Weak Completion Semantics

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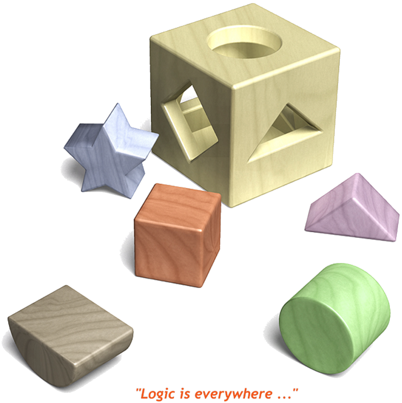
- Some Human Reasoning Tasks
- Logic Programs
- Non-Monotonicity
- Łukasiewicz Logic
- Least Models
- Weak Completion Semantics
- Abduction



Human Reasoning – Two Examples

- **Instructions on boarding card distributed at Amsterdam Schiphol Airport**
  - **If it's thirty minutes before your flight departure, make your way to the gate**
    As soon as the gate number is confirmed, make your way to the gate
- **Notice in London Underground**
  - **If there is an emergency then you press the alarm signal bottom**
    The driver will stop if any part of the train is in a station
- **Observations**
  - **Intended meaning differs from literal meaning**
  - **Rigid adherence to classical logic is no help in modeling the examples**
  - **There seems to be a reasoning process towards more plausible meanings**
    - **The driver will stop the train in a station**
      if the driver is alerted to an emergency
      and any part of the train is in the station
      Kowalski: Computational Logic and Human Life:
      How to be Artificially Intelligent. Cambridge University Press 2011

The Suppression Task

- Byrne: Suppressing Valid Inferences with Conditionals
  Cognition 31, 61-83: 1989
- If she has an essay to write then she will study late in the library
  She has an essay to write
  - Will she study late in the library? ☐ yes ☐ no ☐ I don't know
  - 96% conlude that she will study late in the library; modus ponens
- If she has an essay to write then she will study late in the library
  She has an essay to write
  If she has a textbook to read she will study late in the library
  - 96% conlude that she will study late in the library; alternative arguments
- If she has an essay to write then she will study late in the library
  She has an essay to write
  If the library stays open she will study late in the library
  - 38% conlude that she will study late in the library.
  - Additional arguments lead to suppression of earlier (correct) conclusions

Reasoning Towards an Appropriate Logical Form

- **Context independent rules**
  - **If she has an essay to write and the library is open then she will study late in the library If the library is open and she has a reason for studying in the library then she will study late in the library**
- **Context dependent rule plus exception**
  - **If she has an essay to write then she will study late in the library However, if the library is not open, then she will not study late in the library**
  - **The last sentence is the contrapositive of the converse of the original sentence!**

The Suppression Task – The Search for Models

- **Can we find a logic which adequately models these human reasoning tasks?**
- **How about classical two-valued propositional logic?**
- **Let's consider a direct encoding of conditionals by implications**

$$\{\ell \leftarrow e,\ e\}$$
$$\{\ell \leftarrow e,\ e,\ \ell \leftarrow t\}$$
$$\{\ell \leftarrow e,\ e,\ \ell \leftarrow o\}$$

- **Recall the examples**

$\{\ell \leftarrow e,\ e\}$	$\models$	ℓ	**modus ponens**
$\{\ell \leftarrow e,\ e,\ \ell \leftarrow t\}$	$\models$	ℓ	**classical logic is monotonic**
$\{\ell \leftarrow e,\ e,\ \ell \leftarrow o\}$	$\models$	ℓ	**upps, only 38% of subjects did this**

- **Conclusion** **classical logic is inadequate**
  - **Often mistakenly generalized to "*logic is inadequate*"**

