Lecture 2

CP in a Nutshell

Outline

- Introduce notion of equivalence of CSP's
- Provide intuitive introduction to general methods of Constraint Programming
- Introduce basic framework for Constraint Programming
- Illustrate this framework by 2 examples

Projection

- Given: variables $X := x_1, ..., x_n$ with domains $D_1, ..., D_n$
  Consider
  - $d := (d_1, ..., d_n) \in D_1 \times ... \times D_n$
  - subsequence $Y := x_{i_1}, ..., x_{i_l}$ of X

  Denote $(d_{i_1}, ..., d_{i_l})$ by $d[Y]$: projection of d on Y

  In particular: $d[x_i] = d_i$

- Note: For a CSP

  $$\mathcal{P} := \langle \mathcal{C} ; x_1 \in D_1, ..., x_n \in D_n \rangle$$

  $(d_1, ..., d_n) \in D_1 \times ... \times D_n$ is a solution to $\mathcal{P}$ iff for each constraint C of $\mathcal{P}$ on a sequence of variables Y

  $$d[Y] \in C$$

Equivalence of CSP's

- $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent if they have the same set of solutions
- CSP's $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent w.r.t. X iff

  $$\{d[X] \mid d \text{ is a solution to } \mathcal{P}_1\} = \{d[X] \mid d \text{ is a solution to } \mathcal{P}_2\}$$

- Union of $\mathcal{P}_1, ..., \mathcal{P}_m$ is equivalent w.r.t. X to $\mathcal{P}_0$ if

  $$\{d[X] \mid d \text{ is a solution to } \mathcal{P}_0\} = \bigcup_{i=1}^{m} \{d[X] \mid d \text{ is a solution to } \mathcal{P}_i\}$$

Solved and Failed CSP's

- C a constraint on variables $y_1, ..., y_k$ with domains $D_1, ..., D_k$ (so $C \subseteq D_1 \times ... \times D_k$):
  C is solved if $C = D_1 \times ... \times D_k$
- CSP is solved if
  - all its constraints are solved, and
  - no domain of it is empty
- CSP is failed if
  - it contains the false constraint $\bot$, or
  - some of its domains is empty

CP: Basic Framework

```
procedure solve
var continue := true
begin
    while continue and not happy do
        Preprocess;
        Constraint Propagation;
        if not happy then
            if Atomic then continue := false
            else
                Split; Proceed by Cases
            end-if
    end-while
end
```

Preprocess

Bring to desired syntactic form

- Example: Constraints on reals
  Desired syntactic form: no repeated occurrences of a variable

$$ax^7 + bx^5y + cy^{10} = 0$$

$$\rightarrowtail ax^7 + z + cy^{10} = 0,\ bx^5y = z$$

Happy

- Found a solution
- Found all solutions
- Found a solved form from which one can generate all solutions
- Determined that no solution exists (inconsistency)
- Found best solution
- Found all best solutions
- Reduced all interval domains to sizes < ε

Atomic and Split

- Check whether CSP is amenable for splitting, or
- whether search 'under' this CSP is still needed

Split a domain:

- *D* finite (Enumeration)

$$\frac{x \in D}{x \in \{a\} \mid x \in D - \{a\}}$$

- *D* finite (Labeling)

$$\frac{x \in \{a_1, \ldots, a_k\}}{x \in \{a_1\} \mid \ldots \mid x \in \{a_k\}}$$

- *D* interval of reals (Bisection)

$$\frac{x \in [a..b]}{x \in \left[a..\lfloor \frac{a+b}{2} \rfloor\right] \mid x \in \left[\lfloor \frac{a+b}{2} \rfloor + 1..b\right]}$$

Split, ctd

Split a constraint:

- Disjunctive constraints

$$\frac{C_1 \vee C_2}{C_1 \mid C_2}$$

- Constraints in "compound" form
  Example:

$$\frac{|p(\bar{x})| = a}{p(\bar{x}) = a \mid p(\bar{x}) = -a}$$

Effect of Split

- Each Split replaces current CSP $\mathcal{P}$ by CSP's $\mathcal{P}_1, ..., \mathcal{P}_n$ such that the union of $\mathcal{P}_1, ..., \mathcal{P}_n$ is equivalent to $\mathcal{P}$.
- Example:
  Enumeration replaces

  $$\langle \mathcal{C} \ ; \mathcal{DE}, x \in D \rangle$$

  by

  $$\langle \mathcal{C}' \ ; \mathcal{DE}, x \in \{a\} \rangle$$

  and

  $$\langle \mathcal{C}'' \ ; \mathcal{DE}, x \in D - \{a\} \rangle$$

  where $\mathcal{C}'$ and $\mathcal{C}''$ are restrictions of the constraints from $\mathcal{C}$ to the new domains.

