

Cheap Boolean Role Constructors for Description Logics

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Abstract. We investigate the possibility of incorporating Boolean role constructors on simple roles into some of today’s most popular description logics, focussing on cases where those extensions do not increase complexity of reasoning. We show that the expressive DLs *SHOIQ* and *SROIQ*, serving as the logical underpinning of OWL and the forthcoming OWL 2, can accommodate arbitrary Boolean expressions. The prominent OWL-fragment *SHIQ* can be safely extended by safe role expressions, and the tractable fragments \mathcal{EL}^{++} and DLP retain tractability if extended by conjunction on roles, where in the case of DLP the restriction on role simplicity can even be discarded.

1 Introduction

Research on description logics (DLs) is directed by two main goals: increasing expressivity while preserving desirable computational properties such as decidability (as a factual *conditio sine qua non*) and efficiency of reasoning, the latter qualitatively estimated in terms of worst-case complexities. These antagonistic dimensions gave rise to a great variety of logics: *SROIQ* and *SHOIQ* being of high expressiveness and complexity represent one side of the spectrum, whereas the so called tractable fragments like \mathcal{EL}^{++} and DLP provide lower expressivity yet allow for polynomial time reasoning.

In DL history, Boolean constructors (negation, conjunction, disjunction) on roles have occurred and have been investigated sporadically in many places, but have never been integrated into the mainstream of researched languages nor influenced standardisation efforts. In this paper, we argue that those constructors can – sometimes with appropriate restrictions – be incorporated into several of the most prominent DL languages, thereby significantly enhancing expressivity without increasing reasoning complexity.

To illustrate this gain in expressivity, we give some examples on the modelling capabilities of Boolean role constructors:

Universal role. A role U that connects all individuals of the described domain can e.g. be defined via $U \equiv \neg N$ as the complement of the empty role N , which in turn can be axiomatized by the GCI $\top \sqsubseteq \forall N.\perp$.

Role conjunction. This modelling feature comes in handy if certain non-tree-like properties (namely cases where two individuals are interconnected by more than one role) have to be described. The fact that somebody testifying against a relative is not put under oath can e.g. be formalised by $\exists(\text{testifiesAgainst} \sqcap \text{relativeOf}).\top \sqsubseteq \neg\text{UnderOath}$.

Likewise, role conjunction allows for specifying disjointness of roles, as $\text{Dis}(R, S)$ can be paraphrased as $\top \sqsubseteq \forall(R \sqcap S). \perp$.

Concept products. Thoroughly treated in [1], the concept product statement $C \times D \sqsubseteq R$ expresses that any instance of C is connected with any instance of D via role R . As an example, the fact that alkaline solutions neutralise acid solutions, which could be expressed by the concept product axiom $\text{AlkalineSolution} \times \text{AcidSolution} \sqsubseteq \text{neutralises}$, can equivalently be stated by $\text{AlkalineSolution} \sqsubseteq \forall(\neg \text{neutralises}). \neg \text{AcidSolution}$ by using role negation.

Qualified role inclusion. Likewise, the specialisation of roles due to concept memberships of the involved individuals can be expressed. The rule-like FOL statement $C(x) \wedge R(x, y) \wedge D(y) \rightarrow S(x, y)$ (expressing that any C -instance and D -instance that are interconnected by R are also interconnected by S) can be cast into the GCI $C \sqsubseteq \forall(R \sqcap \neg S). \neg D$. For example, the fact that any person of age having signed a contract which is legal is bound to that contract can be expressed by $\text{OfAge} \sqsubseteq \forall(\text{hasSigned} \sqcap \neg \text{boundTo}). \neg (\text{Contract} \sqcap \text{Legal})$.

The latter two types of statements have recently gained increased interest in the context of identifying rule-like fragments of DLs [2].

The rest of the paper is organised as follows. After providing the necessary definitions, we review existing work on Boolean role constructors. Then, we deal with the extension of SROIQ and SHOIQ by full Boolean role expressions on simple roles. Thereafter, we provide an according result for integrating safe Boolean role expressions into the description logic SHIQ . The subsequent two sections settle the case for the tractable fragments \mathcal{EL}^{++} and DLP, respectively, extending them by role conjunction. Finally, we conclude and elaborate on future work. Due to lack of space, some proofs had to be omitted. Those can be found in [3].

2 Preliminaries

In this section, we give the definition of the expressive description logic SROIQB_s , which is obtained from the well-known description logic SROIQ [4] by allowing arbitrary Boolean constructors on simple roles. We assume that the reader is familiar with description logics [5].

The DLs considered in this paper are based on four disjoint sets of *individual names* N_I , *concept names* N_C , and *simple role names* N_R^s (containing the *universal role* $U \in N_R$) as well as *non-simple role names* N_R^n . Furthermore, we let $N_R := N_R^s \cup N_R^n$.