Human Reasoning – A Computational Logic Approach

- **Can we find a logic which adequately models human reasoning tasks?**
- **Approach**
  - **Reasoning towards an appropriate logical form**
    - **Logic programs**
  - **Weak completion semantics**
    - **Non-monotonicity**
  - **Three-valued Łukasiewicz logic**
    - **Least models**
  - **An appropriate semantic operator**
    - **Least fixed points are least models**
    - **Least fixed points can be computed by iterating the operator**
  - **Reasoning with respect to the least models**
  - **Abduction**

Logic Programs

- **Preliminaries**
  - **An atom is an atomic propositions**
  - **A literal is either an atom or its negation**
  - **$\top$ and $\bot$ denote truth and falsehood, respectively**
- **A (logic) program is a finite set of rules**
  - **A rule is of the form $A \leftarrow \top$, $A \leftarrow \bot$, or $A \leftarrow B_1 \wedge \cdots \wedge B_n$ where $n \geq 1$, A is an atom, and each B_i, $1 \leq i \leq n$, is a literal**
  - **A is called head**
  - **$\top$, $\bot$, and $B_1 \wedge \cdots \wedge B_n$ are called bodies**
  - **Rules of the form $A \leftarrow \top$ are called facts**
  - **Rules of the form $A \leftarrow \bot$ are called assumptions**
- **The language underlying a program $\mathcal{P}$ shall contain precisely the relation symbols occurring in $\mathcal{P}$, and no others**

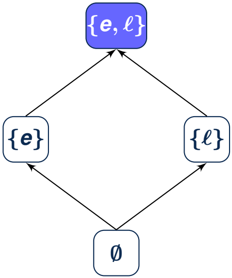
Completion

- Let $\mathcal{P}$ be a program and consider the following transformation
  1. All rules with the same head $A \leftarrow Body_1,\ A \leftarrow Body_2, \ldots$ are replaced by $A \leftarrow Body_1 \vee Body_2 \vee \ldots$
  2. If an atom A is not the head of any rule in $\mathcal{P}$ then add $A \leftarrow \bot$
  3. All occurrences of $\leftarrow$ are replaced by $\leftrightarrow$
     - The resulting set is called **completion** of $\mathcal{P}$ or $c\mathcal{P}$
     - If step 2 is omitted then the resulting set is called **weak completion** of $\mathcal{P}$ or $wc\mathcal{P}$

Completion – Example 1

$$\begin{array}{rcl} & \mathcal{P} & = & \{\ell \leftarrow e,\ e \leftarrow \top\} \\ c\mathcal{P} = & wc\mathcal{P} & = & \{\ell \leftrightarrow e,\ e \leftrightarrow \top\} \end{array}$$

- The models of $\mathcal{P}$, $c\mathcal{P}$ and $wc\mathcal{P}$

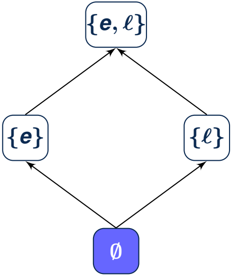


- Hence, $\mathcal{P} \models \ell$, $c\mathcal{P} \models \ell$, and $wc\mathcal{P} \models \ell$

Completion – Example 2

$$\mathcal{P} = \{\ell \leftarrow e,\ e \leftarrow \bot\}$$
$$wc\mathcal{P} = c\mathcal{P} = \{\ell \leftrightarrow e,\ e \leftrightarrow \bot\}$$

- The models of $c\mathcal{P}$ and $wc\mathcal{P}$



- Hence, $c\mathcal{P} \models \neg\ell$ and $wc\mathcal{P} \models \neg\ell$
- But, $\mathcal{P} \not\models \neg\ell$

Completion – Example 3

$$
\begin{aligned}
\mathcal{P} &= \{\ell \leftarrow e,\ e \leftarrow \top,\ \ell \leftarrow t\} \\
c\mathcal{P} &= \{\ell \leftrightarrow e \vee t,\ e \leftrightarrow \top,\ t \leftrightarrow \bot\} \\
wc\mathcal{P} &= \{\ell \leftrightarrow e \vee t,\ e \leftrightarrow \top\}
\end{aligned}
$$

