Is every BDD theory distancing? It might seem that this can be shown using Exercise 16. However, this is not the case, since the path from c to c' in $Ch(\mathcal{T}, \mathbb{D})$ might lead through atoms not containing any constants from the original $\mathbb{D}$. Nevertheless:

Observation 37. *If a BDD theory admits rewritings of linear width, then it is distancing.*

Assuming Conjecture 2, this implies that all theories from previously known BDD classes are distancing. The converse of Observation 37 does not hold, and such theories can be easily found:

Observation 38. *The theory consisting of the single Datalog rule* $A(x), E(x, y) \Rightarrow A(y)$ *is distancing but not BDD.*

So do there exist non-distancing BDD theories at all? Do there exist BDD theories that do not admit rewritings of linear width? The answer is given by Theorem 3, which constitutes the third main result of this paper:

Theorem 3. *There exists a BDD theory that is non-distancing and does not even admit rewritings of polynomial width.*

Definition 39. *Consider a signature with two binary predicates R and G. Let the theory* $\mathcal{T}_d$ *consist of the following rules:*

$$\text{(loop)} \qquad true \Rightarrow \exists x\ R(x, x), G(x, x)$$

$$\text{(pins)} \qquad \forall x(true \Rightarrow \exists z, z'\ R(x, z), G(x, z'))$$

$$\text{(grid)} \qquad R(x, x'), G(x, u), G(u, u') \Rightarrow \exists z\ R(u', z), G(x', z)$$

Note that the rules of $\mathcal{T}_d$ are not single-head and some of them have empty bodies. One could easily reformulate them to avoid this at the cost of readability (see Appendix D in [17]).

We will think of instances over our signature (and of bodies of queries) as graphs with edges colored in red or green. For $n \in \mathbb{N}$, let $G^n(x_0, x_n)$ denote the CQ $\exists x_1 \ldots x_{n-1} G(x_0, x_1), \ldots, G(x_{n-1}, x_n)$ and $R^n(x_0, x_n)$ likewise. Define conjunctive queries $\phi^n_R(x, y)$ by $\exists x', y' R^n(x, x'), R^n(y, y'), G(x', y')$ and let $\mathbb{G}^n(a, b)$ be a path of n green edges, with a as the first vertex and b as the last.

The following technical lemma substantiates Theorem 3:

Lemma 40. *(A) The theory* $\mathcal{T}_d$ *is BDD.*
(B) $G^{2^n}(x, y) \in rew_{\mathcal{T}_d}(\phi^n_R(x, y))$ *holds for every* $n \in \mathbb{N}$.

Let us first prove claim (B) of the theorem, which implies that $\mathcal{T}_d$ is not distancing. The claim follows once we notice that: (i) $Ch(\mathcal{T}_d, \mathbb{G}^{2^n}(a, b)) \models \phi^n_R(a, b)$ and (ii) if $\mathbb{D}$ is a proper subset of $\mathbb{G}^{2^n}(a, b)$ then $Ch(\mathcal{T}_d, \mathbb{D}) \not\models \phi^n_R(a, b)$. Establishing (i) is immediate, as exemplified in Fig. 1 displaying the case $n = 3$. To show (ii), we note that if $\mathbb{D}$ is a proper subset of $\mathbb{G}^{2^n}(a, b)$ then a and b are in two different connected components of $\mathbb{D}$ and, since $\mathcal{T}_d$ is connected, they are in two different connected components of $Ch(\mathcal{T}_d, \mathbb{D})$.

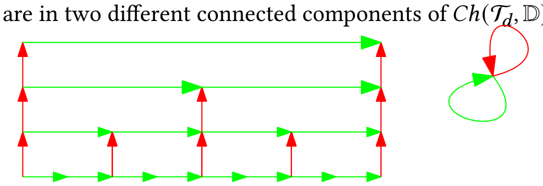

Figure 1: Fragment of $Ch(\mathcal{T}_d, \mathbb{G}^8(a_0, a_8))$ (print in colors!)

The proof of claim (A) is much harder (see Appendix B in [17]). It defines a rewriting procedure in the spirit of [13, 15], whose termination for any given query $\phi(y)$ is shown, via an invariant defined by a complicated multiset ordering.