Definition 1. A SROIQB_s Rbox for N_R is based on a set \mathbf{R} of atomic roles defined as $\mathbf{R} := N_R \cup \{R^- \mid R \in N_R\}$, where we set $\text{Inv}(R) := R^-$ and $\text{Inv}(R^-) := R$ to simplify notation. In turn, we distinguish simple atomic roles $\mathbf{R}^s := N_R^s \cup \text{Inv}(N_R^s)$ and non-simple atomic roles $\mathbf{R}^n := N_R^n \cup \text{Inv}(N_R^n)$.

In the sequel, we will use the symbols R, S , possibly with subscripts, to denote atomic roles.

The set of Boolean role expressions \mathbf{B} is defined as follows:

$$\mathbf{B} ::= \mathbf{R} \mid \neg \mathbf{B} \mid \mathbf{B} \sqcap \mathbf{B} \mid \mathbf{B} \sqcup \mathbf{B}.$$

The set \mathbf{B}_s of simple role expressions comprises all those role expressions containing only simple role names. In the sequel, V and W will denote simple role expressions if not stated otherwise. Moreover, a role expression will be called safe, if in its disjunctive normal form, every disjunct contains at least one non-negated role name.

A generalised role inclusion axiom (RIA) is a statement of the form $V \sqsubseteq W$ with simple role expressions V and W , or of the form

$$S_1 \circ \dots \circ S_n \sqsubseteq R$$

where each S_i is a simple role expression or a non-simple atomic role, and where R is a non-simple atomic role. A set of such RIAs will be called a generalised role hierarchy. A role hierarchy is regular if there is a strict partial order $<$ on the non-simple roles \mathbf{R}^n such that

- $S < R$ iff $\text{Inv}(S) < R$, and
- every RIA is of one of the forms
 - $R \circ R \sqsubseteq R$,
 - $R^- \sqsubseteq R$,
 - $S_1 \circ \dots \circ S_n \sqsubseteq R$,
 - $R \circ S_1 \circ \dots \circ S_n \sqsubseteq R$,
 - $S_1 \circ \dots \circ S_n \circ R \sqsubseteq R$,

such that $R \in \mathbf{N}_R$ is a (non-inverse) role name, and $S_i < R$ for $i = 1, \dots, n$ whenever S_i is non-simple.

A role assertion is a statement of the form $\text{Ref}(R)$ (reflexivity), $\text{Asy}(V)$ (asymmetry), or $\text{Dis}(V, W)$ (role disjointness), where V and W are simple role expressions, and R is a simple role expression or a non-simple role. A SROIQB_s Rbox is the union of a set of role assertions together with a role hierarchy. A SROIQB_s Rbox is regular if its role hierarchy is regular.

Definition 2. Given a SROIQB_s Rbox \mathcal{R} , the set of concept expressions \mathbf{C} is defined as follows:

- $\mathbf{N}_C \subseteq \mathbf{C}$, $\top \in \mathbf{C}$, $\perp \in \mathbf{C}$,
- if $C, D \in \mathbf{C}$, $R \in \mathbf{B}_s \cup \mathbf{R}^n$ a simple role expression or non-simple role, $V \in \mathbf{B}_s$ a simple role expression, $a \in \mathbf{N}_I$, and n a non-negative integer, then $\neg C$, $C \sqcap D$, $C \sqcup D$, $\{a\}$, $\forall R.C$, $\exists R.C$, $\exists V.\text{Self}$, $\leq_n V.C$, and $\geq_n V.C$ are also concept expressions.

Throughout this paper, the symbols C , D will be used to denote concept expressions. A SROIQB_s Tbox is a set of general concept inclusion axioms (GCIs) of the form $C \sqsubseteq D$.

An individual assertion can have any of the following forms: $C(a)$, $R(a, b)$, $\neg V(a, b)$, $a \neq b$, with $a, b \in \mathbf{N}_I$ individual names, $C \in \mathbf{C}$ a concept expression, and $R, S \in \mathbf{R}$ roles with S simple. A SROIQB_s Abox is a set of individual assertions.

A SROIQB_s knowledge base KB is the union of a regular Rbox \mathcal{R} , and an Abox \mathcal{A} and Tbox \mathcal{T} for \mathcal{R} . We use the term axiom to uniformly refer to any single statement contained in \mathcal{R} , \mathcal{A} , or \mathcal{T} .

Name	Syntax	Semantics
inverse role	R^-	$\{(x, y) \in \Delta^I \times \Delta^I \mid (y, x) \in R^I\}$
universal role	U	$\Delta^I \times \Delta^I$
role negation	$\neg V$	$\{(x, y) \in \Delta^I \times \Delta^I \mid (x, y) \notin R^I\}$
role conjunction	$V \sqcap W$	$V^I \cap W^I$
role disjunction	$V \sqcup W$	$V^I \cup W^I$
top	\top	Δ^I
bottom	\perp	\emptyset
negation	$\neg C$	$\Delta^I \setminus C^I$
conjunction	$C \sqcap D$	$C^I \cap D^I$
disjunction	$C \sqcup D$	$C^I \cup D^I$
nominals	$\{a\}$	$\{a^I\}$
univ. restriction	$\forall R.C$	$\{x \in \Delta^I \mid (x, y) \in R^I \text{ implies } y \in C^I\}$
exist. restriction	$\exists R.C$	$\{x \in \Delta^I \mid \text{for some } y \in \Delta^I, (x, y) \in R^I \text{ and } y \in C^I\}$
Self concept	$\exists V.\text{Self}$	$\{x \in \Delta^I \mid (x, x) \in V^I\}$
qualified number	$\leq n V.C$	$\{x \in \Delta^I \mid \#\{y \in \Delta^I \mid (x, y) \in V^I \text{ and } y \in C^I\} \leq n\}$
restriction	$\geq n V.C$	$\{x \in \Delta^I \mid \#\{y \in \Delta^I \mid (x, y) \in V^I \text{ and } y \in C^I\} \geq n\}$