- $\{e, \ell\}$ is the only model of $c\mathcal{P}$
- $\{e, \ell\}$ and $\{e, \ell, t\}$ are the models of $wc\mathcal{P}$
- $c\mathcal{P} \models \neg t$
- $wc\mathcal{P} \not\models \neg t$
- $wc\mathcal{P} \not\models t$

Monotonicity

- Let $\mathcal{P}$ and $\mathcal{P}'$ be sets of formulas and G a formula
  A logic is monotonic if the following holds
  If $\mathcal{P} \models G$ then $\mathcal{P} \cup \mathcal{P}' \models G$
- Classical logic is monotonic
- A logic based on completion semantics is non-monotonic
  - Consider

$$\begin{array}{lcl} \mathcal{P} & = & \{\ell \leftarrow e,\ e \leftarrow \top,\ \ell \leftarrow t\} \\ \mathcal{P}' & = & \mathcal{P} \cup \{t \leftarrow \top\} \end{array}$$

  - Then

$$\begin{array}{lclcl} c\mathcal{P} & = & \{\ell \leftrightarrow e \vee t,\ e \leftrightarrow \top,\ t \leftrightarrow \bot\} & \models & \neg t \\ c\mathcal{P}' & = & \{\ell \leftrightarrow e \vee t,\ e \leftrightarrow \top,\ t \leftrightarrow \top\} & \not\models & \neg t \end{array}$$

Łukasiewicz (Ł) Logic

- Łukasiewicz: O logice trójwartościowej. Ruch Filozoficzny 5, 169-171: 1920 English translation: On Three-Valued Logic. In: *Jan Łukasiewicz Selected Works*. (L. Borkowski, ed.), North Holland, 87-88, 1990

	$\neg$
$\top$	$\bot$
$\bot$	$\top$
U	**U**

$\wedge$	$\top$	**U**	$\bot$
$\top$	$\top$	**U**	$\bot$
U	**U**	**U**	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	**U**	$\bot$
$\top$	$\top$	$\top$	$\top$
U	$\top$	**U**	**U**
$\bot$	$\top$	**U**	$\bot$

$\leftarrow$	$\top$	**U**	$\bot$
$\top$	$\top$	$\top$	$\top$
U	**U**	$\top$	$\top$
$\bot$	$\bot$	**U**	$\top$

$\leftrightarrow$	$\top$	**U**	$\bot$
$\top$	$\top$	**U**	$\bot$
U	**U**	$\top$	**U**
$\bot$	$\bot$	**U**	$\top$

Three-Valued Interpretations and Models

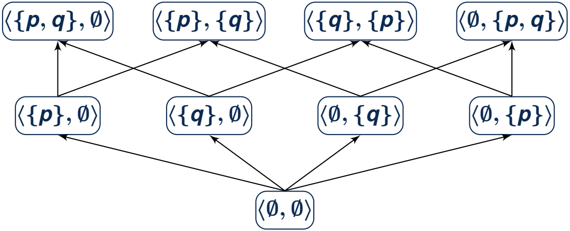
- A (three-valued) interpretation assigns to each formula a value from $\{\top, \bot, \mathsf{U}\}$
- It is represented by $\langle I^\top, I^\bot \rangle$ where
  - $I^\top$ contains all atoms which are mapped to $\top$
  - $I^\bot$ contains all atoms which are mapped to $\bot$
  - $I^\top \cap I^\bot = \emptyset$
  - All atoms which occur neither in $I^\top$ nor $I^\bot$ are mapped to U
- An interpretation $I = \langle I^\top, I^\bot \rangle$ is a model for a program $\mathcal{P}$ under Ł-logic ($I \models_{3Ł} \mathcal{P}$) if each rule occurring in $\mathcal{P}$ is mapped to $\top$ by I under Ł-logic
- Let $I = \langle I^\top, I^\bot \rangle$ and $J = \langle J^\top, J^\bot \rangle$ be two interpretation
  - Their intersection $I \cap J$ is defined as $\langle I^\top \cap J^\top, I^\bot \cap J^\bot \rangle$
  - $I \subseteq J$ iff $I^\top \subseteq J^\top$ and $I^\bot \subseteq J^\bot$.

