Exercise 13.1. Let $G = \langle V, E \rangle$ be an undirected graph. A cut $C = \langle A, B \rangle$ of $G$ is a partition $A \cup B = V$ of $V$ such that there is an edge in $G$ from a vertex $a \in A$ to a vertex $b \in B$. The cut-set of $C$ is $S := \{ \{ a, b \} \mid a \in A, b \in B \}$. The normalised cut value of a cut $C = \langle A, B \rangle$ is

$$\frac{|\{ \{ a, b \} \mid a \in A, b \in B \}|}{|\{ a, v \} \in E \mid a \in A, v \in V \}| + \frac{|\{ a, b \} \mid a \in A, b \in B \}| + \frac{|\{ b, v \} \in E \mid b \in B, v \in V \}|}$$

i.e., the number of endpoints of edges in the cut-set divided by the number of vertices that have an edge with an endpoint in $A$ or $B$.

Find the minimal cuts (i) with respect to the cardinality of the cut-set, and (ii) with respect to the normalised cut value of the following graph. Which of these cuts best describes the community structure of the graph?

![Graph](image)

Exercise 13.2. Let $G = \langle V, E \rangle$ be an undirected graph. A bi-clique in $G$ consists of two disjoint, nonempty sets $A, B \subseteq V$ such that the induced subgraph of $A \cup B$ in $G$ is a complete bipartite graph, i.e., $\{ \{ a, b \} \mid a \in A, b \in B \} \subseteq E$, and no two vertices in $A$ (and $B$, respectively) are adjacent.

Find all bi-cliques in the following graph. Which bi-cliques are maximal?

![Graph](image)
Exercise 13.3. Let $G = (V, E)$ be a directed graph. For two vertices $v, w \in V$, the distance $d(v, w)$ is the length of the shortest directed path from $v$ to $w$ (or $\infty$ if there is no such path). For a set $S \subseteq V$ of vertices, the reachable set is $R(S) := \{ v \in V \mid \exists w \in S. d(v, w) < \infty \}$. A point base of $G$ is a minimal set $B \subseteq V$ such that $R(B) = V$.

Find a point base for the following graph. How does the point base change when adding the edge $\{B, E\}$?

Exercise 13.4. Write a program that takes as input

- a directed graph in METIS format
- and a dictionary file in the format of Exercise 11.4 mapping vertex IDs to labels,

and uses the Girvan-Newman algorithm to print out all communities on level $k$ of the hierarchical clustering of the input graph.

Modify the program from Exercise 11.4 to extract the subgraph from Wikidata that consists of all edges of the form $t \rightarrow s$, where $t$ has a P375 (“space launch vehicle”)-statement with value $s$, and also a P619 (“time of spacecraft launch”)-statement with some value. Use your program to print out the communities on level 5 of the hierarchical clustering of this subgraph.

**Hint:** NetworkX\(^1\) provides an implementation of the GN algorithm.\(^2\)

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\(^1\)https://networkx.github.io/

\(^2\)https://networkx.github.io/documentation/latest/reference/algorithms/generated/networkx.algorithms.community.centrality.girvan_newman.html