FOUNDATIONS OF COMPLEXITY THEORY

Lecture 10: Polynomial Space

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The Class PSpace

We defined PSpace as:

$$\text{PSpace} = \bigcup_{d \geq 1} \text{DSpace}(n^d)$$

and we observed that

$$\text{P} \subseteq \text{NP} \subseteq \text{PSpace} = \text{NPSpace} \subseteq \text{ExpTime}.$$

We can also define a corresponding notion of PSpace-hardness:

Definition 10.1:
- A language **H** is PSpace-hard, if **L** $\leq_p$ **H** for every language **L** $\in$ PSpace.
- A language **C** is PSpace-complete, if **C** is PSpace-hard and **C** $\in$ PSpace.

Quantified Boolean Formulae (QBF)

A QBF is a formula of the following form:

$$Q_1 X_1.Q_2 X_2.\cdots Q_\ell X_\ell.\varphi[X_1,\ldots,X_\ell]$$

where $Q_i \in \{\exists, \forall\}$ are quantifiers, X_i are propositional logic variables, and φ is a propositional logic formula with variables $X_1,\ldots,X_\ell$ and constants $\top$ (true) and $\bot$ (false)

Semantics:
- Propositional formulae without variables (only constants $\top$ and $\bot$) are evaluated as usual
- $\exists X.\varphi[X]$ is true if either $\varphi[X/\top]$ or $\varphi[X/\bot]$ are true
- $\forall X.\varphi[X]$ is true if both $\varphi[X/\top]$ and $\varphi[X/\bot]$ are true

  (where $\varphi[X/\top]$ is "φ with X replaced by $\top$, and similar for $\bot$)

Deciding QBF Validity

True QBF
Input: A quantified Boolean formula φ.
Problem: Is φ true (valid)?

Observation: We can assume that the quantified formula is in CNF or 3-CNF (same transformations possible as for propositional logic formulae)

Consider a propositional logic formula φ with variables $X_1,\ldots,X_\ell$:

Example 10.2: The QBF $\exists X_1.\cdots\exists X_\ell.\varphi$ is true if and only if φ is satisfiable.

Example 10.3: The QBF $\forall X_1.\cdots\forall X_\ell.\varphi$ is true if and only if φ is a tautology.

The Power of QBF

Theorem 10.4: True QBF is PSpace-complete.

Proof:

(1) **True QBF** $\in$ PSpace:
Give an algorithm that runs in polynomial space.

(2) **True QBF** is PSpace-hard:
Proof by reduction from the word problem of any polynomially space-bounded TM.

□

Solving True QBF in PSpace

```
01 TrueQBF(φ) {
02   if φ has no quantifiers :
03     return "evaluation of φ"
04   else if φ = ∃X.ψ :
05     return (TrueQBF(ψ[X/⊤]) OR TrueQBF(ψ[X/⊥]))
06   else if φ = ∀X.ψ :
07     return (TrueQBF(ψ[X/⊤]) AND TrueQBF(ψ[X/⊥]))
08 }
```

- Evaluation in line 03 can be done in polynomial space
- Recursions in lines 05 and 07 can be executed one after the other, reusing space
- Maximum depth of recursion = number of variables (linear)
- Store one variable assignment per recursive call

$\leadsto$ polynomial space algorithm

PSpace-Hardness of True QBF

Express TM computation in logic, similar to Cook-Levin

Given:

An arbitrary polynomially space-bounded NTM, that is:

- a polynomial p
- a p-space bounded 1-tape NTM $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}})$

Intended reduction

Given a word w, define a QBF $\varphi_{p,\mathcal{M},w}$ such that
$\varphi_{p,\mathcal{M},w}$ is true if and only if $\mathcal{M}$ accepts w in space $p(|w|)$.