Three-Valued Interpretations and Models – Examples

$$
\begin{array}{rcl}
\mathcal{P} & = & \{\ell \leftarrow e \wedge \neg ab,\ ab \leftarrow \bot,\ e \leftarrow \top\} \\
wc\mathcal{P} & = & \{\ell \leftrightarrow e \wedge \neg ab,\ ab \leftrightarrow \bot,\ e \leftrightarrow \top\} \\
\langle\{e, ab\}, \emptyset\rangle & \models_{3Ł} & \mathcal{P} \\
\langle\{e, ab\}, \emptyset\rangle & \not\models_{3Ł} & wc\mathcal{P} \\
\langle\{e, \ell\}, \{ab\}\rangle & \models_{3Ł} & wc\mathcal{P}
\end{array}
$$

Three-Valued Interpretations and Models – Semi-lattices

- Fitting: A Kripke-Kleene Semantics for Logic Programs
  Journal of Logic Programming 2, 295-312: 1985
- **Let $\mathcal{I}$ denote the set of all three-valued interpretations**
  $(\mathcal{I}, \subseteq)$ is a complete semi-lattice
- **Example** **Consider $\{p \leftarrow q\}$**

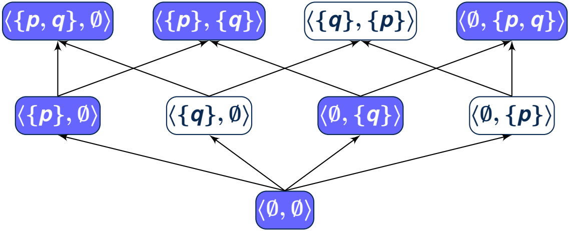


Programs under Ł-Logic

- H., Kencana Ramli:
  Logic Programs under Three-Valued Łukasiewicz's Semantics
  In: Logic Programming. Hill, Warren (eds), LNCS 5649, 464-478: 2009
- **Proposition 1** If $\langle I^\top, I^\bot \rangle \models_{Ł} \mathcal{P}$ then $\langle I^\top, \emptyset \rangle \models_{Ł} \mathcal{P}$
- **Example** Consider $\mathcal{P} = \{p \leftarrow q \wedge \neg r\}$
  - $\langle \{p, q\}, \{r\} \rangle$ is a model for $\mathcal{P}$ and so is $\langle \{p, q\}, \emptyset \rangle$
  - $\langle \{p, r\}, \{q\} \rangle$ is a model for $\mathcal{P}$ and so is $\langle \{p, r\}, \emptyset \rangle$
  - $\langle \{r\}, \{q\} \rangle$ is a model for $\mathcal{P}$ and so is $\langle \{r\}, \emptyset \rangle$
  - $\langle \{p\}, \emptyset \rangle = \langle \{p, q\}, \emptyset \rangle \cap \langle \{p, r\}, \emptyset \rangle$ is a model for $\mathcal{P}$
- **Proposition 2** If $\langle I_1^\top, \emptyset \rangle \models_{Ł} \mathcal{P}$ and $\langle I_2^\top, \emptyset \rangle \models_{Ł} \mathcal{P}$ then $\langle I_1^\top \cap I_2^\top, \emptyset \rangle \models_{Ł} \mathcal{P}$
- **Theorem 3** The model intersection property holds for $\mathcal{P}$
  i.e. $\cap\{I \mid I \models_{Ł} \mathcal{P}\} \models_{Ł} \mathcal{P}$

