Exploring Implications and General Concept Inclusions with High Confidence

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QuantLA Summer Workshop 2014

2014-10-06

http://www.sealifeconservation.org.au/adopt-an-animal/

$\text{Venomous} \sqcap \text{Mammal} \sqsubseteq \bot$?

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Accept this GCI?

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How to decide?

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Accept this GCI?

It depends

- True if all mammals are considered
- But exceptional, i. e. counterexamples rare
- In general not desired

How to decide?

External source of information needed

Attribute Exploration (Formal Concept Analysis)

Expert

Algorithm

$\mathbb{K}$	m_1	$\ldots$	m_n
g_1		$\ldots$	
$\vdots$			
g_k		$\ldots$	

$$\mathcal{S} = \{ A_1 \to B_1, \ldots, A_\ell \to B_\ell \}$$

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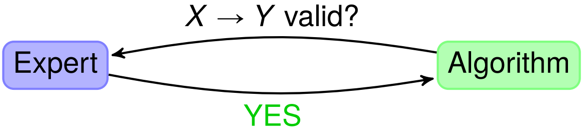

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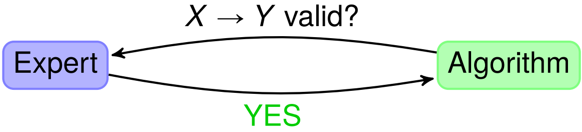

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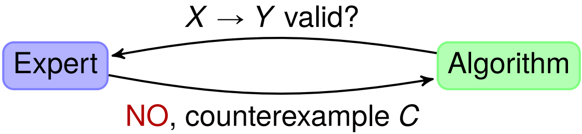

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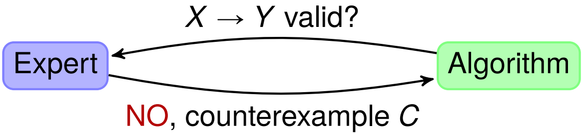

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g_1		$\dots$	
$\vdots$			
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g_{k+1}	——	C	——

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- Upon termination, $\mathrm{Cn}(\mathcal{S}) = \mathrm{Th}(\mathbb{K})$ and $\mathcal{S}$ is a base of the implicational knowledge of the expert

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Expert | Algorithm

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Data may contain errors

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Problem

Data may contain errors $\rightsquigarrow$ ask implications with *high confidence*

Definition

Define

$$\text{conf}_{\mathbb{K}}(X \to Y) := \begin{cases} 1 & X' = \varnothing \\ \frac{|(X \cup Y)'|}{|X'|} & \text{otherwise} \end{cases}$$

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For $c \in [0,1]$ set

$$\mathrm{Th}_c(\mathbb{K}) := \{\, X \to Y \mid \mathrm{conf}_{\mathbb{K}}(X \to Y) \geqslant c \,\}.$$

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Let M be a finite set. Then an *expert* p on M is a mapping

$$p \colon \operatorname{Imp}(M) \to \mathfrak{P}(M) \cup \{\, \top \,\}$$

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Let M be a finite set. Then an *expert* p on M is a mapping

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such that

- $p(X \to Y) = C \neq \top$ implies $X \subseteq C$ and $Y \nsubseteq C$,
- $p(U \to V) = \top, p(X \to Y) = C \neq \top$ implies $U \nsubseteq C$ or $V \subseteq C$.

Definition

To *explore* $\mathrm{Th}_c(\mathbb{K})$ *with expert* p *with background knowledge* $\mathcal{S}$ means to find a base of

$$\mathrm{Th}(p) \cap \mathrm{Th}_c(\mathbb{K})$$

with background knowledge $\mathcal{S}$,

Definition

To *explore* $\mathrm{Th}_c(\mathbb{K})$ *with expert p with background knowledge* $\mathcal{S}$ means to find a base of

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with background knowledge $\mathcal{S}$, i. e. to compute a set $\mathcal{B} \subseteq \mathrm{Imp}(M)$ such that

$$\mathrm{Cn}(\mathcal{B} \cup \mathcal{S}) = \mathrm{Cn}(\mathrm{Th}(p) \cap \mathrm{Th}_c(\mathbb{K})).$$

Classical Attribute Exploration

	a	b	c
1	$\times$	.	.

Classical Attribute Exploration

	a	b	c
1	$\times$	.	.