As our final exercise illustrates, the reasons why $\mathcal{T}_d$ is BDD are quite subtle indeed:

Exercise 41. *Show that without rule (loop),* $\mathcal{T}_d$ *would not be BDD. Hint: Consider the CQ* $\exists x, y\ R(x, y), G(x, y)$.

A remark on Theorem 3. A folklore belief seems to be that the existence of BDD theories that enforce rewritings of unbounded width is a consequence of the the fact that it is undecidable to check if a theory is BDD (see e.g. a recent stackexchange post [16]). Our results call this belief into question, because being BDD is undecidable for theories with a binary signature, and yet such theories, if BDD, are local and thus admit rewritings of linear width.

Remark on distancing and linear width rewritings. One may ask whether distancing is the same as admitting rewritings of linear width. The answer is no. While Observation 37 shows one implication, the converse is not true. Consider theory $\mathcal{T}_d$ from Definition 39, but with every predicate's arity increased by one, and the new variable r occurring in the last position in every atom. Then the new theory is distancing (unlike $\mathcal{T}_d$), but it still requires exponential size rewritings (and, of course, it will remain BDD).

12 CONCLUSIONS AND FUTURE WORK

Our major motivation to embark on this journey was the pending status of the FUS/FES conjecture. On our way, we realized that any progress in that direction requires to significantly advance our understanding of the BDD class, seperating folklore beliefs from hard facts. To this end, we introduced several new notions, characterizing specific properties of theories, and investigated their correspondencies. Most notably, we defined *local theories*, a BDD subclass. Our major results are the following:

- We show that the FUS/FES conjecture holds for all local theories (Theorem 2), which include all theories over binary signatures (Corollary 1). If the conjecture holds in the general case, then our work may provide the basis for a complete proof. If it does not, we now know that we must look for counter-examples of higher arity to disprove it.
- We show that there are BDD theories that are non-distancing and even necessitate rewritings of exponential width (Theorem 3). This result highlights the limitations of existing BDD classes [1, 8, 18], which can only characterise rule sets that admit rewritings of polynomial width.

As for future work, we intend to explore the following:

- Study the relation between distancing and bd-local. More precisely, find out if there are theories that are BDD and bd-local but are not distancing.
- Extend the proof of Theorem 1 to show Conjecture 2, i.e., that all BDD frontier-guarded theories [3] are local and thus the FUS/FES conjecture holds for them. Also, show if the FUS/FES conjecture holds for bd-local theories and then, of course, try to show the conjecture in the general case!
- Define a class of BDD theories that contains rule sets such as the one from Definition 39. Also, define an expressive class that captures the intuitive notion of locality, contains all known BDD classes, and implies BDD membership.

- Even though we extend Theorem 3 in the appendix (see Lemma 40), we wonder if there is a theory that does not admit an elementary bound on the width of its rewritings.

Acknowledgements. Ostropolski-Nalewaja and Marcinkowski were supported by the Polish National Science Centre (NCN) grant 2016/23/B/ST6/01438. Carral was supported by the ANR project CQFD (ANR-18-CE23-0003), the DFG project 389792660 (TRR 248, Center for Perspicuous Systems) and by the BMBF in the Center for Scalable Data Analytics and Artificial Intelligence (ScaDS.AI). Rudolph and Ostropolski-Nalewaja was supported by the European Research Council through the ERC Consolidator Grant DeciGUT (project number 771779).

REFERENCES

[1] Alessandro Artale, Diego Calvanese, Roman Kontchakov, and Michael Zakharyaschev. 2009. The DL-Lite Family and Relations. *J. Artif. Intell. Res.* 36 (2009), 1–69.

[2] Jean-François Baget, Fabien Garreau, Marie-Laure Mugnier, and Swan Rocher. 2014. Revisiting Chase Termination for Existential Rules and their Extension to Nonmonotonic Negation. *CoRR* abs/1405.1071 (2014).

[3] Jean-François Baget, Michel Leclère, Marie-Laure Mugnier, and Eric Salvat. 2011. On rules with existential variables: Walking the decidability line. *Artif. Intell.* 175, 9-10 (2011), 1620–1654.

