Review
Recall our earlier definitions of space complexities:

**Definition 9.1:** Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function.

1. $\text{DSpace}(f(n))$ is the class of all languages $L$ for which there is an $O(f(n))$-space bounded Turing machine deciding $L$.
2. $\text{NSpace}(f(n))$ is the class of all languages $L$ for which there is an $O(f(n))$-space bounded nondeterministic Turing machine deciding $L$.

Being $O(f(n))$-space bounded requires a (nondeterministic) TM

- to halt on every input and
- to use $\leq f(|w|)$ tape cells on every computation path.
Space Complexity Classes

Some important space complexity classes:

\[ L = \text{LogSpace} = \text{DSpace}(\log n) \quad \text{logarithmic space} \]

\[ \text{PSpace} = \bigcup_{d \geq 1} \text{DSpace}(n^d) \quad \text{polynomial space} \]

\[ \text{ExpSpace} = \bigcup_{d \geq 1} \text{DSpace}(2^{n^d}) \quad \text{exponential space} \]

\[ \text{NL} = \text{NLogSpace} = \text{NSpace}(\log n) \quad \text{nondet. logarithmic space} \]

\[ \text{NPSpace} = \bigcup_{d \geq 1} \text{NSpace}(n^d) \quad \text{nondet. polynomial space} \]

\[ \text{NExpSpace} = \bigcup_{d \geq 1} \text{NSpace}(2^{n^d}) \quad \text{nondet. exponential space} \]
The Power of Space

Space seems to be more powerful than time because space can be reused.

Example 9.2: SAT can be solved in linear space: Just iterate over all possible truth assignments (each linear in size) and check if one satisfies the formula.

Example 9.3: Tautology can be solved in linear space: Just iterate over all possible truth assignments (each linear in size) and check if all satisfy the formula.

More generally: $\text{NP} \subseteq \text{PSpace}$ and $\text{coNP} \subseteq \text{PSpace}$
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**Theorem 9.4:** For every function $f : \mathbb{N} \to \mathbb{R}^+$, for all $c \in \mathbb{N}$, and for every $f$-space bounded (deterministic/nondeterministic) Turing machine $M$:

there is a $\max\{1, \frac{1}{c}f(n)\}$-space bounded (deterministic/nondeterministic) Turing machine $M'$ that accepts the same language as $M$. 
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**Proof idea:** Similar to (but much simpler than) linear speed-up.
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This justifies using $O$-notation for defining space classes.
**Theorem 9.5:** For every function $f : \mathbb{N} \rightarrow \mathbb{R}^+$ all $k \geq 1$ and $L \subseteq \Sigma^*$:

If $L$ can be decided by an $f$-space bounded $k$-tape Turing-machine, then it can also be decided by an $f$-space bounded 1-tape Turing-machine.
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If $L$ can be decided by an $f$-space bounded $k$-tape Turing-machine, then it can also be decided by an $f$-space bounded 1-tape Turing-machine.

Proof idea: Combine tapes with a similar reduction as for time. Compress space to avoid linear increase.

Note: We still use a separate read-only input tape to define some space complexities, such as LogSpace.
**Theorem 9.6:** For all functions $f : \mathbb{N} \to \mathbb{R}^+$:

$$\text{DTime}(f) \subseteq \text{DSpace}(f) \quad \text{and} \quad \text{NTime}(f) \subseteq \text{NSpace}(f)$$

**Proof:** Visiting a cell takes at least one time step.
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Theorem 9.7: For all functions $f : \mathbb{N} \rightarrow \mathbb{R}^+$ with $f(n) \geq \log n$:

$$\text{DSpace}(f) \subseteq \text{DTime}(2^{O(f)}) \quad \text{and} \quad \text{NSpace}(f) \subseteq \text{DTime}(2^{O(f)})$$
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**Proof:** Based on configuration graphs and a bound on the number of possible configurations.
Number of Possible Configurations

Let $\mathcal{M} := (Q, \Sigma, \Gamma, q_0, \delta, q_{\text{start}})$ be a 2-tape Turing machine
(1 read-only input tape + 1 work tape)

Recall: A configuration of $\mathcal{M}$ is a quadruple $(q, p_1, p_2, x)$ where
- $q \in Q$ is the current state,
- $p_i \in \mathbb{N}$ is the head position on tape $i$, and
- $x \in \Gamma^*$ is the tape content.