Notes

- We show the reduction for NTMs, which is more than needed, but makes little difference in logic and allows us to reuse our previous formulae from Cook-Levin
- The proof actually shows many reductions, one for every polyspace NTM, showing PSpace-hardness from first principles

Review: Encoding Configurations

Use propositional variables for describing configurations:

Q_q for each $q \in Q$ means "$\mathcal{M}$ is in state $q \in Q$"

P_i for each $0 \leq i < p(n)$ means "the head is at Position i"

$S_{a,i}$ for each $a \in \Gamma$ and $0 \leq i < p(n)$ means "tape cell i contains Symbol a"

Represent configuration $(q, p, a_0 \ldots a_{p(n)})$

by assigning truth values to variables from the set

$$\overline{C} := \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \leq i < p(n)\}$$

using the truth assignment β defined as

$$\beta(Q_s) := \begin{cases} 1 & s = q \\ 0 & s \neq q \end{cases} \qquad \beta(P_i) := \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases} \qquad \beta(S_{a,i}) := \begin{cases} 1 & a = a_i \\ 0 & a \neq a_i \end{cases}$$

Review: Validating Configurations

We define a formula Conf($\overline{C}$) for a set of configuration variables

$$\overline{C} = \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \le i < p(n)\}$$

as follows:

Conf($\overline{C}$) := "the assignment is a valid configuration":

$$\bigvee_{q \in Q} \Big(Q_q \wedge \bigwedge_{q' \neq q} \neg Q_{q'}\Big)$$ "TM in exactly one state $q \in Q$"

$$\wedge \bigvee_{p < p(n)} \Big(P_p \wedge \bigwedge_{p' \neq p} \neg P_{p'}\Big)$$ "head in exactly one position $p < p(n)$"

$$\wedge \bigwedge_{0 \le i < p(n)} \bigvee_{a \in \Gamma} \Big(S_{a,i} \wedge \bigwedge_{b \neq a \in \Gamma} \neg S_{b,i}\Big)$$ "exactly one $a \in \Gamma$ in each cell"

Review: Validating Configurations

For an assignment β defined on variables in $\overline{C}$ define

$$\text{conf}(\overline{C}, \beta) := \left\{ (q, p, w_0 \ldots w_{p(n)}) \;\middle|\; \begin{array}{l} \beta(Q_q) = 1, \\ \beta(P_p) = 1, \\ \beta(S_{w_i,i}) = 1 \text{ for all } 0 \le i < p(n) \end{array} \right\}$$

Note: β may be defined on other variables besides those in $\overline{C}$.

Lemma 10.5: If β satisfies Conf($\overline{C}$) then $|\text{conf}(\overline{C}, \beta)| = 1$.
We can therefore write $\text{conf}(\overline{C}, \beta) = (q, p, w)$ to simplify notation.

Observations:

- $\text{conf}(\overline{C}, \beta)$ is a potential configuration of $\mathcal{M}$, but it may not be reachable from the start configuration of $\mathcal{M}$ on input w.
- Conversely, every configuration $(q, p, w_1 \ldots w_{p(n)})$ induces a satisfying assignment β for which $\text{conf}(\overline{C}, \beta) = (q, p, w_1 \ldots w_{p(n)})$.

Review: Transitions Between Configurations

Consider the following formula Next($\overline{C}, \overline{C}'$) defined as

$$\text{Conf}(\overline{C}) \wedge \text{Conf}(\overline{C}') \wedge \text{NoChange}(\overline{C}, \overline{C}') \wedge \text{Change}(\overline{C}, \overline{C}').$$

$$\text{NoChange} := \bigvee_{0 \le p < p(n)} \Big(P_p \wedge \bigwedge_{i \neq p, a \in \Gamma} (S_{a,i} \to S'_{a,i})\Big)$$

$$\text{Change} := \bigvee_{0 \le p < p(n)} \Big(P_p \wedge \bigvee_{\substack{q \in Q \\ a \in \Gamma}} \big(Q_q \wedge S_{a,p} \wedge \bigvee_{(q',b,D) \in \delta(q,a)} (Q'_{q'} \wedge S'_{b,p} \wedge P'_{D(p)})\big)\Big)$$

where $D(p)$ is the position reached by moving in direction D from p.

