TU Dresden, Fakultät Informatik
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Complexity Theory
**Exercise 6: Diagonalization**
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**Exercise 6.1.** Find the fault in the following proof of $\mathsf{P} \neq \mathsf{NP}$.

Assume that $\mathsf{P} = \mathsf{NP}$. Then $\mathsf{SAT} \in \mathsf{P}$ and thus there exists a $k \in \mathbb{N}$ such that $\mathsf{SAT} \in \mathsf{DTime}(n^k)$. Because every language in NP is polynomial-time reducible to SAT we have $\mathsf{NP} \subseteq \mathsf{DTime}(n^k)$. It follows that $\mathsf{P} \subseteq \mathsf{DTime}(n^k)$. But by the Time Hierarchy Theorem there exist languages in $\mathsf{DTime}(n^{k+1})$ that are not in $\mathsf{DTime}(n^k)$, contradicting $\mathsf{P} \subseteq \mathsf{DTime}(n^k)$. Therefore, $\mathsf{P} \neq \mathsf{NP}$.

**Exercise 6.2.** Show the following.

1. $\textsc{Time}(2^n) = \textsc{Time}(2^{n+1})$
2. $\textsc{Time}(2^n) \subset \textsc{Time}(2^{2n})$
3. $\textsc{NTime}(n) \subset \textsc{PSpace}$

**Exercise 6.3.** Show that there exists **A** function that is not time-constructible.

**Exercise 6.4.** Consider the function $\mathsf{pad}\colon \Sigma^* \times \mathbb{N} \to \Sigma^*\#^*$ defined as $\mathsf{pad}(s, \ell) = s\#^j$, where $j = \max(0, \ell - |s|)$. In other words, $\mathsf{pad}(s, \ell)$ adds enough copies of $\#$ to the end of $s$ so that the length is at least $\ell$.

For some language $\mathbf{A} \subseteq \Sigma^*$ and $f\colon \mathbb{N} \to \mathbb{N}$ define $\mathsf{pad}(\mathbf{A}, f) = \{\, \mathsf{pad}(s, f(|s|)) \mid s \in \mathbf{A} \,\}$.

1. Show that if $\mathbf{A} \in \textsc{DTime}(n^6)$, then $\mathsf{pad}(\mathbf{A}, n^2) \in \textsc{DTime}(n^3)$.
2. Show that if $\textsc{NExpTime} \neq \textsc{ExpTime}$, then $\mathrm{P} \neq \mathrm{NP}$.
3. Show for every $\mathbf{A} \subseteq \Sigma^*$ and every $k \in \mathbb{N}$ that $\mathbf{A} \in \mathrm{P}$ if and only if $\mathsf{pad}(\mathbf{A}, n^k) \in \mathrm{P}$.
4. Show that $\mathrm{P} \neq \textsc{DSpace}(n)$.
5. Show that $\mathrm{NP} \neq \textsc{DSpace}(n)$.