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Negation: Model Theory

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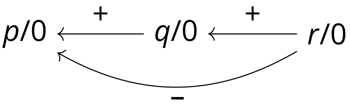
Previously …

- For every normal logic program P, its **completion** $comp(P)$ replaces the logical implications of clauses by equivalences (to disjunctions of bodies).
- SLDNF resolution w.r.t. P is **sound** for entailment w.r.t. $comp(P)$.
- SLDNF resolution is only **complete** (for entailment w.r.t. $comp(P)$) for certain combinations of classes of programs, queries, and selection rules.
- For a normal program P, its **dependency graph** D_P explicitly shows positive and negative dependencies between predicate symbols.
- A normal program P is **stratified** iff D_P has no cycle with a negative edge.

$$
P: \quad \begin{aligned} p &\leftarrow q, \sim r \\ q &\leftarrow r \end{aligned}
$$



Completion of P:

$$
\begin{aligned}
p &\leftrightarrow (q \wedge \neg r) \\
q &\leftrightarrow r \\
r &\leftrightarrow \mathit{false}
\end{aligned}
$$

Overview

Consequence Operator for Normal Programs

Standard Models

Perfect Models and Local Stratification

Well-Supported Models

Consequence Operator for Normal Programs

Consequence Operator for Normal Programs

Definition

Let P be a normal logic program and I be a Herbrand interpretation. Then

$$T_P(I) := \{H \mid H \leftarrow \vec{B} \in ground(P),\ I \Vdash \vec{B}\}$$

In case P is a definite program, we know that

- T_P is monotonic,
- T_P is continuous,
- T_P has the least fixpoint $T_P \uparrow \omega$,
- $T_P \uparrow \omega = \mathcal{M}(P) = \bigcap \{I \mid I$ is a Herbrand model of $P\}$.

For normal programs, all of these properties are lost.

T_P-Characterisation for Normal LPs (1)

Lemma 4.3

Let P be a normal logic program and I be a Herbrand interpretation. Then

$$I \Vdash P \quad \text{iff} \quad T_P(I) \subseteq I$$

Proof.

$$
\begin{aligned}
& I \Vdash P \\
\text{iff} \quad & \text{for every } H \leftarrow \vec{B} \in \mathit{ground}(P)\text{:} \quad I \Vdash \vec{B} \text{ implies } I \Vdash H \\
\text{iff} \quad & \text{for every } H \leftarrow \vec{B} \in \mathit{ground}(P)\text{:} \quad I \Vdash \vec{B} \text{ implies } H \in I \\
\text{iff} \quad & \text{for every ground atom } H\text{:} \\
& \quad H \leftarrow \vec{B} \in \mathit{ground}(P) \text{ and } I \Vdash \vec{B} \text{ implies } H \in I \\
\text{iff} \quad & \text{for every ground atom } H\text{:} \quad H \in T_P(I) \text{ implies } H \in I \\
\text{iff} \quad & T_P(I) \subseteq I
\end{aligned}
$$

□

T_P-Characterisation for Normal LPs (2)

Definition

Let F and Π be ranked alphabets of function symbols and predicate symbols, respectively, let $= \notin \Pi$ be a binary predicate symbol (equality), and let I be a Herbrand interpretation for F and Π. Then

$$I_= \;:=\; I \cup \{=(t,t) \mid t \in HU_F\}$$

is called a **standardised** Herbrand interpretation for F and $\Pi \cup \{=\}$.

Lemma 4.4

Let P be a normal logic program and I a Herbrand interpretation. Then

$$I_= \Vdash comp(P) \quad \text{iff} \quad T_P(I) = I$$

$\rightsquigarrow$ The T_P operator for normal LPs characterises the completion semantics.

T_P-Characterisation for Normal LPs (3)

Proof Idea of Lemma 4.4:

	$I_= \Vdash comp(P)$
iff	(since $I_=$ is a model for standard axioms of equality and inequality) for every ground atom H: $I \Vdash \left(H \leftrightarrow \bigvee_{(H \leftarrow \vec{B}) \in ground(P)} \vec{B}\right)$
iff	for every ground atom H: $H \in I$ iff $I \Vdash \bigvee_{(H \leftarrow \vec{B}) \in ground(P)} \vec{B}$
iff	for every ground atom H: $H \in I$ iff $I \Vdash \vec{B}$ for some $H \leftarrow \vec{B} \in ground(P)$
iff	for every ground atom H: $H \in I$ iff $H \in T_P(I)$
iff	$T_P(I) = I$

$\square$

$\rightsquigarrow$ Is $comp(P)$ the "intended" declarative semantics of P?

