Lecture 14: P vs. NP: Ladner’s Theorem

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Review
Review: Hierarchies and Gaps

Hierarchy theorems tell us that more time/space leads to more power:

\[ L \subseteq \text{NL} \subseteq P \subseteq \text{NP} \subseteq \text{PSpace} \subseteq \text{ExpTime} \subseteq \text{NExpTime} \subseteq \text{ExpSpace} \]

Gap theorems tell us that, for non-constructible functions as time/space bounds, arbitrary (constructible or not) boosts in resources may not lead to more power.
Any natural problems in the hierarchy?

To show that complexity classes are different

- we have defined concrete diagonalisation languages that can show the difference (i.e., our argument was constructive),
- but these diagonalisation languages are rather artificial (i.e., not natural).

Are there, e.g., any natural ExpTime problems that are not in P?
Any natural problems in the hierarchy?

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- but these diagonalisation languages are rather artificial (i.e., not natural).

Are there, e.g., any natural ExpTime problems that are not in P?

Yes, many:

**Theorem 14.1:** If $L$ is ExpTime-hard, then $L \not\in P$. 
Any natural problems in the hierarchy?

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- we have defined concrete diagonalisation languages that can show the difference (i.e., our argument was constructive),
- but these diagonalisation languages are rather artificial (i.e., not natural).

Are there, e.g., any natural ExpTime problems that are not in P?

Yes, many:

**Theorem 14.1:** If \( L \) is ExpTime-hard, then \( L \notin P \).

**Proof:** We have shown that there is a language \( D \in \text{ExpTime} \setminus P \). If \( L \) is ExpTime-hard, then there is a polynomial many-one reduction \( D \leq_p L \). Therefore, if \( L \) were in \( P \), then so would \( D \) – contradiction.

Similar results hold for other classes we separated: A problem that is hard for the larger class cannot be included in the smaller.
Ladner’s Theorem
P vs. NP revisited

We have seen that a great variety of difficult problems in NP turn out to be NP-complete. A natural question to ask is whether this apparent dichotomy is a law of nature:

**Hypothesis:** Every problem in NP is either in P or NP-complete.
P vs. NP revisited

We have seen that a great variety of difficult problems in NP turn out to be NP-complete. A natural question to ask is whether this apparent dichotomy is a law of nature:

**Hypothesis:** Every problem in NP is either in P or NP-complete.

In 1975, Richard E. Ladner showed that this is wrong, unless P = NP (in the latter case, uninterestingly, P would turn out to be exactly the set of NP-complete problems)

**Theorem 14.2 (Ladner, 1975):** If P ≠ NP, then there are problems in NP that are neither in P nor NP-complete.

Such problems are called NP-intermediate.
Theorem 14.2 (Ladner, 1975): If \( P \neq NP \), then there are problems in \( NP \) that are neither in \( P \) nor \( NP \)-complete.

In other words, given the following illustrations of the possible relationships between \( P \) and \( NP \):

- **a)** \( P = NP \)
- **b)** \( NP \)-complete
- **c)** \( NP \)-complete

Ladner tells us that the middle cannot be correct.
**Theorem 14.2 (Ladner, 1975):** If $P \neq NP$, then there are problems in $NP$ that are neither in $P$ nor $NP$-complete.

**Proof idea:** We will directly define an $NP$-intermediate language by defining an NTM $\mathcal{K}$ that recognises it.
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**Proof idea:** We will directly define an NP-intermediate language by defining an NTM $K$ that recognises it.

We want to construct $L(K)$ to be:

1. different from all problems in P
2. different from all problems that SAT can be reduced to

**Observation:** This is similar to two concurrent diagonalisation arguments
Proving the Theorem

**Theorem 14.2 (Ladner, 1975):** If \( P \neq NP \), then there are problems in \( NP \) that are neither in \( P \) nor \( NP \)-complete.

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2. different from all problems that \( SAT \) can be reduced to

**Observation:** This is similar to two concurrent diagonalisation arguments

Moreover, the sets we diagonalise against are effectively enumerable:

- There is an effective enumeration \( M_0, M_1, M_2, \ldots \) of all polynomially time-boundend DTMs, each together with a suitable bounding function
  
  For example, enumerate all pairs of TMs and polynomials, and make the enumeration consist of the TMs obtained by artificially restricting the run of a TM with a suitable countdown.

- There is an effective enumeration \( R_0, R_1, R_2, \ldots \) of all polynomial many-one reductions, each together with a suitable bounding function

  This is similar to enumerating polytime TMs; we can restrict to one input alphabet that we also use for \( SAT \)
The problem with diagonalisation

How can we do two diagonalisations at once?

• On each even number $2i$, show that the $i$th polytime TM $M_i$ is not equivalent to $K$:
  
  there is $w$ such that $M_i(w), K(w)$

• For each odd number $2i + 1$, show that the $i$th reduction $R_i$ does not reduce $K$ to $S$ at:

  there is $w$ such that $K(R_i(w)), S(w)$

Nevertheless, there is a problem: How can we flip the output of $S$ at $K$?