Heuristics

Which

- variable to choose
- value to choose
- constraint to split

Examples:

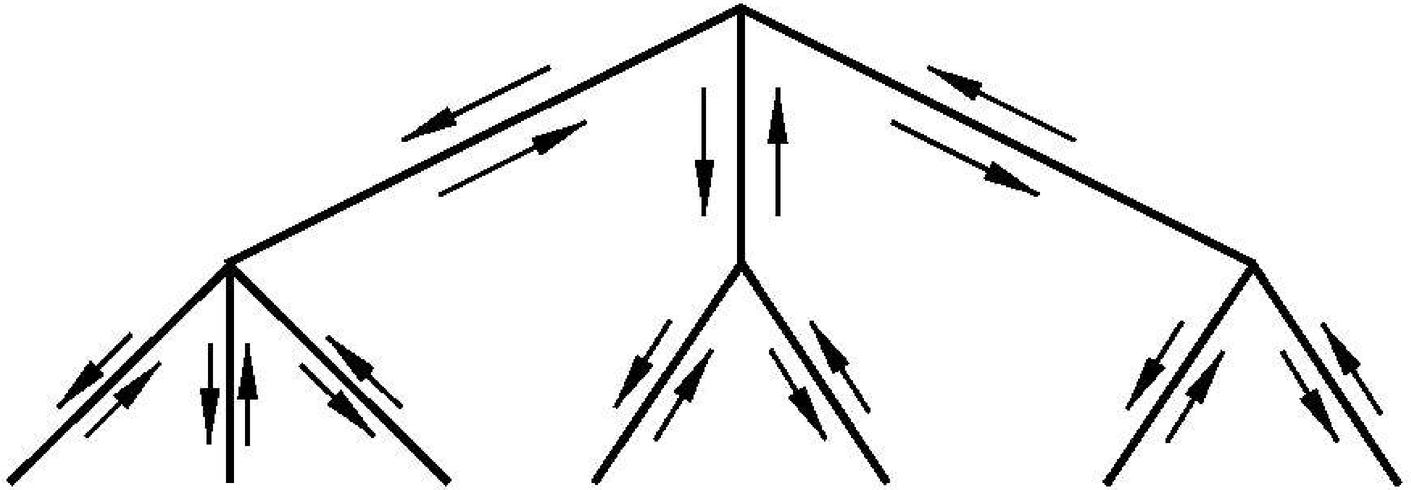
- Select a variable that appears in the largest number of constraints (most constrained variable)
- For a domain being an integer interval: select the middle value

Proceed by Cases

Various search techniques

- Backtracking
- Branch and bound
- Can be combined with Constraint Propagation
- Intelligent backtracking