Let $w \in \Sigma^*$ be an input to $\mathcal{M}$ and $n := |w|$.
- Then also $p_1 \leq n$.
- If $\mathcal{M}$ is $f(n)$-space bounded we can assume $p_2 \leq f(n)$ and $|x| \leq f(n)$
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Hence, there are at most

$$|Q| \cdot n \cdot f(n) \cdot |\Gamma|^{f(n)} = n \cdot 2^{O(f(n))} = 2^{O(f(n))}$$

different configurations on inputs of length $n$ (the last equality requires $f(n) \geq \log n$).
Configuration Graphs

The possible computations of a TM $M$ (on input $w$) form a directed graph:

- **Vertices:** configurations that $M$ can reach (on input $w$)
- **Edges:** there is an edge from $C_1$ to $C_2$ if $C_1 \vdash_M C_2$  
  ($C_2$ reachable from $C_1$ in a single step)

This yields the **configuration graph**:

- Could be infinite in general.
- For $f(n)$-space bounded 2-tape TMs, there can be at most $2^{O(f(n))}$ vertices and $(2^{O(f(n))})^2 = 2^{O(f(n))}$ edges
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A computation of $M$ on input $w$ corresponds to a path in the configuration graph from the start configuration to a stop configuration.

Hence, to test if $M$ accepts input $w$,

- construct the configuration graph and
- find a path from the start to an accepting stop configuration.
Theorem 9.6: For all functions $f : \mathbb{N} \to \mathbb{R}^+$:

$$\text{DTime}(f) \subseteq \text{DSpace}(f) \quad \text{and} \quad \text{NTime}(f) \subseteq \text{NSpace}(f)$$

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Proof: Build the configuration graph (time $2^{O(f(n))}$) and find a path from the start to an accepting stop configuration (time $2^{O(f(n))}$). □
Applying the results of the previous slides, we get the following relations:

\[ L \subseteq NL \subseteq P \subseteq NP \subseteq PSpace \subseteq NPSpace \subseteq \text{ExpTime} \subseteq \text{NExpTime} \]

We also noted \( P \subseteq \text{coNP} \subseteq \text{PSpace} \).

Open questions:

- What is the relationship between space classes and their co-classes?
- What is the relationship between deterministic and non-deterministic space classes?
Nondeterminism in Space

Most experts think that nondeterministic TMs can solve strictly more problems when given the same amount of time than a deterministic TM:

Most believe that \( P \subset NP \)

How about nondeterminism in space-bounded TMs?
Nondeterminism in Space

Most experts think that nondeterministic TMs can solve strictly more problems when given the same amount of time than a deterministic TM:

Most believe that $P \subsetneq NP$

How about nondeterminism in space-bounded TMs?

**Theorem 9.8 (Savitch’s Theorem, 1970):** For any function $f : \mathbb{N} \to \mathbb{R}^+$ with $f(n) \geq \log n$:

$$\text{NSpace}(f(n)) \subseteq \text{DSpace}(f^2(n)).$$

That is: nondeterminism adds almost no power to space-bounded TMs!
Theorem 9.8 (Savitch’s Theorem, 1970): For any function $f : \mathbb{N} \to \mathbb{R}^+$ with $f(n) \geq \log n$:

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Consequences of Savitch’s Theorem

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**Corollary 9.9:** $\text{PSPACE} = \text{NPSPACE}$.

**Proof:** $\text{PSPACE} \subseteq \text{NPSPACE}$ is clear. The converse follows since the square of a polynomial is still a polynomial.

Similarly for “bigger” classes, e.g., $\text{EXPSPACE} = \text{NEXPSPACE}$.
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Similarly for “bigger” classes, e.g., $\text{ExpSpace} = \text{NExpSpace}$.

