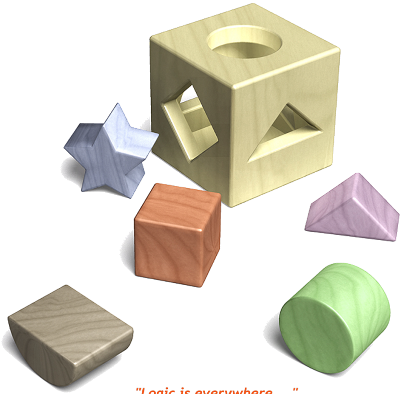
Equational Logic

Steffen Hölldobler
International Center for Computational Logic
Technische Universität Dresden
Germany

- Equational Systems
- Paramodulation
- Term Rewriting Systems
- Unification Theory
- Application: Multisets



"Logic is everywhere …"

Equational Systems

- Consider a first order language with the following precedence hierarchy

$$\{\forall, \exists\} > \neg > \wedge > \vee > \{\leftarrow, \rightarrow\} > \leftrightarrow$$

- Let $\approx$ be a binary predicate symbol written infix
- An equation is an atom of the form $s \approx t$
- An equational system $\mathcal{E}$ is a finite set of universally closed equations
- Notation Universal quantifiers are usually omitted

$\mathcal{E}_1$	$(X \cdot Y) \cdot Z \approx X \cdot (Y \cdot Z)$	(associativity)
	$1 \cdot X \approx X$	(left unit)
	$X \cdot 1 \approx X$	(right unit)
	$X^{-1} \cdot X \approx 1$	(left inverse)
	$X \cdot X^{-1} \approx 1$	(right inverse)

Axioms of Equality

- **The equality relation enjoys some typical properties expressed by the following universally closed axioms of equality $\mathcal{E}_{\approx}$**

$$X \approx X \qquad \text{(reflexivity)}$$

$$X \approx Y \rightarrow Y \approx X \qquad \text{(symmetry)}$$

$$X \approx Y \wedge Y \approx Z \rightarrow X \approx Z \qquad \text{(transitivity)}$$

$$\bigwedge_{i=1}^{n} X_i \approx Y_i \rightarrow f(X_1, \ldots, X_n) \approx f(Y_1, \ldots, Y_n) \qquad \text{(f–substitutivity)}$$

$$\bigwedge_{i=1}^{n} X_i \approx Y_i \wedge r(X_1, \ldots, X_n) \rightarrow r(Y_1, \ldots, Y_n) \qquad \text{(r–substitutivity)}$$

- **Note**
  - **Substitutivity axioms are defined for each function symbol f and each relation symbol r in the underlying alphabet**
  - **Universal quantifiers have been omitted**

Equality and Logical Consequence

- We are interested in computing logical consequences of $\mathcal{E} \cup \mathcal{E}_{\approx}$
  - $\mathcal{E}_1 \cup \mathcal{E}_{\approx} \models (\exists X)\, X \cdot a \approx 1$?
  - $\mathcal{E}_1 \cup \mathcal{E}_{\approx} \cup \{X \cdot X \approx 1\} \models (\forall X, Y)\, X \cdot Y \approx Y \cdot X$?
- One possibility is to apply resolution
  - There are 10^{21} resolution steps needed to solve the examples
  - $\mathcal{E} \cup \mathcal{E}_{\approx}$ causes an extremely large search space
- Idea Remove troublesome formulas from $\mathcal{E} \cup \mathcal{E}_{\approx}$ and build them into the deductive machinery
  - Use additional rule of inference like paramodulation
  - Build the equational theory into the unification computation

Least Congruence Relation

- $\mathcal{E} \cup \mathcal{E}_{\approx}$ is a set of definite clauses
- There exists a least model for $\mathcal{E} \cup \mathcal{E}_{\approx}$
- Example
  - Let the only function symbols be the constants a, b and the binary g
  - Let $\mathcal{E}_2 = \{a \approx b\}$
  - The least model of $\mathcal{E}_2 \cup \mathcal{E}_{\approx}$ is

$$\begin{array}{l} \{t \approx t \mid t \text{ is a ground term}\} \\ \cup \{a \approx b,\ b \approx a\} \\ \cup \{g(a,a) \approx g(b,a),\ g(a,a) \approx g(a,b),\ g(a,a) \approx g(b,b), \ldots\} \end{array}$$