Backtracking

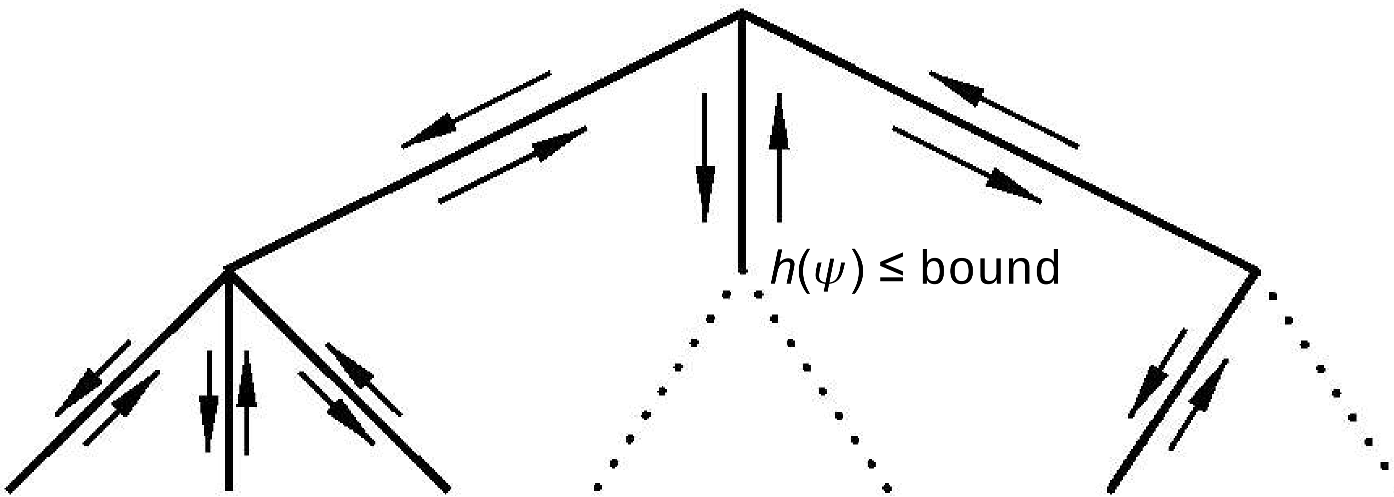


- Nodes generated "on the fly"
- Nodes are CSP's
- Leaves are CSP's that are solved or failed

Branch and Bound

- Modification of backtracking aiming at finding the optimal solution
- Takes into account objective function
- Maintain currently best value of the objective function in variable bound
- bound initialized to $-\infty$ and updated each time a better solution found
- Used in combination with heuristic function
- Conditions on heuristic function *h*:
  - If ψ is a direct descendant of ϕ, then
    $h(\psi) \leq h(\phi)$
  - If ψ is solved CSP with singleton set domains, then
    $obj(\psi) \leq h(\psi)$
- *h* allows us to prune the search tree

Illustration



Constraint Propagation

Replace a CSP by an equivalent one that is “simpler”

Constraint propagation performed by repeatedly reducing

- domains

and/or

- constraints

while maintaining equivalence

Reduce a Domain: Examples

- Projection rule:
  Take a constraint *C* and choose a variable *x* of it with domain *D*.
  Remove from *D* all values for *x* that do not participate in a solution to *C*.

- Linear inequalities on integers:

$$\frac{\langle x < y \,;\, x \in [50..200],\, y \in [0..100] \rangle}{\langle x < y \,;\, x \in [50..99],\, y \in [51..100] \rangle}$$

Repeated Domain Reduction: Example

Consider

$\langle x < y, y < z ; x \in [50..200], y \in [0..100], z \in [0..100] \rangle$

Apply above rule to $x < y$:

$\langle x < y, y < z ; x \in [50..99], y \in [51..100], z \in [0..100] \rangle$

Apply it now to $y < z$:

$\langle x < y, y < z ; x \in [50..99], y \in [51..99], z \in [52..100] \rangle$

Apply it again to $x < y$:

$\langle x < y, y < z ; x \in [50..98], y \in [51..99], z \in [52..100] \rangle$

Reduce Constraints

Usually by introducing new constraints!

- Transitivity of <:

$$\frac{\langle x<y\,,\,y<z\,;\mathcal{D}\,\mathcal{E}\rangle}{\langle x<y\,,\,y<z\,,\,x<z\,;\mathcal{D}\,\mathcal{E}\rangle}$$

  This rule introduces new constraint $x < z$

- Resolution rule:

$$\frac{\langle C_1\vee L\,,\,C_2\vee\bar{L}\,;\mathcal{D}\,\mathcal{E}\rangle}{\langle C_1\vee L\,,\,C_2\vee\bar{L}\,,\,C_1\vee C_2\,;\mathcal{D}\,\mathcal{E}\rangle}$$

  This rule introduces new constraint $C_1 \vee C_2$

Constraint Propagation Algorithms

- Deal with scheduling of atomic reduction steps
- Try to avoid useless applications of atomic reduction steps
- Stopping criterion for general CSP's: a local consistency notion

  Example:
  Local consistency criterion corresponding to the projection rule is Hyper-arc consistency:
  For every constraint *C* and every variable *x* with domain *D*, each value for *x* from *D* participates in a solution to *C*.