**Corollary 9.10:** $\text{NL} \subseteq \text{DSpace}(O(\log^2 n))$.

Note that $\log^2(n) \notin O(\log n)$, so we do not obtain $\text{NL} = \text{L}$ from this.
Simulating nondeterminism with more space:

- Use configuration graph of nondeterministic space-bounded TM
- Check if an accepting configuration can be reached
- Store only one computation path at a time (depth-first search)
Proving Savitch’s Theorem

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This still requires exponential space. We want quadratic space!

*What to do?*
Proving Savitch’s Theorem

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**What to do?**

Things we can do:

- Store one configuration:
  - one configuration requires $\log n + O(f(n))$ space
  - if $f(n) \geq \log n$, then this is $O(f(n))$ space
- Store $\log n$ configurations (remember we have $\log^2 n$ space)
- Iterate over all configurations (one by one)
Proving Savitch’s Theorem: Key Idea

To find out if we can reach an accepting configuration, we solve a slightly more general question:

**YIELDABILITY**

Input: TM configurations $C_1$ and $C_2$, integer $k$
Problem: Can TM get from $C_1$ to $C_2$ in at most $k$ steps?
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**Approach:** check if there is an intermediate configuration $C'$ such that

1. $C_1$ can reach $C'$ in $k/2$ steps and
2. $C'$ can reach $C_2$ in $k/2$ steps

$\leadsto$ **Deterministic:** we can try all $C'$ (iteration)
$\leadsto$ **Space-efficient:** we can reuse the same space for both steps
An Algorithm for Yieldability

```plaintext
01 CanYield(C₁, C₂, k) {
02    if k = 1 :
03        return (C₁ = C₂) or (C₁ ⊢ₘ C₂)
04    else if k > 1 :
05        for each configuration C of M for input size n :
06            if CanYield(C₁, C, k/2) and
07                CanYield(C, C₂, k/2) :
08                return true
09        // eventually, if no success:
10        return false
11 }
```

- We only call CanYield only with $k$ a power of 2, so $k/2 ∈ \mathbb{N}$
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Overall space usage: $O(f(n) \cdot \log k)$
Simulating Nondeterministic Space-Bounded TMs

**Input:** TM $M$ that runs in $\text{NSpace}(f(n))$; input word $w$ of length $n$

**Algorithm:**

- Modify $M$ to have a unique accepting configuration $C_{\text{accept}}$: when accepting, erase tape and move head to the very left
- Select $d$ such that $2^{df(n)} \geq |Q| \cdot n \cdot f(n) \cdot |\Gamma|^f(n)$
- Return $\text{CanYield}(C_{\text{start}}, C_{\text{accept}}, k)$ with $k = 2^{df(n)}$
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**Space requirements:**

$\text{CanYield}$ runs in space

$$O(f(n) \cdot \log k) = O\left(f(n) \cdot \log 2^{df(n)}\right) = O(f(n) \cdot df(n)) = O(f^2(n))$$
"Select $d$ such that $2^{\frac{df(n)}{n}} \geq |Q| \cdot n \cdot f(n) \cdot |\Gamma| f(n)$".

How does the algorithm actually do this?

- $f(n)$ was not part of the input!
- Even if we knew $f(n)$, it might not be easy to compute!

Solution: replace $f(n)$ by a parameter $\ell$ and probe its value.

1. Start with $\ell = 1$
2. Check if $M$ can reach any configuration with more than $\ell$ tape cells (iterate over all configurations of size $\ell + 1$; use CanYield on each)
3. If yes, increase $\ell$ by 1; goto (2)
4. Run algorithm as before, with $f(n)$ replaced by $\ell$

Therefore: we don't need to know $f$ at all. This finishes the proof. □
Did We Really Do It?

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Summary: Relationships of Space and Time

Summing up, we get the following relations:

\[ L \subseteq NL \subseteq P \subseteq NP \subseteq PSpace = NPSpace \subseteq ExpTime \subseteq NExpTime \]

We also noted \( P \subseteq coNP \subseteq PSpace \).

Open questions:

- Is Savitch’s Theorem tight?
- Are there any interesting problems in these space classes?
- We have \( PSpace = NPSpace = coNPSpace \).
  But what about \( L, NL, \) and \( coNL \)?
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\( \rightarrow \) the first: nobody knows (YCTBF); the others: see upcoming lectures