Least Models – Example

- **Example** Consider $\{p \leftarrow q\}$



Weakly Completed Programs under Ł-Logic

- **Theorem 4** The model intersection property holds for $wc\mathcal{P}$ as well
- **Theorem 5** If $I \models_{Ł} wc\mathcal{P}$ then $I \models_{Ł} \mathcal{P}$
- **How can the least model of $wc\mathcal{P}$ be computed?**

Computing the Least Models of Weakly Completed Programs

- Stenning, van Lambalgen: Human Reasoning and Cognititve Science
  MIT Press: 2008

- Consider the following immediate consequence operator
  $\Phi_{\mathcal{P}}(I) = \langle J^{\top}, J^{\bot} \rangle$, where

$$\begin{array}{rcl} J^{\top} & = & \{A \mid \text{there exists } A \leftarrow Body \in \mathcal{P} \text{ with } I(Body) = \top\} \\ J^{\bot} & = & \{A \mid \text{there exists } A \leftarrow Body \in \mathcal{P} \text{ and} \\ & & \quad \text{for all } A \leftarrow Body \in \mathcal{P} \text{ we find } I(Body) = \bot\} \end{array}$$

- **Note** $\Phi_{\mathcal{P}}$ "without the red condition" is the Fitting operator

- **Theorem 6** (H., Kencana Ramli 2009)
  (1) $\Phi_{\mathcal{P}}$ is monotone on $(\mathcal{I}, \subseteq)$
  (2) $\Phi_{\mathcal{P}}$ is continuous
  and, hence, admits a least fixed point denoted by *lfp* $\Phi_{\mathcal{P}}$
  (3) *lfp* $\Phi_{\mathcal{P}}$ can be computed by iterating $\Phi_{\mathcal{P}}$ on $\langle \emptyset, \emptyset \rangle$
  (4) *lfp* $\Phi_{\mathcal{P}}$ is the least model of *wc*$\mathcal{P}$

Computing Least Fixed Points

- **Examples**

$$
\begin{array}{rcl}
\mathcal{P} & = & \{\ell \leftarrow e \wedge \neg ab,\ ab \leftarrow \bot,\ e \leftarrow \top\} \\
wc\mathcal{P} & = & \{\ell \leftrightarrow e \wedge \neg ab,\ ab \leftrightarrow \bot,\ e \leftrightarrow \top\} \\
\Phi_{\mathcal{P}}(\langle \emptyset, \emptyset \rangle) & = & \langle \{e\}, \{ab\} \rangle \\
\Phi_{\mathcal{P}}(\langle \{e\}, \{ab\} \rangle) & = & \langle \{e, \ell\}, \{ab\} \rangle \ = \ lfp\, \Phi_{\mathcal{P}}
\end{array}
$$

$$
\begin{array}{rcl}
\mathcal{P} & = & \{\ell \leftarrow e \wedge \neg ab_1,\ ab_1 \leftarrow \neg o, \\
 & & \ \ell \leftarrow o \wedge \neg ab_2,\ ab_2 \leftarrow \neg e,\ e \leftarrow \bot\} \\
wc\mathcal{P} & = & \{\ell \leftrightarrow (e \wedge \neg ab_1) \vee (o \wedge \neg ab_2), \\
 & & \ ab_1 \leftrightarrow \neg o,\ ab_2 \leftrightarrow \neg e,\ e \leftrightarrow \bot\} \\
\Phi_{\mathcal{P}}(\langle \emptyset, \emptyset \rangle) & = & \langle \emptyset, \{e\} \rangle \\
\Phi_{\mathcal{P}}(\langle \emptyset, \{e\} \rangle) & = & \langle \{ab_2\}, \{e\} \rangle \\
\Phi_{\mathcal{P}}(\langle \{ab_2\}, \{e\} \rangle) & = & \langle \{ab_2\}, \{e, \ell\} \rangle \ = \ lfp\, \Phi_{\mathcal{P}}
\end{array}
$$