Fig. 1. Semantics of concept and role constructors in $SROIQB_s$ for an interpretation I with domain Δ^I .

We further give the semantics of $SROIQB_s$ knowledge bases.

Definition 3. An interpretation I consists of a set Δ^I called domain together with a function \cdot^I mapping individual names to elements of Δ^I , concept names to subsets of Δ^I , and role expressions to subsets of $\Delta^I \times \Delta^I$.

The function \cdot^I is inductively extended to role and concept expressions as shown in Fig. 1. An interpretation I satisfies an axiom φ if we find that $I \models \varphi$:

- $I \models V \sqsubseteq W$ if $V^I \subseteq W^I$,
- $I \models V_1 \circ \dots \circ V_n \sqsubseteq R$ if $V_1^I \circ \dots \circ V_n^I \subseteq R^I$ (\circ being overloaded to denote the standard composition of binary relations here),
- $I \models \text{Ref}(R)$ if R^I is a reflexive relation,
- $I \models \text{Asy}(V)$ if V^I is antisymmetric and irreflexive,
- $I \models \text{Dis}(V, W)$ if V^I and W^I are disjoint,
- $I \models C \sqsubseteq D$ if $C^I \subseteq D^I$,
- $I \models C(a)$ if $a^I \in C^I$,
- $I \models R(a, b)$ if $(a^I, b^I) \in R^I$,
- $I \models \neg V(a, b)$ if $(a^I, b^I) \notin V^I$,
- $I \models a \neq b$ if $a^I \neq b^I$.

An interpretation I satisfies a knowledge base KB (we then also say that I is a model of KB and write $I \models \text{KB}$) if it satisfies all axioms of KB . A knowledge base KB is satisfiable if it has a model. Two knowledge bases are equivalent if they have exactly the same models, and they are equisatisfiable if either both are unsatisfiable or both are satisfiable.

We obtain $SROIQ$ from $SROIQ_{\mathcal{B}_s}$ by disallowing all junctors in role expressions. Further details on $SROIQ$ can be found in [4]. We have omitted here several syntactic constructs that can be expressed indirectly, especially role assertions for transitivity, reflexivity of simple roles, and symmetry. Moreover, the DL $SHOIQ$ is obtained from $SROIQ$ by discarding the universal role as well as reflexivity, asymmetry, role disjointness statements and allowing only RIAs of the form $R \sqsubseteq S$ or $R \circ R \sqsubseteq R$.

3 Related Work

Boolean constructors on roles have been investigated in the context of both description and modal logics. [6] used them extensively for the definition of a DL that is equivalent to the two-variable fragment of FOL.

As a classical result on complexities, it was shown in [7], that augmenting \mathcal{ALC} with full Boolean role constructors (\mathcal{ALCB}) leads to NEXP TIME -completeness of the standard reasoning tasks (while restricting to role negation [7] or role conjunction [8] only retains EXP TIME -completeness). This complexity does not further increase when allowing for inverses, qualified number restrictions, and nominals as was shown in [8] by a polynomial translation of \mathcal{ALCIQB} into \mathcal{C}^2 , the two variable fragment of first order logic with counting quantifiers, which in turn was proven to be NEXP TIME -complete in [9]. Also the recently considered description logic \mathcal{ALBO} [10] falls in that range of NEXP TIME -complete DLs.

On the contrary, it was also shown in [8] that restricting to *safe* Boolean role constructors keeps \mathcal{ALC} 's reasoning complexity in EXP TIME , even when adding inverses and qualified number restrictions (\mathcal{ALCIQb}).

For logics including modelling constructs that deal with role concatenation like transitivity or – more general – complex role inclusion axioms, results on complexities in the presence of Boolean role constructors are more sparse. [11] shows that \mathcal{ALC} can be extended by negation and regular expressions on roles while keeping reasoning within EXP TIME . Furthermore, [12] provided EXP TIME complexity for a similar logic that includes inverses and qualified number restriction but reverts to safe negation on roles.

An extension of $SHIQ$ with role conjunction (denoted $SHIQ^{\square}$) is presented in [13] in the context of conjunctive query answering, the results implying an upper bound of 2EXP TIME .