Completion May Be Inadequate

Consider the following normal logic program P:

$$\begin{aligned} &\mathit{ill} \leftarrow {\sim}\mathit{ill}, \mathit{infection} \\ &\mathit{infection} \leftarrow \end{aligned}$$

Its completion $\mathit{comp}(P) \supseteq \{\mathit{ill} \leftrightarrow (\neg\mathit{ill} \wedge \mathit{infection}), \quad \mathit{infection} \leftrightarrow \mathit{true}\}$ is unsatisfiable (it has no models).

Hence, $\mathit{comp}(P) \models \mathit{healthy}$.

But $I = \{\mathit{infection}, \mathit{ill}\}$ is the only Herbrand model of P (taken as a theory):

$$P \equiv \{\mathit{ill} \leftarrow (\neg\mathit{ill} \wedge \mathit{infection}),\ \mathit{infection}\} \equiv \{\mathit{ill} \vee \neg\neg\mathit{ill} \vee \neg\mathit{infection},\ \mathit{infection}\}$$

Hence, $P \not\models \mathit{healthy}$.

Non-Intended Minimal Herbrand Models

For normal LPs, a unique least Herbrand model is not guaranteed to exist.

Can we at least settle for minimal Herbrand models?

$$P_1 : \quad p \leftarrow {\sim}q$$

P_1 has three Herbrand models:

$$M_1 = \{p\}, M_2 = \{q\}, \text{ and } M_3 = \{p, q\}.$$

P_1 has no least, but two minimal Herbrand models: M_1 and M_2

However: M_1, and not M_2, is the “intended” model of P_1.

Supported Herbrand Interpretations

Definition

A Herbrand interpretation I of P is **supported**
$:\Longleftrightarrow$ for every $H \in I$ there exists some $H \leftarrow \vec{B} \in ground(P)$ such that $I \Vdash \vec{B}$.

If additionally $I \Vdash P$, we say that I is a **supported model** of P.

(Intuitively: $\vec{B}$ is an explanation for H.)

Example

- M_1 is a supported model of P_1. (Literal $\sim q$ is a support for p.)
- M_2 is no supported model of P_1. (Atom $q \in M_2$ has no support.)
- Note (cf. Lemma 4.3) that $T_{P_1}(M_2) = \emptyset \subsetneq M_2$, but in contrast $T_{P_1}(M_1) = M_1$.
- The definite (therefore normal) program $\{p \leftarrow q, \quad q \leftarrow p\}$ has two supported models: $\emptyset$ and $\{p, q\}$. In the second supported model, p is an explanation for q and vice versa. Thus "support" can be cyclic.

T_P-Characterisation for Normal LPs (4)

Lemma 6.2

Let P be a normal program and I be a Herbrand interpretation. Then

$$I \text{ is a supported model of } P \quad \text{iff} \quad T_P(I) = I$$

Proof Idea.

$$\begin{array}{ll}
 & I \Vdash P \text{ and } I \text{ is supported} \\
\text{iff} & \text{for every } (H \leftarrow \vec{B}) \in \mathit{ground}(P)\text{:} \quad I \Vdash \vec{B} \text{ implies } I \Vdash H \\
 & \quad \text{and for every } H \in I\text{: } I \Vdash \bigvee_{(H \leftarrow \vec{B}) \in \mathit{ground}(P)} \vec{B} \\
\text{iff} & \text{for every ground atom } H\text{: } I \Vdash \left(H \leftarrow \bigvee_{(H \leftarrow \vec{B}) \in \mathit{ground}(P)} \vec{B}\right) \\
 & \quad \text{and } I \Vdash \left(H \rightarrow \bigvee_{(H \leftarrow \vec{B}) \in \mathit{ground}(P)} \vec{B}\right) \\
\text{iff} & \text{for every ground atom } H\text{: } I \Vdash \left(H \leftrightarrow \bigvee_{(H \leftarrow \vec{B}) \in \mathit{ground}(P)} \vec{B}\right) \\
\text{iff} & I_{=} \text{ model for } \mathit{comp}(P) \\
\text{iff} & \text{(Lemma 4.4)} \quad T_P(I) = I
\end{array}$$

□

Standard Models

Non-Intended Supported Models

$$P_2: \quad \begin{array}{lcl} p & \leftarrow & \sim q \\ q & \leftarrow & q \end{array}$$

P_2 has three Herbrand models:

$$M_1 = \{p\}, M_2 = \{q\}, \text{and } M_3 = \{p, q\}$$

P_2 has two supported Herbrand models:

$$M_1 \text{ and } M_2$$

However: M_1, and not M_2, is the “intended” model of P_2.

M_1 will be called the standard model of P_2 (cf. slide 19).

Stratifications

Definition

Let P be a normal program with dependency graph D_P.

