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Advanced Topics in Complexity Theory

Exercise 5: Approximation and Complexity

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Exercise 5.1 Let $G = (V, E)$ be an undirected graph. A *cut* in G is a set $S \subseteq G$ such that $S \neq V$ and $S \neq \emptyset$. The *size* of the cut S is the number of edges in G between nodes in S and $V \setminus S$. The problem MaxCut is to find a cut of maximal size in a given undirected graph.

1. Devise an approximation algorithm based on the following idea of *local improvement*: whenever a cut S of G is given and there exists a node v such that swapping membership in S strictly increases the size of the cut, add or remove v to S, respectively. Show that this algorithm stops after at most polynomially many steps.

2. Show that this yields an $1/2$-approximation algorithm for MaxCut. For this consider a partition
$$V = (V_1 \cup V_2 \cup V_3 \cup V_4)$$
such that the cut returned by the approximation algorithm is $(V_1 \cup V_2, V_3 \cup V_4)$, whereas the optimal cut is $(V_1 \cup V_3, V_2 \cup V_4)$. Denote with e_{ij} the number of edges between V_i and V_j. Consider the number of edges between nodes in V_1 and V and show
$$e_{12} \leq e_{13} + e_{14}$$
(and similar inequalities) and conclude
$$e_{12} + e_{34} + e_{14} + e_{23} \leq 2 \cdot (e_{13} + e_{14} + e_{23} + e_{24}).$$
Why is this sufficient to prove the claim?

Exercise 5.2 Let us consider the following optimization problem: let $\Phi = \{ \varphi_1, \dots, \varphi_m \}$ be a set of Boolean expressions in the n variables $x_1, \dots, x_n$, where each expression φ_k involves at most k of the n variables. Denote with k-MaxGSAT (for *maximum generalized satisfiability*) the problem to find an assignment for Φ that maximizes the number of satisfied expressions in Φ.

1. Denote with $X_i \colon \Omega \to \{ 0, 1 \}$ the random variable representing the fact that the expression φ_i is satisfied when uniformly choosing an assignment for the n variables. Let $X = \sum_{i=1}^{n} X_i$. Show
$$\mathrm{E}(X) = \sum_{i=1}^{n} \Pr(X_i = 1).$$
Argue that $\mathrm{E}(X)$ can be computed in linear time in the size of Φ.

2. Let $y \in \{0,1\}$ and denote with $\Phi[x_1 = y]$ the set of Boolean expressions φ'_i, where $\varphi'_i = \varphi_i[x_1 = y]$. Show

$$\mathrm{E}(X) = \frac{1}{2}\big(\mathrm{E}(X \mid x_1 = 1) + \mathrm{E}(X \mid x_1 = 0)\big).$$

Use this to show that an assignment of Φ can be computed that satisfies at least $\mathrm{E}(X)$ of the expressions in Φ.

3. Let $p := \min_{i=1,\ldots,n} \Pr(X_i = 1)$. Show

$$\frac{\mathrm{E}(X)}{\mathrm{OPT}(\Phi)} \geq p \geq 2^{-k}.$$

Conclude that the approximation threshold for k-MaxGSAT is at least $1 - 2^{-k}$.

4. Can one improve the approximation threshold given that all expressions φ_i are clauses? What happens if all those clauses have *at least* k distinct literals?

Exercise 5.3 Consider the following decision (not optimization!) problem $\mathsf{Max2SAT_D}$: given a 2CNF formula φ in n variables and some $K \in \mathbb{N}$, does there exist an assignment β of the n variables such that at least K clauses in φ are satisfied? The goal of this exercise is to show that $\mathsf{Max2SAT_D}$ is NP-complete.

1. Consider the following ten clauses

$$\begin{gathered}(x), (y), (z), (w),\\ (\neg x \vee \neg y), (\neg y \vee \neg z), (\neg z \vee \neg x),\\ (x \vee \neg w), (y \vee \neg w), (z \vee \neg w).\end{gathered}$$

Show that if β is a truth assignment that satisfies $(x \vee y \vee z)$, then β can be extended such that β satisfies exactly seven of the above clauses. Moreover, if β does not satisfy $(x \vee y \vee z)$, then any extension of β will satisfy at most six of the above clauses.

2. Use this fact to show that 3SAT can be reduced in polynomial time to $\mathsf{Max2SAT_D}$. Conclude that $\mathsf{Max2SAT_D}$ is NP-complete.