[4] Pablo Barceló, Gerald Berger, Carsten Lutz, and Andreas Pieris. 2018. First-Order Rewritability of Frontier-Guarded Ontology-Mediated Queries. In *Proc. of the 27th International Joint Conference on Artificial Intelligence, IJCAI 2018,*, Jérôme Lang (Ed.). ijcai.org, 1707–1713.

[5] Pierre Bourhis, Michel Leclère, Marie-Laure Mugnier, Sophie Tison, Federico Ulliana, and Lily Gallois. 2019. Oblivious and Semi-Oblivious Boundedness for Existential Rules. In *Proc. of the 28th International Joint Conference on Artificial Intelligence, IJCAI 2019,*, Sarit Kraus (Ed.). ijcai.org, 1581–1587.

[6] Andrea Calì, Georg Gottlob, and Thomas Lukasiewicz. 2009. Datalog$^{\pm}$: a unified approach to ontologies and integrity constraints. In *Proc. of the 12th International Conference on Database Theory, ICDT 2009, (ACM International Conference Proceeding Series)*, Ronald Fagin (Ed.), Vol. 361. ACM, 14–30.

[7] Andrea Calì, Georg Gottlob, and Thomas Lukasiewicz. 2012. A general Datalog-based framework for tractable query answering over ontologies. *J. Web Semant.* 14 (2012), 57–83.

[8] Andrea Calì, Georg Gottlob, and Andreas Pieris. 2010. Advanced Processing for Ontological Queries. *Proc. VLDB Endow.* 3, 1 (2010), 554–565.

[9] Cristina Civili and Riccardo Rosati. 2015. On the first-order rewritability of conjunctive queries over binary guarded existential rules. In *Proc. of the 30th Italian Conference on Computational Logic, (CEUR Workshop Proceedings)*, Davide Ancona, Marco Maratea, and Viviana Mascardi (Eds.), Vol. 1459. CEUR-WS.org, 25–30.

[10] Stathis Delivorias, Michel Leclère, Marie-Laure Mugnier, and Federico Ulliana. 2018. On the k-Boundedness for Existential Rules. In *Proc. of 2nd International Joint Conference on Rules and Reasoning, RuleML+RR 2018, (Lecture Notes in Computer Science)*, Christoph Benzmüller, Francesco Ricca, Xavier Parent, and Dumitru Roman (Eds.), Vol. 11092. Springer, 48–64.

[11] Alin Deutsch, Alan Nash, and Jeffrey B. Remmel. 2008. The chase revisited. In *Proc. of the 27th ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems, PODS 2008,*, Maurizio Lenzerini and Domenico Lembo (Eds.). ACM, 149–158.

[12] Haim Gaifman, Harry G. Mairson, Yehoshua Sagiv, and Moshe Y. Vardi. 1993. Undecidable Optimization Problems for Database Logic Programs. *J. ACM* 40, 3 (1993), 683–713.

[13] Georg Gottlob, Giorgio Orsi, and Andreas Pieris. 2014. Query Rewriting and Optimization for Ontological Databases. *ACM Trans. Database Syst.* 39, 3 (2014), 25:1–25:46.

[14] Pavol Hell and Jaroslav Nesetril. 1992. The core of a graph. *Discret. Math.* 109, 1-3 (1992), 117–126.

[15] Mélanie König, Michel Leclère, Marie-Laure Mugnier, and Michaël Thomazo. 2015. Sound, complete and minimal UCQ-rewriting for existential rules. *Semantic Web* 6, 5 (2015), 451–475.

[16] Markus Krötzsch. 2020. What is First-Order Rewritable (and FO-Query)? https://cstheory.stackexchange.com/questions/4859/what-is-first-order-rewritable-and-fo-query. Accessed: 2020-11-27.

[17] Piotr Ostropolski-Nalewaja, Jerzy Marcinkowski, David Carral, and Sebastian Rudolph. 2022. A Journey to the Frontiers of Query Rewritability (Extended Technical Report). In *https://hal.archives-ouvertes.fr/hal-03599601v1*.

[18] Michaël Thomazo. 2013. *Conjunctive Query Answering Under Existential Rules - Decidability, Complexity, and Algorithms.* Ph.D. Dissertation. Montpellier 2 University, France.