- Define $s \approx_{\mathcal{E}} t$ iff $\mathcal{E} \cup \mathcal{E}_{\approx} \models \forall\, s \approx t$
  - $g(a,a) \approx_{\mathcal{E}_2} g(a,b)$
  - $g(X,a) \approx_{\mathcal{E}_2} g(X,b)$
  - $\approx_{\mathcal{E}}$ is the least congruence relation on terms generated by $\mathcal{E}$

Paramodulation

- $L\lceil s \rceil$ literal which contains an occurrence of the term s

  $L\lceil s/t \rceil$ literal obtained from L by replacing an occurrence of s by t

- **Paramodulation**

$$\frac{[L_1\lceil s \rceil, L_2, \ldots, L_n] \qquad [l \approx r, L_{n+1}, \ldots, L_m]}{[L_1\lceil s/r \rceil, L_2, \ldots, L_m]\theta} \; \theta = \text{mgu}(s, l)$$

- **Notation** Instead of $\neg s \approx t$ we write $s \not\approx t$

- **Remember**

$$\begin{array}{lll} \mathcal{E} \cup \mathcal{E}_{\approx} \models \forall\, s \approx t & \text{iff} & \bigwedge_{\mathcal{E} \cup \mathcal{E}_{\approx}} \to \forall\, s \approx t \text{ is valid} \\ & \text{iff} & \neg(\bigwedge_{\mathcal{E} \cup \mathcal{E}_{\approx}} \to \forall\, s \approx t) \text{ is unsatisfiable} \\ & \text{iff} & \mathcal{E} \cup \mathcal{E}_{\approx} \cup \{\neg\forall\, s \approx t\} \text{ is unsatisfiable} \\ & \text{iff} & \mathcal{E} \cup \mathcal{E}_{\approx} \cup \{\exists\, s \not\approx t\} \text{ is unsatisfiable} \end{array}$$

- **Theorem 1** $\mathcal{E} \cup \mathcal{E}_{\approx} \cup \{\exists\, s \not\approx t\}$ is unsatisfiable iff there is a refutation of $\mathcal{E} \cup \{X \approx X\} \cup \{\exists\, s \not\approx t\}$ wrt paramodulation, resolution and factoring

An Example

$$\mathcal{E}_1 \cup \{X \approx X,\ X \cdot X \approx 1\} \models (\forall X, Y)\, X \cdot Y \approx Y \cdot X$$

1	$a \cdot b \not\approx b \cdot a$	initial query		$\cdot$	hypothesis
2	$1 \cdot X_1 \approx X_1$	left unit		$a \cdot b \not\approx ((X_3 \cdot X_3) \cdot b) \cdot (a \cdot (X_4 \cdot X_4))$	
3	$X_2 \approx X_2$	reflexivity		$\cdot$	associativity
4	$X_1 \approx 1 \cdot X_1$	pm(2,3)		$a \cdot b \not\approx (X_3 \cdot ((X_3 \cdot b) \cdot (a \cdot X_4))) \cdot X_4$	
5	$a \cdot b \not\approx (1 \cdot b) \cdot a$	pm(1,4)		$\cdot$	hypothesis
6	$X_3 \cdot X_3 \approx 1$	hypothesis		$a \cdot b \not\approx (a \cdot 1) \cdot b$	
7	$X_4 \approx X_4$	reflexivity		$\cdot$	right unit
8	$1 \approx X_3 \cdot X_3$	pm(6,7)	n	$a \cdot b \not\approx a \cdot b$	
9	$a \cdot b \not\approx ((X_3 \cdot X_3) \cdot b) \cdot a$	pm(5,8)	n'	$X_5 \approx X_5$	reflexivity
	$\cdot$	right unit	n''	$[\,]$	res (n, n')
	$a \cdot b \not\approx ((X_3 \cdot X_3) \cdot b) \cdot (a \cdot 1)$				

The Example in Shorthand Notation

$$\mathcal{E}_1 \cup \{X \approx X,\; X \cdot X \approx 1\} \models (\forall X, Y)\, X \cdot Y \approx Y \cdot X$$

1	$a \cdot b \not\approx b \cdot a$	initial query
2	$1 \cdot X_1 \approx X_1$	left unit
3	$X_2 \approx X_2$	reflexivity
4	$X_1 \approx 1 \cdot X_1$	pm(2,3)
5	$a \cdot b \not\approx (1 \cdot b) \cdot a$	pm(1,4)
6	$X_3 \cdot X_3 \approx 1$	hypothesis
7	$X_4 \approx X_4$	reflexivity
8	$1 \approx X_3 \cdot X_3$	pm(6,7)
9	$a \cdot b \not\approx ((X_3 \cdot X_3) \cdot b) \cdot a$	pm(5,8)
	$\cdot$	right unit
	$a \cdot b \not\approx ((X_3 \cdot X_3) \cdot b) \cdot (a \cdot 1)$	