$\emptyset$

$\{a\}$

$\{b\}$

$\{a, b\}$

$\{c\}$

$\{a, c\}$

$\{b, c\}$

Classical Attribute Exploration

	a	b	c
1	$\times$	.	.

$$\emptyset \to \emptyset''$$
$$\{a\} \to \{a\}''$$
$$\{b\} \to \{b\}''$$
$$\{a,b\} \to \{a,b\}''$$
$$\{c\} \to \{c\}''$$
$$\{a,c\} \to \{a,c\}''$$
$$\{b,c\} \to \{b,c\}''$$

Classical Attribute Exploration

	a	b	c
1	$\times$	.	.

$$\begin{aligned}
\emptyset &\to \emptyset'' & &= \{\, a \,\} \\
\{\, a \,\} &\to \{\, a \,\}'' \\
\{\, b \,\} &\to \{\, b \,\}'' \\
\{\, a, b \,\} &\to \{\, a, b \,\}'' \\
\{\, c \,\} &\to \{\, c \,\}'' \\
\{\, a, c \,\} &\to \{\, a, c \,\}'' \\
\{\, b, c \,\} &\to \{\, b, c \,\}''
\end{aligned}$$

Classical Attribute Exploration

	a	b	c
1	$\times$	.	.

$$\emptyset \to \emptyset'' \qquad = \{a\} \checkmark$$
$$\{a\} \to \{a\}''$$
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$$\{ a \} \to \{ a \}''$$

$$\sout{\{ b \} \to \{ b \}''}$$

$$\{ a, b \} \to \{ a, b \}''$$

$$\sout{\{ c \} \to \{ c \}''}$$

$$\{ a, c \} \to \{ a, c \}''$$

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Classical Attribute Exploration

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1	×	.	.

$\varnothing \to \varnothing'' = \{a\}$ ✓

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~~$\{b\} \to \{b\}''$~~

$\{a,b\} \to \{a,b\}'' = \{a,b,c\}$ ✗

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Classical Attribute Exploration

	a	b	c
1	×	.	.
2	×	×	.

$$\emptyset \to \emptyset'' = \{a\} ✓$$
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$$\{a, b\} \to \{a, b\}'' = \{a, b, c\} ✗$$
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1	×	.	.
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Note

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Note

- Order of premises extends $\subseteq$

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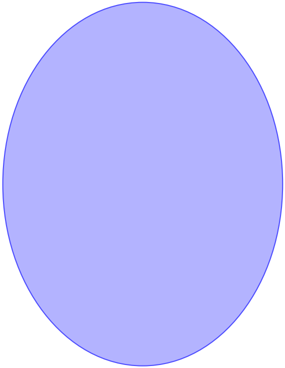
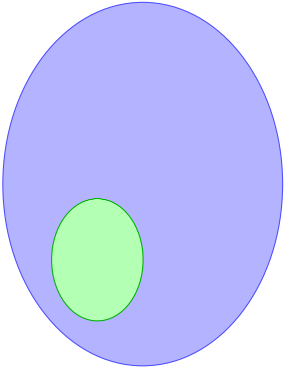
Note

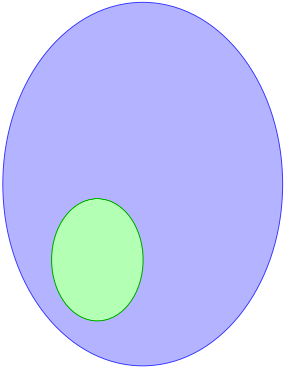
- Order of premises extends $\subseteq$
- Premises are always closed under known implications



- *Interesting Implications* $\mathcal{L}$



- *Interesting Implications $\mathcal{L}$*
- *Known Implications $\mathcal{S}$*



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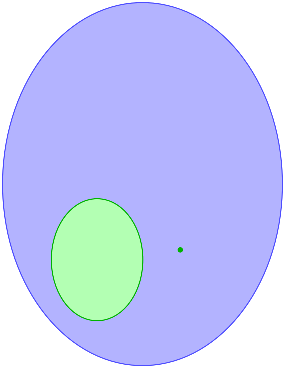
Goal

Which interesting but unknown implications are *valid* in our domain?