Example: Boolean Constraints

Happy: found all solutions

Desired syntactic form (for preprocessing):

- $x = y$
- $\neg x = y$
- $x \wedge y = z$
- $x \vee y = z$

- Preprocessing:

$$\frac{x \wedge s = z}{x \wedge y = z \, , s = y}$$

- Constraint propagation:

$$\frac{\langle x \wedge y = z \, ; x \in D_x \, , y \in D_y \, , z \in \{1\} \rangle}{\langle ; x \in D_x \cap \{1\} , y \in D_y \cap \{1\} , z \in \{1\} \rangle}$$

(write as $x \wedge y = z$, $z = 1 \rightarrowtail x = 1$, $y = 1$)

Boolean Constraints, ctd

- $x = y,\ x = 1 \rightarrowtail y = 1$
- $x = y,\ y = 1 \rightarrowtail x = 1$
- $x = y,\ x = 0 \rightarrowtail y = 0$
- $x = y,\ y = 0 \rightarrowtail x = 0$

- $\neg x = y,\ x = 1 \rightarrowtail y = 0$
- $\neg x = y,\ x = 0 \rightarrowtail y = 1$
- $\neg x = y,\ y = 1 \rightarrowtail x = 0$
- $\neg x = y,\ y = 0 \rightarrowtail x = 1$

- $x \wedge y = z,\ x = 1,\ y = 1 \rightarrowtail z = 1$
- $x \wedge y = z,\ x = 1,\ z = 0 \rightarrowtail y = 0$
- $x \wedge y = z,\ y = 1,\ z = 0 \rightarrowtail x = 0$
- $x \wedge y = z,\ x = 0 \rightarrowtail z = 0$
- $x \wedge y = z,\ y = 0 \rightarrowtail z = 0$
- $x \wedge y = z,\ z = 1 \rightarrowtail x = 1,\ y = 1$

- $x \vee y = z,\ x = 1 \rightarrowtail z = 1$
- $x \vee y = z,\ x = 0,\ y = 0 \rightarrowtail z = 0$
- $x \vee y = z,\ x = 0,\ z = 1 \rightarrowtail y = 1$
- $x \vee y = z,\ y = 0,\ z = 1 \rightarrowtail x = 1$
- $x \vee y = z,\ y = 1 \rightarrowtail z = 1$
- $x \vee y = z,\ z = 0 \rightarrowtail x = 0,\ y = 0$

Boolean Constraints, ctd

Split:

- Choose the most constrained variable
- Apply the labeling rule:

$$\frac{x \in \{0,1\}}{x \in \{0\} \mid x \in \{1\}}$$

Proceed by cases: backtrack

Example: Polynomial Constraints on Integer Intervals

Domains: integer intervals $[a..b]$

$$[a..b] \coloneqq \{x \in \mathbb{Z} \mid a \leq x \leq b\}$$

Constraints:

$$s = 0$$

s is a polynomial (possibly in several variables) with integer coefficients

Example:

$$2 \cdot x^5 \cdot y^2 \cdot z^4 + 3 \cdot x \cdot y^3 \cdot z^5 - 4 \cdot x^4 \cdot y^6 \cdot z^2 + 10 = 0$$

Objective function: a polynomial

Example

Find a solution to

$$x^3 + y^2 - z^3 = 0$$

in [1..1000] such that

$$2 \cdot x \cdot y - z$$

is maximal.

Answer:

$x = 112$, $y = 832$, $z = 128$

Polynomial Constraints on Integer Intervals, ctd

Desired syntactic form:

- $\sum_{i=1}^{n} a_i x_i = b$
- $x \cdot y = z$

Preprocess:

Use appropriate transformation rules

Example:

$$\frac{\langle \sum_{i=1}^{n} m_i = 0 \, ; \mathcal{DE} \rangle}{\langle \sum_{i=1}^{n} v_i = 0, m_1 = v_1, \ldots, m_n = v_n ; \mathcal{DE}, v_1 \in \mathbb{Z}, \ldots, v_n \in \mathbb{Z} \rangle}$$

where

- some m_i is not of the form ax_i
- $v_1, \ldots, v_n$ do not appear in $\mathcal{DE}$

Happy: found an optimal solution w.r.t. the objective function

Polynomial Constraints on Integer Intervals, ctd

Constraint propagation: uses interval arithmetic

X, Y sets of integers

- addition:

  $X + Y := \{x + y \mid x \in X, y \in Y\}$

- subtraction:

  $X - Y := \{x - y \mid x \in X, y \in Y\}$

- multiplication:

  $X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}$

- division:

  $X/Y := \{u \in \mathbb{Z} \mid \exists x \in X \exists y \in Y \; u \cdot y = x\}$

Interval Arithmetic, ctd

Given: X, Y integer intervals, a an integer

- $X \cap Y$, $X + Y$, $X - Y$ are integer intervals
- $X/\{a\}$ is an integer interval
- $X \cdot Y$ does not have to be an integer interval, even if $X = \{a\}$ or $Y = \{a\}$
- X/Y does not have to be an integer interval

Examples:

[2..4] + [3..8] = [5..12]