	$\cdot$	hypothesis
	$a \cdot b \not\approx ((X_3 \cdot X_3) \cdot b) \cdot (a \cdot (X_4 \cdot X_4))$	
	$\cdot$	associativity
	$a \cdot b \not\approx (X_3 \cdot ((X_3 \cdot b) \cdot (a \cdot X_4))) \cdot X_4$	
	$\cdot$	hypothesis
	$a \cdot b \not\approx (a \cdot 1) \cdot b$	
	$\cdot$	right unit
n	$a \cdot b \not\approx a \cdot b$	
n'	$X_5 \approx X_5$	reflexivity
n''	[ ]	res (n, n')

The Example in Shorthand Notation Again

$b \cdot a$	$\approx$	$(1 \cdot b) \cdot a$	left unit
	$\approx$	$((X_3 \cdot X_3) \cdot b) \cdot a$	hypothesis
	$\approx$	$((X_3 \cdot X_3) \cdot b) \cdot (a \cdot 1)$	right unit
	$\approx$	$((X_3 \cdot X_3) \cdot b) \cdot (a \cdot (X_4 \cdot X_4))$	hypothesis
	$\approx$	$(X_3 \cdot ((X_3 \cdot b) \cdot (a \cdot X_4))) \cdot X_4$	associativity
	$\approx$	$(a \cdot 1) \cdot b$	hypothesis
	$\approx$	$a \cdot b$	right unit

- **Now, the search space is 10^{11} instead of 10^{21} steps**
  - **Symmetry can be simulated, which leads to cycles**
  - **All terms s occurring in L_1 are candidates**
  - **$L_1\lceil s \rceil$ may be a variable and can be unified with any term**
- **There are still many redundant and useless steps**
- **Idea Use equations only from left to right $\leadsto$ term rewriting systems**

Term Rewriting Systems

- An expression of the form $s \rightarrow t$ is called **rewrite rule**
- A **term rewriting system** is a finite set of rewrite rules
- In the sequel, $\mathcal{R}$ shall denote a term rewriting system
- $s\lceil u \rceil$ denotes a term s which contains an occurrence of u
  $s\lceil u/v \rceil$ denotes the term obtained from s by replacing an occ. of u by v
- The **rewrite relation** $\rightarrow_{\mathcal{R}}$ on terms is defined as follows: $s\lceil u \rceil \rightarrow_{\mathcal{R}} t$ iff there exist $l \rightarrow r \in \mathcal{R}$ and θ such that $u = l\theta$ and $t = s\lceil u/r\theta \rceil$
- **Example**

$$\mathcal{R}_3 = \{ \begin{array}{lcl} append([\,], X) & \rightarrow & X, \\ append([X|Y], Z) & \rightarrow & [X|append(Y, Z)] \end{array} \}$$

$$\begin{array}{lcl} append([1,2],[3,4]) & \rightarrow_{\mathcal{R}_3} & [1|append([2],[3,4])] \\ & \rightarrow_{\mathcal{R}_3} & [1,2|append([\,],[3,4])] \\ & \rightarrow_{\mathcal{R}_3} & [1,2,3,4] \end{array}$$

Matching

- **Matching problem**
  Given terms u and l, does there exist a substitution θ such that $u = l\theta$?
  If such a substitution exists, then it is called a **matcher**
- If a matching problem is solvable, then there exists a most general matcher
- If can be computed by a variant of the unification algorithm,
  where variables occurring in u are treated as (different new) constant symbols
- Whereas unification is in the complexity class $\mathcal{P}$, matching is in $\mathcal{NC}$

