliability parameter Δp and number m of choices:

$$n \geq \frac{2(m-1)}{\Delta p^2 (1 - P_{\min})}.$$

Example: For $\Delta p = 0.5$,

- for $m = 11$ and $P_{\min} = 0.9$, number of required voters is 800
- for $m = 101$ and $P_{\min} = 0.99$, number of required voters is $80,000$

$\Rightarrow$ guarantees still unsatisfactory (especially for high $P_{\min}$ and/or m)

Better Bounds for High Values of $P_{\min}$ and/or m

From **Hoeffding's inequality**, we obtain the following improved bound

$$n \geq \frac{2}{\Delta p^2} \ln \frac{2(m-1)}{1-P_{\min}}$$

Example: For $\Delta p = 0.5$,

- for $m = 11$ and $P_{\min} = 0.9$, number of required voters is 42 (was: 800)
- for $m = 101$ and $P_{\min} = 0.99$, number of required voters is 80 (was: $80,000$)

Better Bounds for Large Δp

Using some more tools (inequalities by Jensen and Chebyshev-Cantelli) we get better estimate for large values of Δp for number of independent agents needed to surpass a given success probability of $P_{\min}$:

$$n \geq 1 + 2\Big(\frac{1}{\Delta p^2} - 1\Big)\Big(\frac{m-1}{1-P_{\min}}\Big).$$

None of the two improved estimates dominates the other for all values: determine the minimum of the two in every case.

Final Bound

Theorem: In a Δp-group reliable setting with m choices, the worst case approval vote success probability is at least $P_{\min}$ whenever the number of agents is equal or higher than

$$\min\left(\frac{2}{\Delta p^2}\ln Q, 1 + \left(\frac{1}{\Delta p^2} - 1\right)Q\right),$$

where $Q = 2\frac{m-1}{1-P_{\min}}$ is the twofold ratio between the number of incorrect alternatives and the admissible error probability.

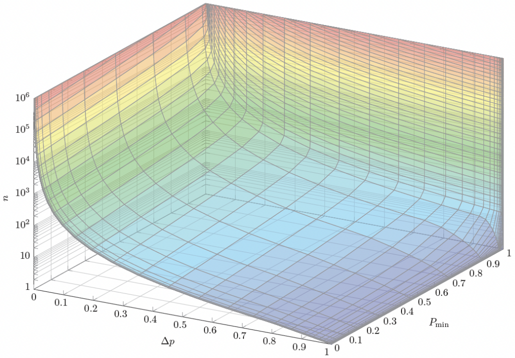

Figure: Lower bound for n (logscale), given Δp and $P_{\min}$ for fixed $m = 2$.

Summary and Future Work

Summary

Our setting allows

- heterogeneous competence levels among agents;
- approval voting for any (finite) number of alternatives.

For this setting, we

- derived practical estimates for the number of independent agents necessary to guarantee a prescribed minimal probability of success;
- proved failure of non-asymptotic part of the CJT.

Future Work

- generalization for weakened independence assumption: allow for certain degree of **joint external** or **mutual influence** among the voters;
- generalization towards more **fine-grained voter feedback**;
- application of results in the context of logic-based **belief fusion**;
- **experiments** comparing theoretically established guarantees with actual behaviour in simulations.