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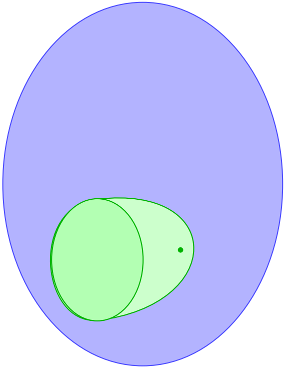

- *Interesting Implications* $\mathcal{L}$
- *Known Implications* $\mathcal{S}$
- Expert can *confirm* implications, extending $\mathcal{S}$

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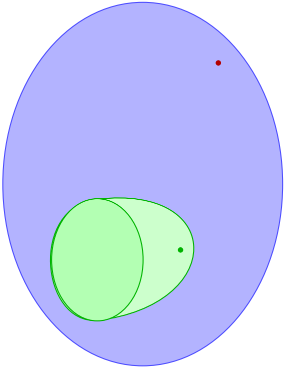

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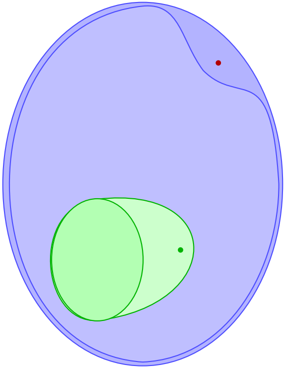

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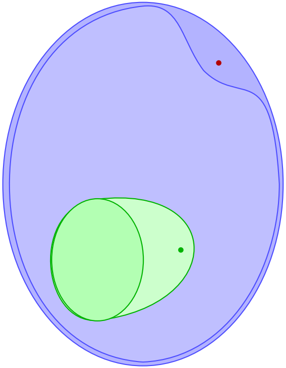

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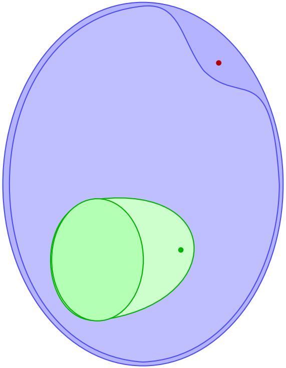

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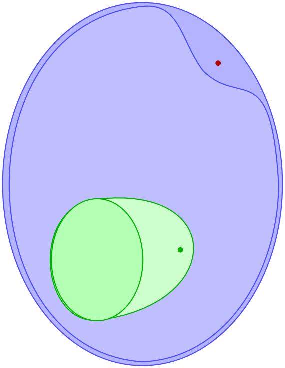
- Classical Attribute Exploration: $\mathcal{L} = \mathrm{Th}(\mathbb{K})$



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Which interesting but unknown implications are *valid* in our domain?

- Classical Attribute Exploration: $\mathcal{L} = \mathrm{Th}(\mathbb{K})$
- Exploration by Confidence: $\mathcal{L} = \mathrm{Th}_c(\mathbb{K})$

First Idea

Replace occurrence of $(\cdot)''$ with $\mathrm{Th}_c(\mathbb{K})(\cdot)$.

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- It doesn't work, i. e. exploration is only "approximative":

$$\mathrm{Th}(p) \cap \mathrm{Cn}(\mathrm{Th}_c(\mathbb{K})) \supseteq \mathrm{Cn}(\mathcal{B} \cup \mathcal{S}) \supsetneq \mathrm{Cn}(\mathrm{Th}(p) \cap \mathrm{Th}_c(\mathbb{K}))$$

First Idea

Replace occurrence of $(\cdot)''$ with $\mathsf{Th}_c(\mathbb{K})(\cdot)$.

Problems

- It doesn't work, i. e. exploration is only "approximative":

$$\mathsf{Th}(p) \cap \mathsf{Cn}(\mathsf{Th}_c(\mathbb{K})) \supseteq \mathsf{Cn}(\mathcal{B} \cup \mathcal{S}) \supsetneq \mathsf{Cn}(\mathsf{Th}(p) \cap \mathsf{Th}_c(\mathbb{K}))$$

- Closures under $\mathsf{Th}_c(\mathbb{K})$ are (potentially) expensive

Idea

Instead of

$$X \to \mathrm{Th}_c(\mathbb{K})(X),$$

ask implications of the form

$$X \to \{\, m \in M \mid \mathrm{conf}_{\mathbb{K}}(X \to \{\, m \,\}) \geqslant c \,\}.$$

Idea

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ask implications of the form

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Problem

Doesn't work either.