• $K$ is required to run in NP
• Computing the actual result of $S$ at is NP-hard
• To show $K(R_i(w)), S(w)$, one might have to show $w < S(w)$, which is presumably not in NP

{the required computation seems too hard!}
The problem with diagonalisation

How can we do two diagonalisations at once? — Simply interleave the enumerations:

- On each even number $2i$, show that the $i$th polytime TM $M_i$ is not equivalent to $K$:
  there is $w$ such that $M_i(w) \neq K(w)$

- For each odd number $2i + 1$, show that the $i$th reduction $R_i$ does not reduce $K$ to $\text{Sat}$:
  there is $w$ such that $K(R_i(w)) \neq \text{Sat}(w)$

Nevertheless, there is a problem: How can we flip the output of $\text{Sat}$?

- $K$ is required to run in NP
- Computing the actual result of $\text{Sat}$ is NP-hard
- To show $K(R_i(w))$, $\text{Sat}(w)$, one might have to show $w < \text{Sat}(w)$, which is presumably not in NP
  {the required computation seems too hard!}
The problem with diagonalisation

How can we do two diagonalisations at once? — Simply interleave the enumerations:

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  there is $w$ such that $M_i(w) \neq K(w)$
- For each odd number $2i + 1$, show that the $i$th reduction $R_i$ does not reduce $K$ to $\text{SAT}$:
  there is $w$ such that $K(R_i(w)) \neq \text{SAT}(w)$

Nevertheless, there is a problem: How can we flip the output of $\text{SAT}$?

- $K$ is required to run in NP
- Computing the actual result of $\text{SAT}$ is NP-hard
- To show $K(R_i(w)) \neq \text{SAT}(w)$, one might have to show $w \notin \text{SAT}$, which is presumably not in NP

$\Rightarrow$ the required computation seems too hard!
Solution: Lazy diagonalisation

Idea: Do not attempt to show too much on small inputs, but wait patiently until inputs are large enough to show the required differences

Main ingredients:
- A very slow growing but polynomially computable function $f$
- A problem in NP that is NP-hard: $\text{SAT}$
- A problem in NP that is not NP-hard:
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Intuition: the NP-intermediate language $L(K)$ is $\text{Sat}$ with "holes punched out of it"
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**Main ingredients:**
- A very slow growing but polynomially computable function $f$
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We will define a TM $\mathcal{K}$ that does the following on input $w$:

1. Compute the value $f(|w|)$
2. If $f(|w|)$ is even: return whether $w \in \text{Sat}$
3. If $f(|w|)$ is odd: return whether $w \in \emptyset$, i.e., reject

**Intuition:** the NP-intermediate language $L(\mathcal{K})$ is $\text{Sat}$ with “holes punched out of it” (namely for all inputs where $f$ is odd)
We can sketch the behaviour of $\mathcal{K}$ as follows:

\[
f(|w|) = \begin{cases} 
0 & \text{for } |w| = 0, \\
1 & \text{for } 1 \leq |w| < 4, \\
2 & \text{for } |w| = 4, \\
3 & \text{for } |w| > 4. 
\end{cases}
\]
What is $f$?

**Reminder:** $K(w)$ is $\text{Sat}(w)$ if $f(|w|)$ is even, and $false$ if $f(|w|)$ is odd.

The key to the proof is the definition of $f$ – this is where the diagonalisation happens.
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The key to the proof is the definition of $f$ – this is where the diagonalisation happens.

**Intuition:** Keep the current value of $f$ until progress has been made in diagonalisation

- Keep an even value $f(|w|) = 2i$ until you can show in polynomial time (in $|w|$) that there is $v$ such that $M_i(v) \neq \mathcal{K}(v)$
- Keep an odd value $f(|w|) = 2i + 1$ until you can show in polynomial time (in $|w|$) that there is $v$ such that $\mathcal{K}(R_i(v)) \neq \text{Sat}(v)$
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If we can do this in NP, it will be enough already:

- If $K$ were equivalent to any $M_i$, then $f$ would eventually become an even constant, and $K$ would solve $\text{Sat}$ on all but finitely many instances
  $\Rightarrow K$ would be NP-hard, and equivalent to a polytime TM $\Rightarrow P = NP$
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- If $K$ would allow $\text{SAT}$ to be reduced to it by some reduction $R_i$, then $f$ would eventually become an odd constant, and $L(K)$ would be a finite language
  $\leadsto K$ would be in P, and $\text{SAT}$ would reduce to it $\leadsto P = NP$

In each case, this contradicts our assumption that $P \neq NP$
What is $f$?

We consider some fixed deterministic TM $S$ with $\mathbb{L}(S) = \mathbb{SAT}$, and an enumeration $v_0, v_1, \ldots$ of all words ordered by length, and lexicographic for words of equal length.