4 $SROIQ_{\mathcal{B}_s}$ and $SHOIQ_{\mathcal{B}_s}$

In this section, we show that adding arbitrary (i.e. also unsafe) Boolean role expressions to the widely known description logics $SROIQ$ and $SHOIQ$ does not harm their reasoning complexities – N2EXP TIME [14] and NEXP TIME [8], respectively – if this extension is restricted to simple roles.

Note that in the sequel, $SHOIQ$ (resp. $SHOIQ_{\mathcal{B}_s}$) will be treated as a special case of $SROIQ$ (resp. $SROIQ_{\mathcal{B}_s}$), as most considerations hold for both cases.

As shown in [14], any $SROIQ$ ($SHOIQ$) knowledge base can be transformed into an equisatisfiable knowledge base containing only axioms of the form:

Table 1. Additional transformation for $SROIQB_s$ and $SHOIQB_s$. A, B are concept names. V, W are simple role expressions. V_i are simple role expressions or non-simple roles. \hat{V} is a simple role expression that is not just a role. R is a non-simple role name. \bar{S} is a new simple role name.

$A \sqsubseteq \forall \hat{V}.B$	$\mapsto \{A \sqsubseteq \forall \bar{S}.B, \hat{V} \sqsubseteq \bar{S}\}$
$A \sqsubseteq \geq n \hat{V}.B$	$\mapsto \{A \sqsubseteq \geq n \bar{S}.B, \bar{S} \sqsubseteq \hat{V}\}$
$A \sqsubseteq \leq n \hat{V}.B$	$\mapsto \{A \sqsubseteq \leq n \bar{S}.B, \hat{V} \sqsubseteq \bar{S}\}$
$A \sqsubseteq \exists \hat{V}.\text{Self}$	$\mapsto \{A \sqsubseteq \exists \bar{S}.\text{Self}, \bar{S} \sqsubseteq \hat{V}\}$
$\text{Dis}(V, W)$	$\mapsto \{V \sqcap W \sqsubseteq \bar{S}, \top \sqsubseteq \forall \bar{S}.\perp\}$
$V_1 \circ \dots \circ \hat{V} \circ \dots \circ V_n \sqsubseteq R$	$\mapsto \{V_1 \circ \dots \circ \bar{S} \circ \dots \circ V_n \sqsubseteq R, \hat{V} \sqsubseteq \bar{S}\}$

$$\begin{array}{lll}
A \sqsubseteq \forall R.B & \sqcap A_i \sqsubseteq \sqcup B_j & S_1 \sqsubseteq S_2 \\
A \sqsubseteq \geq n S.B & A \equiv \{a\} & S_1 \sqsubseteq S_2^- \\
A \sqsubseteq \leq n S.B & A \equiv \exists S.\text{Self} & \text{Dis}(S_1, S_2) \\
R_1 \circ \dots \circ R_n \sqsubseteq R. & &
\end{array}$$

Trivially, this normalization can be applied to $SROIQB_s$ ($SHOIQB_s$) as well, yielding the same types of axioms whereas simple role expressions may occur in the place of simple roles. A second transformation carried out by exhaustively applying the transformation steps depicted in Table 1 yields an equisatisfiable knowledge base containing only the original axiom types depicted above (i.e. again only simple role names in places of $S_{(i)}$ and role names in places of R_i) and just one additional axiom type $W \sqsubseteq V$ with W, V simple role expressions. As shown in [14], any of these original axiom types except the one containing role concatenation can be translated into C^2 , the two-variable fragment of first order logic with counting quantifiers. The additionally introduced type of axiom can clearly also be transformed into C^2 statements namely into the proposition $\forall xy(\Phi(W) \rightarrow \Phi(V))$ where Φ is inductively defined by:

$$\begin{aligned}
\Phi(S) &= S(x, y) \\
\Phi(S^-) &= S(y, x) \\
\Phi(\neg V) &= \neg\Phi(V) \\
\Phi(V \sqcap W) &= \Phi(V) \wedge \Phi(W) \\
\Phi(V \sqcup W) &= \Phi(V) \vee \Phi(W)
\end{aligned}$$

Further following the argumentation from [14], the remaining complex role inclusions not directly convertible into C^2 can be taken into account by cautiously materializing the consequences resulting from their interplay with axioms of the type $A \sqsubseteq \forall R.B$ through automata encoding techniques – see also [15]. This way, one obtains a C^2 theory that is satisfiable exactly if the original knowledge base is. In the case of $SROIQ$ (and hence $SROIQB_s$), this can result in an exponential blowup of the knowledge base while for $SHOIQB_s$ (and hence $SHOIQ$) the transformation is time polynomial. Thus we see that the upper complexity bounds for $SROIQ$ and $SHOIQ$ carry over to $SROIQB_s$ and $SHOIQB_s$ by just a slight extension of the according proofs from [14] while the lower bounds follow directly from those of $SROIQ$ and $SHOIQ$. Hence, we can establish the following theorem.