Choose

- $c := \frac{1}{2}$
- $\mathcal{S} := \{ \{ \text{a} \} \to \{ \text{b} \} \}$
- p constantly $\top$

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

Choose

- $c := \frac{1}{2}$
- $\mathcal{S} := \{\, \{\, \mathsf{a} \,\} \to \{\, \mathsf{b} \,\} \,\}$
- p constantly $\top$

Then

$$\mathrm{conf}_{\mathbb{K}}(\{\, \mathsf{a} \,\} \to \{\, \mathsf{c} \,\}) \geqslant c,$$

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

Choose

- $c := \frac{1}{2}$
- $\mathcal{S} := \{ \{ \mathsf{a} \} \to \{ \mathsf{b} \} \}$
- p constantly $\top$

Then

$$\mathrm{conf}_{\mathbb{K}}(\{ \mathsf{a} \} \to \{ \mathsf{c} \}) \geqslant c,$$

but $\{ \mathsf{a} \}$ is not closed,

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

Choose

- $c := \frac{1}{2}$
- $\mathcal{S} := \{\{\mathsf{a}\} \to \{\mathsf{b}\}\}$
- p constantly $\top$

Then

$$\mathrm{conf}_{\mathbb{K}}(\{\mathsf{a}\} \to \{\mathsf{c}\}) \geqslant c,$$

but $\{\mathsf{a}\}$ is not closed, and

$$\mathrm{conf}_{\mathbb{K}}(\{\mathsf{b}\} \to \{\mathsf{c}\}) = \frac{2}{5} < c,$$

$$\mathrm{conf}_{\mathbb{K}}(\{\mathsf{a},\mathsf{b}\} \to \{\mathsf{c}\}) = \frac{2}{5} < c,$$

$$\mathrm{conf}_{\mathbb{K}}(\varnothing \to \{\mathsf{c}\}) = \frac{4}{10} < c.$$

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

Solution

Ensure: when

$$X \to \{\, m \in M \mid \mathrm{conf}_{\mathbb{K}}(X \to \{\, m \,\}) \geqslant c \,\}$$

is asked in iteration i,

Solution

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is asked in iteration i, then for $Y'' \subsetneq X$

$$\mathrm{conf}_{\mathbb{K}}(Y'' \to \{\, n \,\}) \geqslant c \implies \mathcal{B}_i \cup \mathcal{S}_i \models (Y'' \to \{\, n \,\}).$$

Solution

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Types of Implications asked

Solution

Ensure: when

$$X \to \{ m \in M \mid \mathrm{conf}_{\mathbb{K}}(X \to \{ m \}) \geqslant c \}$$

is asked in iteration i, then for $Y'' \subsetneq X$

$$\mathrm{conf}_{\mathbb{K}}(Y'' \to \{ n \}) \geqslant c \implies \mathcal{B}_i \cup \mathcal{S}_i \models (Y'' \to \{ n \}).$$

Types of Implications asked

- $X \to X''$, where X is closed under $\mathcal{B}_i \cup \mathcal{S}_i$, but not an intent

Solution

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$$X \to \{\, m \in M \mid \mathrm{conf}_{\mathbb{K}}(X \to \{\, m \,\}) \geqslant c \,\}$$

is asked in iteration i, then for $Y'' \subsetneq X$

$$\mathrm{conf}_{\mathbb{K}}(Y'' \to \{\, n \,\}) \geqslant c \implies \mathcal{B}_i \cup \mathcal{S}_i \models (Y'' \to \{\, n \,\}).$$

Types of Implications asked

- $X \to X''$, where X is closed under $\mathcal{B}_i \cup \mathcal{S}_i$, but not an intent
- $X \to \{\, m \in M \mid \mathrm{conf}_{\mathbb{K}}(X \to \{\, m \,\}) \geqslant c \,\} \setminus (\mathcal{B}_i \cup \mathcal{S}_i)(X)$, where X is an intent

Exploration by Confidence

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

- $c := \frac{1}{2}$
- $\mathcal{S} = \{\,\{\,a\,\} \to \{\,b\,\}\,\}$
- p constantly $\top$

Exploration by Confidence

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

- $c := \frac{1}{2}$
- $\mathcal{S} = \{\, \{\, \text{a} \,\} \to \{\, \text{b} \,\} \,\}$
- p constantly $\top$

$$\varnothing$$
$$\{\, \text{a} \,\}$$
$$\{\, \text{b} \,\}$$
$$\{\, \text{a}, \text{b} \,\}$$
$$\{\, \text{c} \,\}$$
$$\{\, \text{a}, \text{c} \,\}$$
$$\{\, \text{b}, \text{c} \,\}$$