Closures

- $\stackrel{*}{\rightarrow}_{\mathcal{R}}$ denotes the reflexive and transitive closure of $\rightarrow_{\mathcal{R}}$
  - $append([1,2],[3,4]) \stackrel{*}{\rightarrow}_{\mathcal{R}_3} [1,2,3,4]$
- $s \leftrightarrow_{\mathcal{R}} t$ iff $s \leftarrow_{\mathcal{R}} t$ or $s \rightarrow_{\mathcal{R}} t$
  - Let $\mathcal{R}_4 = \{a \rightarrow b,\ c \rightarrow b\}$,
    then $a \rightarrow_{\mathcal{R}_4} b \leftarrow_{\mathcal{R}_4} c$ and, consequently, $a \leftrightarrow_{\mathcal{R}_4} b \leftrightarrow_{\mathcal{R}_4} c$
- $\stackrel{*}{\leftrightarrow}_{\mathcal{R}}$ denotes the reflexive and transitive closure of $\leftrightarrow_{\mathcal{R}}$
  - $a \stackrel{*}{\leftrightarrow}_{\mathcal{R}_4} c$
- We sometimes simply write $\rightarrow$ or $\leftrightarrow$ instead of $\rightarrow_{\mathcal{R}}$ or $\leftrightarrow_{\mathcal{R}}$, respectively

Term Rewriting Systems and Equational Systems

- Let $\mathcal{R}$ be a term rewriting system
- $\mathcal{E}_\mathcal{R} := \{l \approx r \mid l \rightarrow r \in \mathcal{R}\} \cup \mathcal{E}_\approx$
  - For $\mathcal{R}_4 = \{a \rightarrow b,\ c \rightarrow b\}$ we obtain $\mathcal{E}_{\mathcal{R}_4} = \{a \approx b, c \approx b\} \cup \mathcal{E}_\approx$
- Theorem 2 (i) $s \xrightarrow{*}_\mathcal{R} t$ implies $s \approx_{\mathcal{E}_\mathcal{R}} t$
  (ii) $s \approx_{\mathcal{E}_\mathcal{R}} t$ iff $s \overset{*}{\leftrightarrow}_\mathcal{R} t$
- Proof $\rightsquigarrow$ Exercise
  - $g(X, a) \rightarrow_{\mathcal{R}_4} g(X, b)$ and $g(X, a) \approx_{\mathcal{E}_{\mathcal{R}_4}} g(X, b)$
  - $g(X, a) \approx_{\mathcal{E}_{\mathcal{R}_4}} g(X, c)$ and $g(X, a) \rightarrow_{\mathcal{R}_4} g(X, b) \leftarrow_{\mathcal{R}_4} g(X, c)$

Reducibility and Normal Forms

- s is **reducible** wrt $\mathcal{R}$ iff there exists t such that $s \rightarrow_{\mathcal{R}} t$
  - otherwise it is **irreducible**
- t is a **normal form** of s wrt $\mathcal{R}$ iff $s \xrightarrow{*}_{\mathcal{R}} t$ and t is irreducible
  - $[1, 2, 3, 4]$ is the normal form of $append([1, 2], [3.4])$ wrt $\mathcal{R}_3$
- Normal forms are not necessarily unique. Consider

$$\mathcal{R}_5 = \{ \begin{array}{lcl} \text{neg}(\text{neg}(X)) & \rightarrow & X, \\ \text{neg}(or(X, Y)) & \rightarrow & and(\text{neg}(X), \text{neg}(Y)), \\ \text{neg}(and(X, Y)) & \rightarrow & or(\text{neg}(X), \text{neg}(Y)), \\ and(X, or(Y, Z)) & \rightarrow & or(and(X, Y), and(X, Z)), \\ and(or(X, Y), Z) & \rightarrow & or(and(Y, Z), and(Z, X)) \end{array} \}$$

$and(or(X, Y), or(U, V))$ has the normal forms
$or(or(and(Y, U), and(U, X)), or(and(Y, V), and(V, X)))$ and
$or(or(and(Y, U), and(Y, V)), or(and(V, X), and(X, U)))$ wrt $\mathcal{R}_5$

Confluent Term Rewriting Systems

- $s \downarrow_{\mathcal{R}} t$ iff there exists u such that $s \overset{*}{\rightarrow}_{\mathcal{R}} u \overset{*}{\leftarrow}_{\mathcal{R}} t$
- $s \uparrow_{\mathcal{R}} t$ iff there exists u such that $s \overset{*}{\leftarrow}_{\mathcal{R}} u \overset{*}{\rightarrow}_{\mathcal{R}} t$
  - Consider $\mathcal{R}_6 = \{b \rightarrow a,\ b \rightarrow c\}$. Then $a \not\downarrow_{\mathcal{R}_6} c$, but $a \uparrow_{\mathcal{R}_6} c$
- $\mathcal{R}$ is confluent iff for all terms s and t we find $s \uparrow_{\mathcal{R}} t$ implies $s \downarrow_{\mathcal{R}} t$
  - $\mathcal{R}_7 = \mathcal{R}_6 \cup \{a \rightarrow c\}$ is confluent
- $\mathcal{R}$ is Church-Rosser iff for all terms s and t we find $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$ iff $s \downarrow_{\mathcal{R}} t$
- Theorem 3 $\mathcal{R}$ is Church-Rosser iff $\mathcal{R}$ is confluent
- Remember $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$ iff $s \approx_{\mathcal{E}_{\mathcal{R}}} t$
  - If a term rewriting system is confluent,
    then rewriting has only to be applied in one direction, viz. from left to right !