Lemma 10.6: For any assignment β defined on $\overline{C} \cup \overline{C}'$:

β satisfies Next($\overline{C}, \overline{C}'$) if and only if $\text{conf}(\overline{C}, \beta) \vdash_{\mathcal{M}} \text{conf}(\overline{C}', \beta)$

Review: Start and End

Defined so far:

- Conf($\overline{C}$): $\overline{C}$ describes a potential configuration
- Next($\overline{C}, \overline{C}'$): $\text{conf}(\overline{C}, \beta) \vdash_{\mathcal{M}} \text{conf}(\overline{C}', \beta)$

Start configuration: Let $w = w_0 \cdots w_{n-1} \in \Sigma^*$ be the input word

$$\text{Start}_{\mathcal{M},w}(\overline{C}) := \text{Conf}(\overline{C}) \wedge Q_{q_0} \wedge P_0 \wedge \bigwedge_{i=0}^{n-1} S_{w_i,i} \wedge \bigwedge_{i=n}^{p(n)-1} S_{\llcorner,i}$$

Then an assignment β satisfies $\text{Start}_{\mathcal{M},w}(\overline{C})$ if and only if $\overline{C}$ represents the start configuration of $\mathcal{M}$ on input w.

Accepting stop configuration:

$$\text{Acc-Conf}(\overline{C}) := \text{Conf}(\overline{C}) \wedge Q_{q_{\text{accept}}}$$

Then an assignment β satisfies Acc-Conf($\overline{C}$) if and only if $\overline{C}$ represents an accepting configuration of $\mathcal{M}$.

Simulating Polynomial Space Computations

For Cook-Levin, we used one set of configuration variables for every computating step:
polynomially time ⇝ polynomially many variables

Problem: For polynomial space, we have $2^{O(p(n))}$ possible steps . . .

What would Savitch do?

Define a formula $\mathsf{CanYield}_i(\overline{C}_1, \overline{C}_2)$ to state that $\overline{C}_2$ is reachable from $\overline{C}_1$ in at most 2^i steps:

$$\mathsf{CanYield}_0(\overline{C}_1, \overline{C}_2) := (\overline{C}_1 = \overline{C}_2) \vee \mathsf{Next}(\overline{C}_1, \overline{C}_2)$$
$$\mathsf{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) := \exists \overline{C}.\mathsf{Conf}(\overline{C}) \wedge \mathsf{CanYield}_i(\overline{C}_1, \overline{C}) \wedge \mathsf{CanYield}_i(\overline{C}, \overline{C}_2)$$

But what is $\overline{C}_1 = \overline{C}_2$ supposed to mean here? It is short for:

$$\bigwedge_{q \in Q} Q_q^1 \leftrightarrow Q_q^2 \wedge \bigwedge_{0 \leq i < p(n)} P_i^1 \leftrightarrow P_i^2 \wedge \bigwedge_{a \in \Gamma, 0 \leq i < p(n)} S_{a,i}^1 \leftrightarrow S_{a,i}^2$$

Putting Everything Together

We define the formula $\varphi_{p,\mathcal{M},w}$ as follows:

$$\varphi_{p,\mathcal{M},w} := \exists \overline{C}_1.\exists \overline{C}_2.\mathsf{Start}_{\mathcal{M},w}(\overline{C}_1) \wedge \mathsf{Acc\text{-}Conf}(\overline{C}_2) \wedge \mathsf{CanYield}_{dp(n)}(\overline{C}_1, \overline{C}_2)$$

where we select d to be the least number such that $\mathcal{M}$ has less than $2^{dp(n)}$ configurations in space $p(n)$.

Lemma 10.7: $\varphi_{p,\mathcal{M},w}$ is satisfiable if and only if $\mathcal{M}$ accepts w in space $p(|w|)$.

Did we do it?