Exploration by Confidence

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

- $c := \frac{1}{2}$
- $\mathcal{S} = \{ \{ \mathrm{a} \} \to \{ \mathrm{b} \} \}$
- p constantly $\top$

$\emptyset$

$\{ \mathrm{a} \}$

$\{ \mathrm{b} \}$

$\{ \mathrm{a}, \mathrm{b} \}$

$\{ \mathrm{c} \}$

$\{ \mathrm{a}, \mathrm{c} \}$

$\{ \mathrm{b}, \mathrm{c} \}$

Exploration by Confidence

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

- $c := \frac{1}{2}$
- $\mathcal{S} = \{ \{ \mathrm{a} \} \to \{ \mathrm{b} \} \}$
- p constantly $\top$

$$\emptyset \to \{ \mathrm{a} \}$$

$\{ \mathrm{a} \}$

$\{ \mathrm{b} \}$

$\{ \mathrm{a}, \mathrm{b} \}$

$\{ \mathrm{c} \}$

$\{ \mathrm{a}, \mathrm{c} \}$

$\{ \mathrm{b}, \mathrm{c} \}$

Exploration by Confidence

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

- $c := \frac{1}{2}$
- $\mathcal{S} = \{\{\mathsf{a}\} \to \{\mathsf{b}\}\}$
- p constantly $\top$

$\emptyset \to \{\mathsf{a}\}$ ✓

$\{\mathsf{a}\}$

$\{\mathsf{b}\}$

$\{\mathsf{a}, \mathsf{b}\}$

$\{\mathsf{c}\}$

$\{\mathsf{a}, \mathsf{c}\}$

$\{\mathsf{b}, \mathsf{c}\}$

Exploration by Confidence

	a	b	c
1	×	×	.
2	×	×	.
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- $\mathcal{S} = \{ \{ \text{a} \} \to \{ \text{b} \} \}$
- p constantly $\top$

$\emptyset \to \{ \text{a} \}$ ✓

$\{ \text{a} \}$

~~$\{ \text{b} \}$~~

$\{ \text{a}, \text{b} \}$

~~$\{ \text{c} \}$~~

$\{ \text{a}, \text{c} \}$

~~$\{ \text{b}, \text{c} \}$~~

Exploration by Confidence

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

- $c := \frac{1}{2}$
- $\mathcal{S} = \{\, \{\, a \,\} \to \{\, b \,\} \,\}$
- p constantly $\top$

$\varnothing \to \{\, a \,\}$ ✓

$\{\, a \,\} \to \{\, a, b, c \,\}$

~~$\{\, b \,\}$~~

$\{\, a, b \,\}$

~~$\{\, c \,\}$~~

$\{\, a, c \,\}$

~~$\{\, b, c \,\}$~~

Exploration by Confidence

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
6	×	.	×
7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

- $c := \frac{1}{2}$
- $\mathcal{S} = \{\,\{\,a\,\} \to \{\,b\,\}\,\}$
- p constantly $\top$

$\emptyset \to \{\,a\,\}$ ✓

$\{\,a\,\} \to \{\,a, b, c\,\}$ ✓

~~$\{\,b\,\}$~~

$\{\,a, b\,\}$

~~$\{\,c\,\}$~~

$\{\,a, c\,\}$

~~$\{\,b, c\,\}$~~

Exploration by Confidence

	a	b	c
1	×	×	.
2	×	×	.
3	×	×	.
4	×	×	×
5	×	×	×
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7	×	.	×
8	.	.	.
9	.	.	.
10	.	.	.

- $c := \frac{1}{2}$
- $\mathcal{S} = \{ \{ \mathrm{a} \} \to \{ \mathrm{b} \} \}$
- p constantly $\top$

$\emptyset \to \{ \mathrm{a} \}$ ✓

$\{ \mathrm{a} \} \to \{ \mathrm{a}, \mathrm{b}, \mathrm{c} \}$ ✓

~~{ b }~~

~~{ a, b }~~

~~{ c }~~

~~{ a, c }~~

~~{ b, c }~~

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- Used attribute exploration to handle exceptional counterexamples
- Generalized attribute exploration
- Developed “Exploration by Confidence”

Outlook to GCIs

- Same ideas can be applied
- Counterexamples need to be *connected subinterpretations*
- Extra expert interaction required (due to growing set of “attributes”)

Fact?

$$\mathsf{Venomous} \sqcap \mathsf{Bird} \sqsubseteq \bot$$