A Note on Locality

- $\simeq$ does not change if we switch from single symbols to sequences local $\mapsto$ global
- $\simeq_\omega$, however, does
- $P \cong_0 Q$ for all processes P and Q
- $P \cong_{i+1} Q$ if, for all $w \in \mathsf{Act}^\star$,
  1. $P \xrightarrow{w} P'$ implies $Q \xrightarrow{w} Q'$ and $P' \cong_i Q'$;
  2. $Q \xrightarrow{w} Q'$ implies $P \xrightarrow{w} P'$ and $P' \cong_i Q'$;
- the limit is $\cong_\omega := \bigcap_{i \geq 0} \cong_i$ and coincides with $\simeq$ for image-finite processes

Recall our Counters?

$$C_0 \xrightarrow{i} C_1$$

$$C_n \xrightarrow{i} C_{n+1}$$

$$C_n \xrightarrow{d} C_{n-1}$$

$$C \xrightarrow{i} C \mid d.\mathbf{0}$$

How to prove $C_0 \simeq C$?

$$\mathcal{R} = \{(C_n, C \mid \Pi_{i=0}^{n} d.\mathbf{0}) \mid n \in \mathbb{N}\}$$

Bisimulations up-to $\simeq$

Definition 31 A process relation $\mathcal{R}$ is a *bisimulation up-to* $\simeq$ if, whenever $p \mathrel{\mathcal{R}} q$, for all $\mu \in \mathsf{Act}$, we have

1. $p \xrightarrow{\mu} p'$ implies a q' such that $q \xrightarrow{\mu} q'$ and $p' \simeq \mathcal{R} \simeq q'$;
2. $q \xrightarrow{\mu} q'$ implies a p' such that $p \xrightarrow{\mu} p'$ and $p' \simeq \mathcal{R} \simeq q'$.

$p' \simeq \mathcal{R} \simeq q'$ iff there are p'', q'' such that $p' \simeq p''$, $p'' \mathrel{\mathcal{R}} q''$, and $q'' \simeq q'$.

Lemma 32 If $\mathcal{R}$ is a bisimulation up-to $\simeq$, then $\simeq \mathcal{R} \simeq$ is a bisimulation.

Recall our Counters 2.0?

$$C_0 \xrightarrow{i} \boldsymbol{\nu}\ell_1\,(C_1 \mid \ell_1.C_0)$$

$$C_1 \xrightarrow{i} \boldsymbol{\nu}\ell_2\,(C_2 \mid \ell_2.C_1)$$

$$C_2 \xrightarrow{i} \boldsymbol{\nu}\ell_1\,(C_1 \mid \ell_1.C_2)$$

The *lhs* in every process context takes care of the *next* counter value, being either *odd* (C_1) or *even* (C_2). The *rhs* waits for the decrement operation to have taken place to *unguard* the counter's original value. Consequently,

$$C_1 \xrightarrow{d} \overline{\ell_1}.\mathbf{0}$$

$$C_2 \xrightarrow{d} \overline{\ell_2}.\mathbf{0}$$

Weak Transitions and Bisimilarity

Definition 33 For $\mathcal{T} = (\mathsf{Pr}, \mathsf{Act}, \longrightarrow)$, define

1. $\Longrightarrow$ as the reflexive and transitive closure of $\xrightarrow{\tau}$;
2. for all $\mu \in \mathsf{Act}$, $p \overset{\mu}{\Longrightarrow} p'$ if there are processes $p_1, p_2 \in \mathsf{Pr}$ such that $p \Longrightarrow p_1 \xrightarrow{\mu} p_2 \Longrightarrow p'$.

Definition 34 A process relation $\mathcal{R}$ is a *weak bisimulation* if for all $p \mathrel{\mathcal{R}} q$,

1. for all $\ell \in \mathsf{Act} \setminus \{\tau\}$, $p \overset{\ell}{\Longrightarrow} p'$ implies a q' such that $q \overset{\ell}{\Longrightarrow} q'$ and $p' \mathrel{\mathcal{R}} q'$;
2. $p \overset{\tau}{\Longrightarrow} p'$ implies a q' such that $q \Longrightarrow q'$ and $p' \mathrel{\mathcal{R}} q'$;
3. the coverse on steps of q.

If a weak bisimulation $\mathcal{R}$ with $p \mathrel{\mathcal{R}} q$ exists, we say that p and q are *weakly bisimilar*, written as $p \approxeq q$.

Axiomatizing $\simeq$ for CCS_{fin}

Decidability implies an algebraic characterization of bisimilarity in the shape of *axiomatizations*.

Axiomatizations are axioms that, incorporating equational reasoning, are sufficient to decide the equivalence.