Note: we used only existential quantifiers when defining $\varphi_{p,\mathcal{M},w}$:

$$\mathsf{CanYield}_0(\overline{C}_1, \overline{C}_2) := (\overline{C}_1 = \overline{C}_2) \vee \mathsf{Next}(\overline{C}_1, \overline{C}_2)$$
$$\mathsf{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) := \exists \overline{C}.\mathsf{Conf}(\overline{C}) \wedge \mathsf{CanYield}_i(\overline{C}_1, \overline{C}) \wedge \mathsf{CanYield}_i(\overline{C}, \overline{C}_2)$$
$$\varphi_{p,\mathcal{M},w} := \exists \overline{C}_1.\exists \overline{C}_2.\mathsf{Start}_{\mathcal{M},w}(\overline{C}_1) \wedge \mathsf{Acc\text{-}Conf}(\overline{C}_2) \wedge \mathsf{CanYield}_{dp(n)}(\overline{C}_1, \overline{C}_2)$$

Now that's quite interesting . . .

- With only (non-negated) ∃ quantifiers, **True QBF** coincides with **Sat**
- **Sat** is in NP
- So we showed that the word problem for PSpace NTMs to be in NP

So we found that NP = PSpace!

Strangely, most textbooks claim that this is not known to be true . . .
Are we up for the next Turing Award, or did we make a mistake?

Size

How big is $\varphi_{p,\mathcal{M},w}$?

$$\mathsf{CanYield}_0(\overline{C}_1, \overline{C}_2) := (\overline{C}_1 = \overline{C}_2) \vee \mathsf{Next}(\overline{C}_1, \overline{C}_2)$$
$$\mathsf{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) := \exists \overline{C}.\mathsf{Conf}(\overline{C}) \wedge \mathsf{CanYield}_i(\overline{C}_1, \overline{C}) \wedge \mathsf{CanYield}_i(\overline{C}, \overline{C}_2)$$
$$\varphi_{p,\mathcal{M},w} := \exists \overline{C}_1.\exists \overline{C}_2.\mathsf{Start}_{\mathcal{M},w}(\overline{C}_1) \wedge \mathsf{Acc\text{-}Conf}(\overline{C}_2) \wedge \mathsf{CanYield}_{dp(n)}(\overline{C}_1, \overline{C}_2)$$

Size of $\mathsf{CanYield}_{i+1}$ is more than twice the size of $\mathsf{CanYield}_i$
⇝ Size of $\varphi_{p,\mathcal{M},w}$ is in $2^{O(p(n))}$. Oops.

A correct reduction: We redefine CanYield by setting

$$\mathsf{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) :=$$
$$\exists \overline{C}.\mathsf{Conf}(\overline{C}) \wedge$$
$$\forall \overline{Z}_1.\forall \overline{Z}_2.(((\overline{Z}_1 = \overline{C}_1 \wedge \overline{Z}_2 = \overline{C}) \vee (\overline{Z}_1 = \overline{C} \wedge \overline{Z}_2 = \overline{C}_2)) \to \mathsf{CanYield}_i(\overline{Z}_1, \overline{Z}_2))$$

Size

Let's analyse the size more carefully this time:

$$\text{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) :=$$
$$\exists\overline{C}.\text{Conf}(\overline{C}) \land$$
$$\forall\overline{Z}_1.\forall\overline{Z}_2.(((\overline{Z}_1 = \overline{C}_1 \land \overline{Z}_2 = \overline{C}) \lor (\overline{Z}_1 = \overline{C} \land \overline{Z}_2 = \overline{C}_2)) \rightarrow \text{CanYield}_i(\overline{Z}_1, \overline{Z}_2))$$

- $\text{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2)$ extends $\text{CanYield}_i(\overline{C}_1, \overline{C}_2)$ by parts that are linear in the size of configurations $\leadsto$ growth in $O(p(n))$
- Maximum index i used in $\varphi_{p,\mathcal{M},w}$ is $dp(n)$, that is in $O(p(n))$
- Therefore: $\varphi_{p,\mathcal{M},w}$ has size $O(p^2(n))$ – and thus can be computed in polynomial time

Exercise:

Why can we just use $dp(n)$ in the reduction? Don't we have to compute it somehow? Maybe even in polynomial time?

The Power of QBF

Theorem 10.4: **TRUE QBF** is PSpace-complete.

Proof:

(1) **TRUE QBF** $\in$ PSpace:
Give an algorithm that runs in polynomial space.