Weak Completion Semantics

- Let $\mathcal{P}$ be a logic program, F a formula, and A an atom
- Let $\mathcal{M}_{\mathcal{P}}$ denote the least Ł-model of $wc\mathcal{P}$
- $\mathcal{P} \models_{wcs} F$ iff $\mathcal{M}_{\mathcal{P}}(F) = \top$

- $def(A, \mathcal{P}) = \{A \leftarrow Body \mid A \leftarrow Body \in \mathcal{P}\}$
- Example Let $\mathcal{P} = \{\ell \leftarrow e \wedge \neg ab_1,\ ab_1 \leftarrow \bot,\ \ell \leftarrow t \wedge \neg ab_2,\ ab_2 \leftarrow \bot\}$
  - $def(\ell, \mathcal{P}) = \{\ell \leftarrow e \wedge \neg ab_1,\ \ell \leftarrow t \wedge \neg ab_2\}$
  - $def(t, \mathcal{P}) = \emptyset = def(e, \mathcal{P})$

Abduction

- **Consider the abductive framework $\langle \mathcal{P}, \mathcal{A}_\mathcal{P}, \mathcal{IC}, \models_{wcs} \rangle$, where**
  - $\mathcal{A}_\mathcal{P} = \{A \leftarrow \top \mid def(A, \mathcal{P}) = \emptyset\} \cup \{A \leftarrow \bot \mid def(A, \mathcal{P}) = \emptyset\}$ **is the set of abducibles**
  - $\mathcal{IC}$ **is a finite set of integrity constraints,** **i.e., expressions of the form** $U \leftarrow B_1 \wedge \ldots \wedge B_n$
- **An observation $\mathcal{O}$ is a set of ground literals**
  - $\mathcal{O}$ **is explainable in** $\langle \mathcal{P}, \mathcal{A}_\mathcal{P}, \mathcal{IC}, \models_{wcs} \rangle$
    iff there exists a minimal $\mathcal{E} \subseteq \mathcal{A}_\mathcal{P}$ called explanation such that $\mathcal{M}_{\mathcal{P} \cup \mathcal{E}}$ satisfies $\mathcal{IC}$ and $\mathcal{P} \cup \mathcal{E} \models_{wcs} L$ for each $L \in \mathcal{O}$
  - F **follows creduluously from $\mathcal{P}$ and $\mathcal{O}$**
    iff there exists an explanantion $\mathcal{E}$ such that $\mathcal{P} \cup \mathcal{E} \models_{wcs} F$
  - F **follows skeptically from $\mathcal{P}$ and $\mathcal{O}$**
    iff for all explanantions $\mathcal{E}$ we find $\mathcal{P} \cup \mathcal{E} \models_{wcs} F$

Abduction – Example

- Let $\mathcal{P} = \{\ell \leftarrow e \wedge \neg ab_1,\ ab_1 \leftarrow \bot,\ \ell \leftarrow t \wedge \neg ab_2,\ ab_2 \leftarrow \bot\}$
- $\mathcal{A}_\mathcal{P} = \{e \leftarrow \top,\ e \leftarrow \bot,\ t \leftarrow \top,\ t \leftarrow \bot\}$
- Let $\mathcal{O} = \{\ell\}$
- Two minimal explanations $\{e \leftarrow \top\}$ and $\{t \leftarrow \top\}$
- Reasoning credulously we conclude e
- Reasoning skeptically we cannot conclude e

- H., Philipp, Wernhard: An Abductive Model for Human Reasoning
  In: Proceedings of the 10th International Symposium on Logical Formalizations of Commonsense Reasoning (CommonSense): 2011
- Dietz, H., Ragni: A Computational Logic Approach to the Suppression Task
  In: Proceedings of the 34th Annual Conference of the Cognitive Science Society, Miyake et.al. (eds.), 1500-1505: 2012