1. use reflexivity, symmetry, and transitivity
2. use substitutivity by equivalent subterms

The System $\mathcal{SB}$

S1	$P + \mathbf{0} = P$
S2	$P + Q = Q + P$
S3	$P + (Q + R) = (P + Q) + R$
S4	$P + P = P$
R1	$\boldsymbol{\nu} a\, \mathbf{0} = \mathbf{0}$
R2 if $\mu \in \{a, \overline{a}\}$	$\boldsymbol{\nu} a\, \mu.P = \mathbf{0}$
R3 if $\mu \notin \{a, \overline{a}\}$	$\boldsymbol{\nu} a\, \mu.P = \mu.\boldsymbol{\nu} a\, P$
R4	$\boldsymbol{\nu} a\, (P + Q) = \boldsymbol{\nu} a\, P + \boldsymbol{\nu} a\, Q$
E	

If $P \stackrel{\text{def}}{=} \sum_{0 \leq i \leq m} \mu_i.P_i$ and $P \stackrel{\text{def}}{=} \sum_{0 \leq j \leq n} \mu_j.P_j$, infer

$$P \mid P' = \sum_{0 \leq i \leq m} \mu_i.(P_i \mid P') + \sum_{0 \leq j \leq n} \mu_j.(P \mid P_j) + \sum_{\mu_i = \overline{\mu_j}} \tau.(P_i \mid P_j)$$

.

Detour: The Bisimulation Game

Let $\mathcal{T} = (\mathsf{Pr}, \mathsf{Act}, \longrightarrow)$ be an LTS. We call $\mathcal{B} := \mathsf{Pr} \times \mathsf{Pr}$ the game board of the *bisimulation game*, being a 2-player game between R (the *refuter*) and V (the *verifier*), played pairs $(P, Q) \in \mathcal{B}$.

A *play for* (P_0, Q_0) is a finite or infinite sequence of pairs

$$(P_0, Q_0), (P_1, Q_1), \ldots, (P_i, Q_i), \ldots$$

in which R tries to show that P_0 and Q_0 are not equal while V tries to show the opposite.

When the play has reached a pair (P_i, Q_i),

1. R challenges V by choosing any transition $P_i \xrightarrow{\mu} P'$ or $Q_i \xrightarrow{\mu} Q'$;
2. V has to find a matching transition, either $Q_i \xrightarrow{\mu} Q'$ or $P_i \xrightarrow{\mu} P'$.

The play continues with the $(i+1)^{\text{th}}$ pair (P', Q').

If, at some point, V is unable to answer, R *wins*. If the situation never occurs, V *wins*.

Detour: (Winning) Strategies

A strategy for R specifies, for all possible plays

$$(P_0, Q_0), (P_1, Q_1), \ldots, (P_i, Q_i)$$

which transition to choose as the next challenge.

A strategy for V specifies, for all possible plays

$$(P_0, Q_0), (P_1, Q_1), \ldots, (P_i, Q_i)$$

and challenges (by R), which transition to choose as the next answer.

A strategy (for R or V) is called a *winning strategy* if it leads to a win in all possible plays.[1]

---

[1]The use of the term *possible* is very important here because it also entails the use of the strategy in question. Therefore, a play is only considered possible, if the pairs adhere to the rules of the game and the chosen strategy.

Detour: (Winning) Strategies

Theorem 35 $P \simeq Q$ if and only if V has a winning strategy for (P, Q).

Theorem 36 $P \not\simeq Q$ if and only if R has a winning strategy for (P, Q).

Reduction from MCVP C and $i \in \{0,1\}^n$

The LTS we consider is the smallest LTS $\mathcal{T}(C,i) = (\mathcal{Q}, \{\ell, r, a, 0\}, \longrightarrow)$ such that

1. $P_{\text{end}} \in \mathcal{Q}$;
2. $P_v, Q_v \in \mathcal{Q}$ for every node v of C, and additionally,
3. $P'_v, Q^\ell_v, Q^r_v \in \mathcal{Q}$ for every node v of C labeled with $\vee$.

The transition relation $\longrightarrow$ contains the following transitions:

1. $P_v \xrightarrow{\ell} P_{v_1}, P_v \xrightarrow{r} P_{v_2}, Q_v \xrightarrow{\ell} Q_{v_1}, Q_v \xrightarrow{r} Q_{v_2}$ for every node v of C with label $\wedge$;
2. $P_v \xrightarrow{a} P'_v, P_v \xrightarrow{a} Q^\ell_v, P_v \xrightarrow{a} Q^r_v$, and
   $P'_v \xrightarrow{\ell} P_{v_1}, P'_v \xrightarrow{r} P_{v_2}$, and
   $Q_v \xrightarrow{a} Q^\ell_v, Q_v \xrightarrow{a} Q^r_v$, and
   $Q^\ell_v \xrightarrow{\ell} Q_{v_1}, Q^\ell_v \xrightarrow{r} P_{v_2}, Q^r_v \xrightarrow{r} Q_{v_2}, Q^r_v \xrightarrow{\ell} P_{v_1}$ for every node v of C with label $\vee$;
3. $P_v \xrightarrow{0} P_{\text{end}}$ for every input node v of C with assigned value 0.

The construction of $\mathcal{T}(C,i)$ can clearly be computed in log-*space*.