(2) **TRUE QBF** is PSpace-hard:
Proof by reduction from the word problem of any polynomially space-bounded TM.

□

A More Common Logical Problem in PSpace

Recall standard first-order logic:

- Instead of propositional variables, we have atoms (predicates with constants and variables)
- Instead of propositional evaluations we have first-order structures (or interpretations)
- First-order quantifiers can be used on variables
- Sentences are formulae where all variables are quantified
- A sentence can be satisfied or not by a given first-order structure

FOL MODEL CHECKING

Input:	A first-order sentence φ and a finite first-order structure $\mathcal{I}$.
Problem:	Is φ satisfied by $\mathcal{I}$?

First-Order Logic is PSpace-complete

Theorem 10.8: **FOL MODEL CHECKING** is PSpace-complete.

Proof:

(1) **FOL MODEL CHECKING** $\in$ PSpace:
Give algorithm that runs in polynomial space.

(2) **FOL MODEL CHECKING** is PSpace-hard:
Proof by reduction **TRUE QBF** $\leq_p$ **FOL MODEL CHECKING**.

□

Checking FOL Models in Polynomial Space (Sketch)

```
Eval(φ, I) {
  switch (φ) :
    case p(c_1, ..., c_n) :  return ⟨c_1, ..., c_n⟩ ∈ p^I
    case ¬ψ :  return NOT Eval(ψ, I)
    case ψ_1 ∧ ψ_2 :  return Eval(ψ_1, I) AND Eval(ψ_2, I)
    case ∃x.ψ :
      for c ∈ Δ^I :
        if Eval(ψ[x ↦ c], I) :  return TRUE
      // eventually, if no success:
      return FALSE
}
```

- We can assume φ only uses $\neg$, $\wedge$ and $\exists$ (easy to get)
- We use $\Delta^{\mathcal{I}}$ to denote the (finite!) domain of $\mathcal{I}$
- We allow domain elements to be used like constants in the formula

Hardness of **FOL Model Checking**

Given: a QBF $\varphi = \mathsf{Q}_1 X_1. \cdots \mathsf{Q}_\ell X_\ell.\psi$

FOL Model Checking Problem:

- Interpretation domain $\Delta^{\mathcal{I}} := \{0, 1\}$
- Single predicate symbol true with interpretation $\mathrm{true}^{\mathcal{I}} = \{\langle 1 \rangle\}$
- FOL formula φ' is obtained by replacing variables in input QBF with corresponding first-order expressions:

$$\mathsf{Q}_1 x_1. \cdots \mathsf{Q}_\ell x_\ell.\psi[X_1 \mapsto \mathrm{true}(x_1), \ldots, X_\ell \mapsto \mathrm{true}(x_\ell)]$$

Lemma 10.9: $\langle \mathcal{I}, \varphi' \rangle \in$ **FOL Model Checking** if and only if $\varphi \in$ **True QBF**.

First-Order Logic is PSpace-complete

Theorem 10.8: **FOL Model Checking** is PSpace-complete.

Proof:

(1) **FOL Model Checking** $\in$ PSpace:
Give algorithm that runs in polynomial space.

(2) **FOL Model Checking** is PSpace-hard:
Proof by reduction **True QBF** $\leq_p$ **FOL Model Checking**.

□

FOL Model Checking: Practical Significance

Why is **FOL Model Checking** a relevant problem?

Correspondence with database query answering:

- Finite first-order interpretation = database
- First-order logic formula = database query
- Satisfying assignments (for non-sentences) = query results

Known correspondence:
As a query language, FOL has the same expressive power as (basic) SQL (relational algebra).

Corollary 10.10: Answering SQL queries over a given database is PSpace-complete.

Games

Games as Computational Problems

Many single-player games relate to NP-complete problems:

- Sudoku
- Minesweeper
- Tetris
- ...

Decision problem: Is there a solution?
(For Tetris: is it possible to clear all blocks?)

What about two-player games?

- Two players take moves in turns
- The players have different goals
- The game ends if a player wins

Decision problem: Does Player 1 have a winning strategy?
In other words: can Player 1 enforce winning, whatever Player 2 does?