Theorem 1. *Knowledge base satisfiability checking, instance retrieval, and computing class subsumptions for $SROIQ\mathcal{B}_s$ ($SHOIQ\mathcal{B}_s$) knowledge bases is N2ExpTime-complete (NExpTime-complete).*

While the results established in this section are rather straightforward consequences of known results, their implications for practice might be more significant: they show that the DLs underlying OWL and OWL 2 can be extended by arbitrary Boolean constructors on simple roles without increasing the worst case complexity of reasoning.

5 $SHIQ\mathcal{B}_s$

$SHIQ$ is a rather expressive fragment obtained from $SHOIQ$ by disallowing nominals, where (in contrast to the NExpTime-complete $SHOIQ$) reasoning is known to be ExpTime-complete [8].

In this section we will introduce the extension of $SHIQ$ by safe role expressions on simple roles. Thereafter, we will present a technique for removing transitivity statements from $SHIQ\mathcal{B}_s$ knowledge bases in a satisfiability preserving way. This yields two results: on the one hand, we provide a way how existing reasoning procedures for $\mathcal{ALCHIQb}$ like e.g. those described in [8, 12, 16] can be used to solve $SHIQ\mathcal{B}_s$ reasoning tasks. On the other hand, as the transformation procedure can be done in polynomial time, the known upper bound for the complexity of reasoning in $\mathcal{ALCHIQb}$ – namely ExpTime – carries over to $SHIQ\mathcal{B}_s$.

Definition 4. *A $SHIQ\mathcal{B}_s$ knowledge base is a $SHOIQ\mathcal{B}_s$ knowledge base that contains no nominals and only safe role expressions.*

Based on a fixed knowledge base \mathbf{KB} , we define \sqsubseteq^ as the smallest binary relation on the non-simple atomic roles \mathbf{R}_n such that:*

- $R \sqsubseteq^* R$ for every atomic role R ,
- $R \sqsubseteq^* S$ and $\text{Inv}(R) \sqsubseteq^* \text{Inv}(S)$ for every Rbox axiom $R \sqsubseteq S$, and
- $R \sqsubseteq^* T$ whenever $R \sqsubseteq^* S$ and $S \sqsubseteq^* T$.

Given an atomic non-simple role R , we write $\text{Trans}(R) \in \mathbf{KB}$ as an abbreviation for: $R \circ R \sqsubseteq R \in \mathbf{KB}$ or $\text{Inv}(R) \circ \text{Inv}(R) \sqsubseteq \text{Inv}(R) \in \mathbf{KB}$.

Slightly generalising according results from [8, 17] (as we allow safe boolean expressions – in GCIs and role inclusion axioms – already for the original logic), we now show that any $SHIQ\mathcal{B}_s$ knowledge base can be transformed into an equisatisfiable knowledge base not containing transitivity statements.

Definition 5. *Given a $SHIQ\mathcal{B}_s$ knowledge base \mathbf{KB} , let $\text{clos}(\mathbf{KB})$ denote the smallest set of concept expressions where*

- $\text{NNF}(\neg C \sqcup D) \in \text{clos}(\mathbf{KB})$ for any Tbox axiom $C \sqsubseteq D$,
- $D \in \text{clos}(\mathbf{KB})$ for every subexpression D of some concept $C \in \text{clos}(\mathbf{KB})$,
- $\text{NNF}(\neg C) \in \text{clos}(\mathbf{KB})$ for any $\leq n R.C \in \text{clos}(\mathbf{KB})$,
- $\forall S.C \in \text{clos}(\mathbf{KB})$ whenever $\text{Trans}(S) \in \mathbf{KB}$ and $S \sqsubseteq^* R$ for a role R with $\forall R.C \in \text{clos}(\mathbf{KB})$.

Moreover, let $\Omega(\text{KB})$ denote the knowledge base obtained from KB by

- removing all transitivity axioms $R \circ R \sqsubseteq R$ and
- adding the axiom $\forall R.C \sqsubseteq \forall S.(\forall S.C)$ for every $\forall R.C \in \text{clos}(\text{KB})$ with $\text{Trans}(S) \in \text{KB}$ and $S \sqsubseteq^* R$.

Proposition 1. *Let KB be a SHIQ_s knowledge base. Then, KB and $\Omega(\text{KB})$ are equisatisfiable.*

Taking into account that the presented transformation is time polynomial, this result can now be employed to determine the complexity of SHIQ_s .

Theorem 2. *Knowledge base satisfiability checking, instance retrieval, and computing class subsumptions for SHIQ_s knowledge bases is ExpTime -complete.*

Proof. Clearly, all standard reasoning problems can be reduced to knowledge base satisfiability checking as usual.