[3..7] – [1..8] = [– 5..6]

[3..3] · [1..2] = {3, 6}

[3..5]/[–1..2] = {–5, –4, –3, 2, 3, 4, 5}

[–3..5]/[–1..2] = $\mathbb{Z}$

Turning Sets to Intervals

$$int(X) := \begin{cases} \text{smallest int. interval} \supseteq X & \text{if } X \text{ finite} \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

Examples:

$int([3..3] \cdot [1..2]) = [3..6]$

$int([3..5]/[-1..2]) = [-5..5]$

$int([-3..5]/[-1..2]) = \mathbb{Z}$

Rule for Linear Equality

$$\frac{\left\langle \sum_{i=1}^{n} a_i x_i = b \,;\, x_1 \in D_{1,} \ldots, x_n \in D_n \right\rangle}{\left\langle \sum_{i=1}^{n} a_i x_i = b \,;\, \ldots, x_j \in D_j^{'}, \ldots \right\rangle}$$

where $j \in [1..n]$, and

$$D_j^{'} \coloneqq D_j \cap \frac{b - \sum_{i \in [1..n]-\{j\}} int(a_i \cdot D_i)}{a_j}$$

Multiplication Rules

Multiplication 1

$$\frac{\langle x \cdot y = z \,;\, x \in D_x ,\, y \in D_y ,\, z \in D_z \rangle}{\langle x \cdot y = z \,;\, x \in D_x ,\, y \in D_y ,\, z \in D_z \cap int(D_x \cdot D_y) \rangle}$$

Multiplication 2

$$\frac{\langle x \cdot y = z \,;\, x \in D_x ,\, y \in D_y ,\, z \in D_z \rangle}{\langle x \cdot y = z \,;\, x \in D_x \cap int(D_z / D_y) ,\, y \in D_y ,\, z \in D_z \rangle}$$

Multiplication 3

$$\frac{\langle x \cdot y = z \,;\, x \in D_x ,\, y \in D_y ,\, z \in D_z \rangle}{\langle x \cdot y = z \,;\, x \in D_x ,\, y \in D_y \cap int(D_z / D_x) ,\, z \in D_z \rangle}$$

Effect of Multiplication Rules

Consider

$$\langle x \cdot y = z \,;\, x \in [1..20],\ y \in [9..11],\ z \in [155..161] \rangle$$

Using Multiplication Rules we can transform this to

$$\langle x \cdot y = z \,;\, x \in [16..16],\ y \in [10..10],\ z \in [160..160] \rangle$$

Polynomial Constraints on Integer Intervals, ctd

Split:

- Choose the variable with the smallest interval domain
- Apply the bisection rule:

$$\frac{x \in [a..b]}{x \in \left[a..\lfloor\frac{a+b}{2}\rfloor\right] \mid x \in \left[\lfloor\frac{a+b}{2}\rfloor+1..b\right]}$$

  where $a < b$

- Proceed by cases: branch and bound

More on Interval Arithmetic

Given objective function *obj*.

obj^+: extension of *obj* to function from sets of integers to sets of integers.

Example: $obj(x,y) \coloneqq x^2 \cdot y - 3x \cdot y^2 + 5$

Then $obj^+(X,Y) = X \cdot X \cdot Y - 3 \cdot X \cdot Y \cdot Y + 5$

Lemma

Consider integer intervals $X_1, ..., X_n$

- $obj^+(X_1, ...,X_n)$ is a finite set of integers
- For all $a_i \in X_i$, $i \in [1..n]$

  $obj(a_1, ...,a_n) \in obj^+(X_1, ...,X_n)$
- For all $Y_i \subseteq X_i$, $i \in [1..n]$

  $obj^+(Y_1, ...,Y_n) \subseteq obj^+(X_1, ...,X_n)$

Heuristic Function

Take

- $\mathcal{P} := \langle \mathcal{C} \,;\, x_1 \in D_1, ..., x_n \in D_n \rangle$, with $D_1, ..., D_n$ integer intervals
- *obj*: polynomial with variables $x_1, ..., x_n$

Define

$$h(\mathcal{P}) := max(obj^{+}(D_1, ..., D_n))$$

Thanks to the preceding lemma, *h* satisfies the conditions for the heuristic function (cf. Slide 15).

Objectives

- Introduce notion of equivalence of CSP's
- Provide intuitive introduction to general methods of Constraint Programming
- Introduce a basic framework for Constraint Programming
- Illustrate this framework by 2 examples