Example: The Formula Game

A contrived game, to illustrate the idea:

- Given: a propositional logic formula φ with consecutively numbered variables $X_1, \ldots X_\ell$.
- Two players take turns in selecting values for the next variable:
  - Player 1 sets X_1 to true or false
  - Player 2 sets X_2 to true or false
  - Player 1 sets X_3 to true or false
  - ...

  until all variables are set.
- Player 1 wins if the assignment makes φ true.
  Otherwise, Player 2 wins.

Deciding the Formula Game

Formula Game

Input: A formula φ.

Problem: Does Player 1 have a winning strategy on φ?

Theorem 10.11: Formula Game is PSpace-complete.

Proof sketch: Formula Game is essentially the same as **True QBF**.

Having a winning strategy means: there is a truth value for X_1, such that, for all truth values of X_2, there is a truth value of $X_3, \ldots$ such that φ becomes true.

If we have a QBF where quantifiers do not alternate, we can add dummy quantifiers and variables that do not change the semantics to get the same alternating form as for the Formula Game. □

Example: The Geography Game

A children's game:

- Two players are taking turns naming cities.
- Each city must start with the last letter of the previous.
- Repetitions are not allowed.
- The first player who cannot name a new city looses.

A mathematicians' game:

- Two players are marking nodes on a directed graph.
- Each node must be a successor of the previous one.
- Repetitions are not allowed.
- The first player who cannot mark a new node looses.

Decision problem **(Generalised) Geography**:
given a graph and start node, does Player 1 have a winning strategy?

Geography is PSpace-complete

Theorem 10.12: Generalised Geography is PSpace-complete.

Proof:

(1) **Geography** $\in$ PSpace:
Give algorithm that runs in polynomial space.
It is not difficult to provide a recursive algorithm similar to the one for **True QBF** or **FOL Model Checking**.

(2) **Geography** is PSpace-hard:
Proof by reduction **Formula Game** $\leq_p$ **Geography**.

□

Geography is PSpace-hard

Let φ with variables $X_1, \ldots, X_\ell$ be an instance of **Formula Game**.
Without loss of generality, we assume:

- ℓ is odd (Player 1 gets the first and last turn)
- φ is in CNF

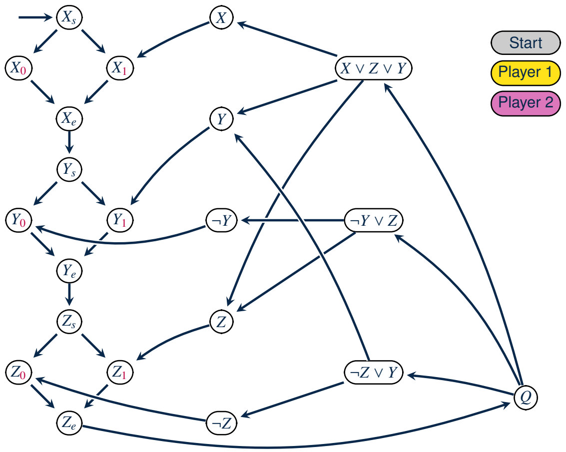
We now build a graph that encodes **Formula Game** in terms of **Geography**

- The left-hand side of the graph is a chain of diamond structures that represent the choices that players have when assigning truth values
- The right-hand side of the graph encodes the structure of φ: Player 2 may choose a clause (trying to find one that is not true under the assignment); Player 1 may choose a literal (trying to find one that is true under the assignment).

(see board or [Sipser, Theorem 8.14]) □

Geography is PSpace-hard: Example

We consider the formula $\exists X.\forall Y.\exists Z.(X \vee Z \vee Y) \wedge (\neg Y \vee Z) \wedge (\neg Z \vee Y)$



Summary and Outlook

True QBF is PSpace-complete

FOL Model Checking and the related problem of SQL query answering are PSpace-complete

Some games are PSpace-complete

What's next?

- Some more remarks on games
- Logarithmic space
- Complements of space classes