Now, by Proposition 1, any given SHIQ_s knowledge base KB can be transformed into an \mathcal{ALCHIQ} knowledge base $\Omega(\text{KB})$ in polynomial time. Furthermore, all role inclusion axioms can be removed from $\Omega(\text{KB})$ as follows. First, all role names contained in $\Omega(\text{KB})$ can be declared to be simple without violating the syntactic constraints. Second, every role inclusion axiom $V \sqsubseteq W$ (with V, W being safe by definition) can be equivalently transformed into the GCI $\top \sqsubseteq \forall(V \sqcap \neg W).\perp$. Note that then $V \sqcap \neg W$ is safe as well and therefore admissible. Moreover the transformation is obviously time linear. So we end up with an \mathcal{ALCIQ} knowledge base whose satisfiability checking is ExpTime -complete due to [8]. \square

So we have shown that allowing safe Boolean expressions on simple roles does not increase the ExpTime reasoning complexity of SHIQ . On the other hand, the recent results on SHIQ^\top [13] seem to indicate that the role simplicity condition is essential for staying within ExpTime even though no definite hardness result for general SHIQ^\top was provided. The safety condition on role expressions, in turn, is clearly needed: dropping it would lead to a DL comprising \mathcal{ALCB} which is known to be NExpTime -complete [7].

6 $\mathcal{EL}^{++}(\sqcap_s)$

In this section, we investigate role conjunction for the DL \mathcal{EL}^{++} [18], for which many typical inference problems can be solved in polynomial time. We simplify our presentation by omitting concrete domains from \mathcal{EL}^{++} – they are not affected by our extension and can be treated as shown in [18].

Definition 6. *An atomic role of $\mathcal{EL}^{++}(\sqcap_s)$ is a (non-inverse) role name. An $\mathcal{EL}^{++}(\sqcap_s)$ role expression is a simple role expression that contains only role conjunctions. An $\mathcal{EL}^{++}(\sqcap_s)$ Rbox is a set of generalised role inclusion axioms (using $\mathcal{EL}^{++}(\sqcap_s)$ role expressions and non-simple atomic roles). An $\mathcal{EL}^{++}(\sqcap_s)$ Tbox is a SROIQB_s Tbox that contains only the concept constructors: $\sqcap, \exists, \top, \perp$ and only $\mathcal{EL}^{++}(\sqcap_s)$ role expressions.*

Note that we do not have any requirement for regularity of roles but we have to introduce the notion of role simplicity in the context of \mathcal{EL}^{++} . In a first step, we observe that any $\mathcal{EL}^{++}(\Pi_s)$ knowledge base can be converted into a normal form.

Definition 7. An $\mathcal{EL}^{++}(\Pi_s)$ knowledge base KB is in normal form if it contains only axioms of one of the following forms:

$$\begin{array}{lll} A \sqsubseteq C & A \sqcap B \sqsubseteq C & R \sqsubseteq T \\ \exists R.A \sqsubseteq B & A \sqsubseteq \exists R.B & R \circ S \sqsubseteq T \\ & R \sqcap S \sqsubseteq T & \end{array}$$

where $A, B \in \mathbf{N}_C \cup \{\{a\} \mid a \in \mathbf{N}_I\} \cup \{\top\}$, $C \in \mathbf{N}_C \cup \{\{a\} \mid a \in \mathbf{N}_I\} \cup \{\perp\}$, and $R, S, T \in \mathbf{N}_R$.

Proposition 2. Any $\mathcal{EL}^{++}(\Pi_s)$ knowledge base can be transformed into an equisatisfiable $\mathcal{EL}^{++}(\Pi_s)$ knowledge base in normal form. The transformation can be done in linear time.

Subsequently, we show that the only axiom type of this normal form not covered by \mathcal{EL}^{++} can be removed from an $\mathcal{EL}^{++}(\Pi_s)$ knowledge base while preserving satisfiability if the relevant consequences are materialized before.

Definition 8. Given an $\mathcal{EL}^{++}(\Pi_s)$ knowledge base KB in normal form, let $\Theta^\square(\text{KB})$ denote the knowledge base obtained from KB by

- adding $R_1 \sqsubseteq R_2$ for all $R_2 \in R_1^\square$ where $S^\square \subseteq \mathbf{N}_R$ denotes the smallest set of role names containing S and satisfying
 - $T \in S^\square$, whenever $R \in S^\square$ and $R \sqsubseteq T \in \text{KB}$ as well as
 - $T \in S^\square$, whenever $R_1, R_2 \in S^\square$ and $R_1 \sqcap R_2 \sqsubseteq T \in \text{KB}$,
- removing every axiom of the form $S_1 \sqcap S_2 \sqsubseteq R$ and instead adding the axioms $\exists S_1.\{o\} \sqcap \exists S_2.\{o\} \sqsubseteq \exists R.\{o\}$ for every individual name $\{o\}$.

Note that $\Theta^\square(\text{KB})$ can be computed in polynomial time. In particular, finding the closed sets R^\square can be done in linear time w.r.t. the size of KB , e.g. using the *linclature* algorithm from [19].

Proposition 3. Let KB be an $\mathcal{EL}^{++}(\Pi_s)$ knowledge base. Then, KB and $\Theta^\square(\text{KB})$ are equisatisfiable.

The shown reduction – besides providing a way of using existing \mathcal{EL}^{++} reasoning algorithms for reasoning in $\mathcal{EL}^{++}(\Pi_s)$ – now gives rise to the complexity result for $\mathcal{EL}^{++}(\Pi_s)$.