Canonical Term Rewriting Systems

- $\mathcal{R}$ is terminating iff it has no infinite rewriting sequences
  - The question whether $\mathcal{R}$ is terminating is undecidable
- $\mathcal{R}$ is canonical iff $\mathcal{R}$ is confluent and terminating
  - If $\mathcal{R}$ is canonical, then $s \approx_{\mathcal{E}_\mathcal{R}} t$ iff $s \downarrow_\mathcal{R} t$
  - If $\mathcal{R}$ is canonical, then $\mathcal{E}_\mathcal{R}$ is decidable
- Given $\mathcal{E}$. If $\approx_\mathcal{E} = \approx_{\mathcal{E}_\mathcal{R}}$ for some canonical term rewriting system $\mathcal{R}$, then the application of paramodulation can be restricted:
  - $L_1\lceil\pi\rceil$ may not be a variable
  - Symmetry can no longer be simulated
  - Equations, i.e., rewrite rules, are only applied from left to right
  - Further restrictions concerning $\pi \in \mathcal{P}_{L_1}$ are possible
  - This restricted form of paramodulation is called narrowing

Termination

- **Is a given term rewriting system $\mathcal{R}$ terminating?**
- Let $\succeq$ be a partial order on the set of terms, i.e., $\succeq$ is reflexive, transitive, and antisymmetric
  - $s \succ t$ **iff** $s \succeq t$ and $s \neq t$
  - $s \succ t$ is **well-founded** **iff** there is no infinite sequence $s_1 \succ s_2 \succ \ldots$
- **Idea** Search for a well-founded ordering $\succ$ such that $s \to_{\mathcal{R}} t$ implies $s \succ t$
- A **termination ordering** $\succ$ is a well-founded, transitive, and antisymmetric relation on the set of terms satisfying the following properties:
  - **full invariance property** if $s \succ t$ then $s\theta \succ t\theta$ for all θ
  - **replacement property** if $s \succ t$ then $u\lceil s\rceil \succ u\lceil s/t\rceil$
- **Theorem 4**
  Let $\mathcal{R}$ be a term rewriting system and $\succ$ a termination ordering.
  If for all rules $l \to r \in \mathcal{R}$ we find that $l \succ r$ then $\mathcal{R}$ is terminating

Termination Orderings: Two Examples

- Let $|s|$ denote the length of the term s
  $s \succ t$ iff for all grounding substitutions θ we find that $|s\theta| > |t\theta|$
  - $f(X, Y) \succ g(X)$
  - $f(X, Y)$ and $g(X, X)$ can not be ordered
- Polynomial ordering assign to each function symbol a polynomial with coefficients taken from $\mathbb{N}^+$
  - Let $$\begin{aligned} f(X, Y)^I &= 2X + Y \\ g(X, Y)^I &= X + Y \end{aligned}$$
  - Define $s \succ t$ iff $s^I > t^I$
  - Then, $f(X, Y) \succ g(X, X)$
- There are many other termination orderings!
- $\succ'$ is more powerful than $\succ$ iff $s \succ t$ implies $s \succ' t$ but not vice versa

Confluence

- **Is a given terminating term rewriting system confluent?**
- **$\mathcal{R}$ is locally confluent**
  iff for all terms r, s, t we find: If $t \leftarrow_{\mathcal{R}} r \rightarrow_{\mathcal{R}} s$ then $s \downarrow_{\mathcal{R}} t$
- **Theorem 5** **Let $\mathcal{R}$ be a terminating term rewriting system.**
  $\mathcal{R}$ is confluent iff it is locally confluent

Local Confluence

- **Is a given terminating term rewriting system locally confluent?**
- A subterm u of t is called a **redex**
  iff there exists θ and $l \to r \in \mathcal{R}$ such that $u = l\theta$
- Let $l_1 \to r_1 \in \mathcal{R}$ and $l_2 \to r_2 \in \mathcal{R}$ be applicable to t $\leadsto$ two redeces
  - **Case analysis**
    - (a) They are disjoint
    - (b) one redex is a subterm of the other one and corresponds to a variable position in the left-hand-side of the other rule
    - (c) one redex is a subterm of the other one but does not correspond to a variable position in the left-hand-side of the other rule (the redeces **overlap**)