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**Definition:** The value of $f$ on input $w$ with $|w| = n$ is defined recursively
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1. Perform the computations of $f(0), f(1), f(2), \ldots$ in order until $n$ computing steps have been performed in total. Store the largest value $f(\ell) = k$ that could be computed in this time (set $k = 0$ if no value was computed).
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2. Determine if \( f(n) \) should remain \( k \) or increase to \( k + 1 \):
   1. If \( k = 2i \) is even: Iterate over all words \( v \), simulate \( M_i(v) \), \( S(v) \), and (recursively) compute \( f(|v|) \). Terminate this effort after \( n \) steps. If a word is found such that \( \mathcal{K}(v) \neq M_i(v) \), then return \( k + 1 \); else return \( k \).
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Definition: The value of $f$ on input $w$ with $|w| = n$ is defined recursively

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2. Determine if $f(n)$ should remain $k$ or increase to $k + 1$:
   
   2.a) If $k = 2i$ is even: Iterate over all words $v$, simulate $M_i(v), S(v)$, and (recursively) compute $f(|v|)$. Terminate this effort after $n$ steps. If a word is found such that $K(v) \neq M_i(v)$, then return $k + 1$; else return $k$.

   2.b) If $k = 2i + 1$ is odd: Iterate over all words $v$, simulate $R_i(v)$ (this produces a word), $S(v), S(R_i(v))$, and (recursively) compute $f(|R_i(v)|)$. Terminate this effort after $n$ steps. If a word is found such that $K(R_i(v)) \neq S(v)$, then return $k + 1$; else return $k$. 

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Is $f$ well-defined?

Our definition of $f$ computes values for $f$ recursively. Is this ok?

- Yes, the computation that needs to be done for each $f(n)$ is fully defined
- All the simulated TMs are known or computable
- Since computation is time-limited to the input value $n$, there is no danger of endless recursion
- For example, $f(0) = 0$: nothing will be achieved in 0 steps
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Indeed, $f$ grows very slowly!

- A large input $n$ might be needed to find the next counterexample word $v$ needed in diagonalisation
- Even if such $v$ was found in $n$ steps (making progress from $n$ to $n + 1$), it will be only much later that $f(n)$ can be computed in step (1) and $f$ will even start to look for a way of getting to $n + 2$.
- In fact, already the requirement to recompute all previous values of $f$ before considering an increase ensures that $f \in O(\log \log n)$. 

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Proof: Let $\mathcal{K}$ be defined as before.
Concluding the Proof

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**Proof:** Let $\mathcal{K}$ be defined as before.

$\mathcal{K}$ runs in nondeterministic polynomial time:

- The computation of $f$ is in polynomial deterministic time (since it is artificially bounded to a short time)
- The computation of $\text{SAT}$ for the cases where $f(|w|)$ is even is possible in NP
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$L(\mathcal{K})$ is not in $P$: As argued before: if it were in $P$, it would be equivalent to some polytime TM $M_i$, and $f$ would eventually be constant at $2i$, making $\mathcal{K}$ equivalent to $\text{SAT}$ (up to finite variations), which contradicts $P \neq NP$. 
Concluding the Proof

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$L(\mathcal{K})$ is not in NP-hard: As argued before: if it were NP-hard, there would be a polynomial many-one reduction $R_i$ from $\text{SAT}$, and $f$ would eventually be constant at $2i + 1$, making $\mathcal{K}$ equivalent to $\emptyset$ (up to finite variations), which contradicts $P \neq NP$. □
Discussion: Proof of Ladner’s Theorem

**Note 1:** It is interesting to meditate on the following facts:

- We have defined a rather “busy” computation of $f$ that checks that diagonalisation (over two different sets) must happen
- This definition of computation is essential to prove the result
- Nevertheless, diagonalisation remained “internal”: from the outside, $K$ is just a TM that sometimes solves $\text{SAT}$ (for a long range of inputs), and at other times just rejects every input (again for very long ranges of inputs)

**Note 2:** The constructed language is very artificial

- It is very “non-uniform” in terms of how hard it is, alternating between long stretches of NP-hardness and long stretches of triviality

**Note 3:** Are there any natural problems that are known to be NP-intermediate?

- No: finding one would prove $P \neq \text{NP}$
- Candidate problems (link) include, e.g., $\text{Graph Isomorphism}$ and $\text{Factoring}$
- Beware: the latter is not about deciding if a number is prime, but about checking something specific about its factors, e.g., whether the largest factor contains at least one 7 when written in decimal
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Ladner’s theorem tells us that, in the intuitive case that $P \neq NP$, there must be (counterintuitively?) many problems in NP that are neither polynomially solvable nor NP-complete.

The proof is based on a technique of lazy diagonalisation.

**What’s next?**

- Generalising Ladner’s Theorem
- Computing with oracles (reprise)
- The limits of diagonalisation, proved by diagonalisation