Theorem 3. Knowledge base satisfiability checking, instance retrieval, and computing class subsumptions for $\mathcal{EL}^{++}(\Pi_s)$ knowledge bases is P-complete w.r.t. the size of the knowledge base.

Proof. Given an arbitrary $\mathcal{EL}^{++}(\Pi_s)$ knowledge base KB , Proposition 2 ensures that it can be transformed in polynomial time into an equisatisfiable knowledge base KB' in normal form. Again in polynomial time, we can compute the knowledge base $\Theta^\square(\text{KB}')$

that – by Proposition 3 – is equisatisfiable with KB' (and hence also with KB). Finally, as $\Theta^\square \text{KB}'$ is an \mathcal{EL}^{++} knowledge base, we can check satisfiability in polynomial time.

P-hardness for $\mathcal{EL}^{++}(\sqcap_s)$ follows from the well known P-hardness of \mathcal{EL} (also being a straightforward consequence of the P-completeness of Horn satisfiability). \square

We finish this section with some general remarks. On the one hand, note that conjunction on roles enhances expressivity of \mathcal{EL}^{++} significantly. For example, it allows for the following modelling features:

- Disjointness of two simple roles S, R . This feature, also provided by *SROIQ* as $\text{Dis}(S, R)$, can be modelled in $\mathcal{EL}^{++}(\sqcap_s)$ by the axiom $\exists(S \sqcap R). \top \sqsubseteq \perp$.
- At least cardinality constraints on the right hand side of a GCI. The axiom $A \sqsubseteq \geq n R.B$ can be modelled by the axiom set $\{R_i \sqsubseteq R, A \sqsubseteq \exists R_i.B \mid 1 \leq i \leq n\} \cup \{\exists(R_i \sqcap R_j). \top \sqsubseteq \perp \mid 1 \leq i < j \leq n\}$ where R_1, \dots, R_n are new simple role names.

On the other hand, it is easy to see that incorporating more than just conjunction on simple roles into \mathcal{EL}^{++} would render the respective fragment intractable at best: Allowing conjunction on non-simple roles would even lead to undecidability as stated in Theorem 1 of [20]. Allowing disjunction or negation on simple roles would allow to model disjunction on concepts: for instance, the GCI $A \sqsubseteq B \sqcup C$ can be expressed by the axiom set $\{A \sqsubseteq \exists(R \sqcup S). \top, \exists R. \top \sqsubseteq B, \exists S. \top \sqsubseteq C\}$ or the axiom set $\{A \sqcap \exists R. \{o\} \sqsubseteq C, A \sqcap \exists \neg R. \{o\} \sqsubseteq B\}$ for new roles R, S and a new individual name o . Hence, any extension of \mathcal{EL}^{++} into this direction would be ExpTime-hard [18].

7 DLP(\sqcap)

Description Logic Programs (DLP) constitutes a tractable knowledge representation formalism in the spirit of (Horn) logic programming [21]. Essentially, it consists of those *SHOIQ* axioms which can be naively translated into (non-disjunctive) Datalog, such that the original knowledge base and its translation are semantically equivalent. As such it represents the fragment of *SHOIQ* that can entail neither disjunctive information nor the existence of anonymous individuals as extensively studied in the context of Horn description logics [22]. Though rather complex syntactic definitions can be given to characterise all admissible axioms of such logics, we use a simpler definition comprising all essential expressive features of DLP without including all their syntactic varieties.

Definition 9. *Atomic roles of DLP are defined as in SROIQ, including inverse roles. A DLP body concept is any SROIQ concept expression that includes only concept names, nominals, \sqcap , \exists , \top , and \perp . A DLP head concept is any SROIQ concept expression that includes only concept names, nominals, \sqcap , \forall , \top , \perp , and expressions of the form $\leq 1.C$ where C is a DLP body concept.*

A DLP knowledge base is a set of Rbox axioms of the form $R \sqsubseteq S$ and $R \circ R \sqsubseteq R$, Tbox axioms of the form $C \sqsubseteq D$, and Abox axioms of the form $D(a)$ and $R(a, b)$, where $C \in \mathbf{C}$ is a body concept, $D \in \mathbf{C}$ is a head concept, and $a, b \in \mathbf{N}_I$ are individual names.

DLP(\sqcap) knowledge bases are defined just as DLP knowledge bases, with the addition that conjunctions of roles may occur in DLP(\sqcap) in all places where roles occur in DLP.

Note that we do not have to distinguish between simple and non-simple roles for DLP.

In [22] it is shown that DLP is of polynomial worst-case complexity. This can be seen most easily by realising that DLP knowledge bases can be transformed in polynomial time (in the size of the knowledge base) into an equisatisfiable set of function-free first-order Horn rules (i.e. non-disjunctive Datalog rules) with at most three variables per formula. On the basis of this result, it is easy to show that $DLP(\sqcap)$ is also of polynomial complexity. We give a brief account of the argument.