Example

- Let $t = (g(a) \cdot f(b)) \cdot c$

  (a) $\mathcal{R}_8 = \{a \to c,\ b \to c\}$
  - a and b are disjoint redeces in t
  - $\mathcal{R}_8$ is locally confluent

  (b) $\mathcal{R}_9 = \{a \to c,\ g(X) \to f(X)\}$
  - a and $g(a)$ are redeces in t
  - a corresponds to the variable position in $g(X)$
  - $\mathcal{R}_9$ is locally confluent

  (c) $\mathcal{R}_{10} = \{(X \cdot Y) \cdot Z \to X,\ g(a) \cdot f(b) \to c\}$
  - $(g(a) \cdot f(b)) \cdot c$ and $g(a) \cdot f(b)$ are overlapping redeces in t
  - This is the problematic case!

Critical Pairs

- Let
  - $l_1 \to r_1,\ l_2 \to r_2$ be two new variants of rules in $\mathcal{R}$
  - u be a non-variable subterm of l_1 and
  - u and l_2 be unifiable with mgu θ
- Then, the pair $\langle (l_1\lceil u/r_2\rceil)\theta,\ r_1\theta\rangle$ is said to be critical
- It is obtained by superimposing l_1 with l_2
  - Superimposing $(X \cdot Y) \cdot Z \to X$ with $g(a) \cdot f(b) \to c$ yields the critical pair $\langle c \cdot Z, g(a)\rangle$
- Theorem 6 A term rewriting system $\mathcal{R}$ is locally confluent iff for all critical pairs $\langle s, t\rangle$ of $\mathcal{R}$ we find $s \downarrow_{\mathcal{R}} t$

Completion

- **Can a terminating and non-confluent $\mathcal{R}$ be turned into a confluent one?**
- **Two term rewriting systems $\mathcal{R}$ and $\mathcal{R}'$ are equivalent iff $\approx_{\mathcal{E}_\mathcal{R}} = \approx_{\mathcal{E}_{\mathcal{R}'}}$**
- **Idea if $\langle s, t\rangle$ is a critical pair then add either $s \to t$ or $t \to s$ to $\mathcal{R}$**
  - **This is called completion**
  - **The equational theory remains unchanged**

Completion Procedure

- **Given a terminating $\mathcal{R}$ together with a termination ordering $\succ$**
  1. **If for all critical pairs $\langle s, t\rangle$ of $\mathcal{R}$ we find that $s \downarrow_{\mathcal{R}} t$ then return "success"; $\mathcal{R}$ is canonical**
  2. **If $\mathcal{R}$ has a critical pair whose elements do not rewrite to a common term, then transform the elements of the critical pair to some normal form. Let $\langle s, t\rangle$ be the normalized critical pair:**
     - **If $s \succ t$ then add the rule $s \to t$ to $\mathcal{R}$ and goto 1**
     - **If $t \succ s$ then add the rule $t \to s$ to $\mathcal{R}$ and goto 1**
     - **If neither $s \succ t$ nor $t \succ s$ then return "fail"**
- **The completion procedure may either succeed or fail or loop**
- **During completion the ordering $\succ$ may be extended to a more powerful one**
- **The completion procedure may be extended to unfailing completion**

Completion: An Example

- Consider
$$\mathcal{R}_{11} = \{c \to b,\ f \to b,\ f \to a,\ e \to a,\ e \to d\}$$
- Let $f \succ e \succ d \succ c \succ b \succ a$
- The critical pairs are $\langle b, a \rangle$ and $\langle d, a \rangle$
- They can be oriented into the new rules $b \to a$ and $d \to a$
- We obtain
$$\mathcal{R}'_{11} = \{c \to b,\ f \to b,\ f \to a,\ e \to a,\ e \to d,\ b \to a,\ d \to a\}$$
- $\mathcal{R}'_{11}$ is canonical
- $s \approx_{\mathcal{E}_\mathcal{R}} t \quad$ iff $\quad s \approx_{\mathcal{E}_{\mathcal{R}'}} t$
- All proofs for $s \approx_{\mathcal{E}_{\mathcal{R}'_{11}}} t$ are in so-called valley form