Consider a $DLP(\sqcap)$ knowledge base K . We now perform the following transformation of K : For any role conjunction $R_1 \sqcap \dots \sqcap R_n$ occurring in the knowledge base, replace the conjunction by a new role R , and add the axioms $R_1 \sqcap \dots \sqcap R_n \sqsubseteq R$ and $R \sqsubseteq R_i$, for all $i = 1, \dots, n$, to the knowledge base.

The resulting knowledge base is obviously equisatisfiable with K . It consists of two types of axioms: Axioms which are in DLP and axioms of the form $R_1 \sqcap \dots \sqcap R_n \sqsubseteq R$. The latter axioms correspond to function-free Horn rules with only two variables. Hence, any $DLP(\sqcap)$ knowledge base can be transformed in polynomial time into an equisatisfiable set of function-free Horn rules.

Theorem 4. *Knowledge base satisfiability checking, instance retrieval, and computing class subsumptions for $DLP(\sqcap)$ knowledge bases is P-complete w.r.t. the size of the knowledge base.*

Proof. First note that instance retrieval and class subsumption can be reduced to satisfiability checking: Retrieval of instances for a class C is done by checking for all individuals a if they are in C – which in turn is reduced to satisfiability checking by adding the axioms $C \sqcap E \sqsubseteq \perp$ and $E(a)$ to the knowledge base, where E is a new atomic class name. Class subsumption $C \sqsubseteq D$ is reduced by adding the axioms $C(a)$, $E(a)$ and $D \sqcap E \sqsubseteq \perp$, for a new individual a and a new atomic class name E .

Now to check satisfiability of a $DLP(\sqcap)$ knowledge base, it is first transformed into an equisatisfiable set of function-free first-order Horn rules as mentioned above. The satisfiability of such a set of formulae can be checked in polynomial time, since any Horn logic program is semantically equivalent to its *grounding* (the set of all possible ground instances of the given rules based on the occurring individual names). For a program with a bounded number n of variables per rule, this grounding is bounded by $r \times i^n$, where i is the number of individual names and r is the number of rules in the program. Finally, the evaluation of ground Horn logic programs is known to be in P .

P-hardness for DLP directly follows from the P-completeness of satisfiability checking for sets of propositional Horn clauses. \square

8 Conclusion

In our work, we have thoroughly investigated the reasoning complexities of DLs allowing for Boolean constructors on simple roles. We found that the expressive DLs *SROIQ* (being the basis of the forthcoming OWL 2 standard) and *SHOIQ* (the logical underpinning of OWL) can accommodate full Boolean role operators while keeping their reasoning complexities $N2ExpTime$ and $NExpTime$, respectively. Likewise, the

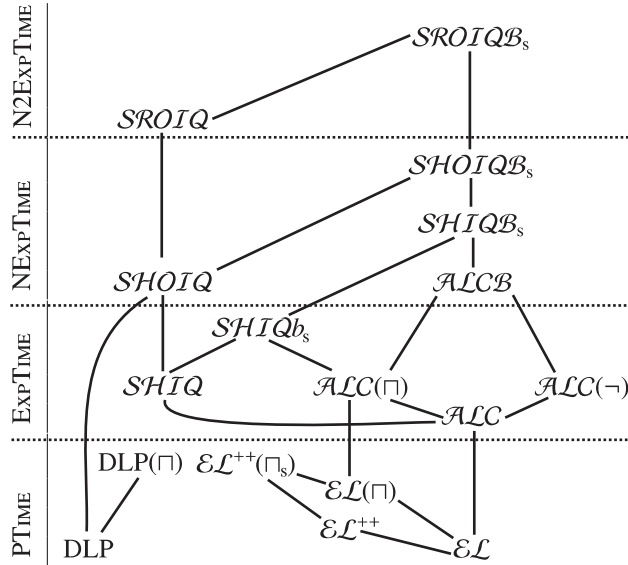


Fig. 2. Overview of complexities and expressivity relationships of DLs in the context of this paper.

EXP TIME-complete $SHIQ$ can be safely extended by safe Boolean expressions. Finally, both the tractable fragments \mathcal{EL}^{++} and DLP retain polynomial time reasoning complexity when adding just role conjunction, where in the case of DLP the role simplicity condition is not necessary. Figure 2 shows our findings integrated with other well-known complexity results relevant in this respect.

In particular, we want to draw the reader’s attention to the fact that – as opposed to hitherto proposed ways – the modelling of concept products and qualified role inclusions as presented in Section 1 does *not* automatically render the inferred roles non-simple. Moreover, due to the safety of the respective axiom, qualified role inclusions can even be modelled in $SHIQ_b_s$.

Future work on that topic includes the further integration of the established results with our work on DL Rules [2], as well as the further investigation of the effects on complexity and decidability when allowing for Boolean constructors on non-simple roles.

Finally note that our results for $SHIQ_b_s$, $\mathcal{EL}^{++}(\cap_s)$, and $DLP(\cap)$ provide direct ways for adapting existing reasoning algorithms for $SHIQ$, \mathcal{EL}^{++} , and DLP , respectively. For $SROIQ_b_s$ and $SHOIQ_b_s$, however, setting up efficient algorithms seems less straightforward and represents another interesting direction of future research.

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