- A predicate symbol p is **defined** in P
  $:\Longleftrightarrow \quad P$ contains a clause $p(t_1, \ldots, t_n) \leftarrow \vec{B}$.
- $P_1 \cup \ldots \cup P_n = P$ is a **stratification** of P
  $:\Longleftrightarrow$
  1. $P_i \neq \emptyset$ for every $i \in [1, n]$
  2. $P_i \cap P_j = \emptyset$ for every $i, j \in [1, n]$ with $i \neq j$

     } a partition of P
  3. for every p defined in P_i and edge $q \xrightarrow{+} p$ in D_P: q is not defined in $\bigcup_{j=i+1}^{n} P_j$
  4. for every p defined in P_i and edge $q \xrightarrow{-} p$ in D_P: q is not defined in $\bigcup_{j=i}^{n} P_j$

Lemma 6.5

A normal program P is stratified iff there exists a stratification of P.

Note: A stratified program may have different stratifications.

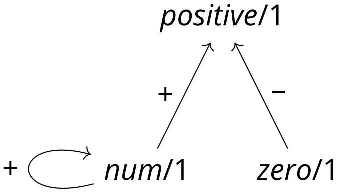
Example (1)

The normal logic program P is the following:

$$
\begin{aligned}
zero(0) &\leftarrow \\
positive(x) &\leftarrow num(x),\ \sim zero(x) \\
num(0) &\leftarrow \\
num(s(x)) &\leftarrow num(x)
\end{aligned}
$$

$P_1 \cup P_2 \cup P_3$ is a stratification of P, where

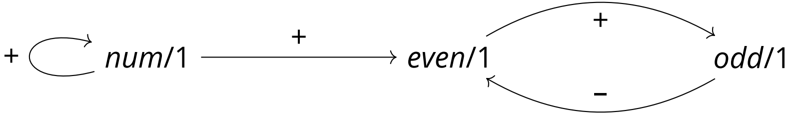
$$
\begin{aligned}
P_3 &= \{positive(x) \leftarrow num(x),\ \sim zero(x)\} \\
P_2 &= \{zero(0) \leftarrow\} \\
P_1 &= \{num(0) \leftarrow,\quad num(s(x)) \leftarrow num(x)\}
\end{aligned}
$$



Example (2)

$$
\begin{aligned}
num(0) &\leftarrow \\
num(s(x)) &\leftarrow num(x) \\
even(0) &\leftarrow \\
even(x) &\leftarrow {\sim}odd(x),\ num(x) \\
odd(s(x)) &\leftarrow even(x)
\end{aligned}
$$

***P* admits no stratification.**



Quiz: Stratifications

Recall: A normal logic program P is *stratified* iff its dependency graph D_P has no cycle involving a negative edge.

Quiz

Consider the normal logic program P where x is the only variable: …

Standard Models (Stratified Programs)

Definition

Let I be an Herbrand interpretation, Π be a set of predicate symbols.

$$I \mid \Pi \;:=\; I \cap \{p(t_1, \ldots, t_n) \mid p \in \Pi,\ t_1, \ldots, t_n \text{ ground terms}\}$$

Let $P_1 \cup \ldots \cup P_n$ be a stratification of the normal program P. Define:

$$\begin{aligned}
M_1 &:= \text{least Herbrand model of } P_1 \text{ such that} \\
&\qquad M_1 \mid \{p \mid p \text{ not defined in } P\text{not defined in } P_1 \cup \ldots \cup P_n\} = \emptyset \\
M_2 &:= \text{least Herbrand model of } P_2 \text{ such that} \\
&\qquad M_2 \mid \{p \mid p \text{ defined nowhere or in } P_1\text{not defined in } P_2 \cup \ldots \cup P_n\} = M_1 \\
&\vdots \\
M_n &:= \text{least Herbrand model of } P_n \text{ such that} \\
&\qquad M_n \mid \{p \mid p \text{ defined nowhere or in } P_1 \cup \ldots \cup P_{n-1}\text{not defined in } P_n\} = M_{n-1}
\end{aligned}$$

We call $M_P = M_n$ the **standard model** of P.

Example (1)

Let $P_1 \cup P_2 \cup P_3$ with

$$\begin{aligned} P_1 &= \{num(0) \leftarrow, \quad num(s(x)) \leftarrow num(x)\} \\ P_2 &= \{zero(0) \leftarrow\} \\ P_3 &= \{positive(x) \leftarrow num(x), \sim zero(x)\} \end{aligned}$$

be a stratification of P. Then:

$$\begin{aligned} M_1 &= \{num(t) \mid t \in HU_{\{s,0\}}\} & &= \{num(0), num(s(0)), \ldots\} \\ M_2 &= \{num(t) \mid t \in HU_{\{s,0\}}\} \cup \{zero(0)\} & &= \{zero(0), num(0), num(s(0)), \ldots\} \\ M_3 &= \{num(t) \mid t \in HU_{\{s,0\}}\} \cup \{zero(0)\} & &= \{zero(0), num(0), num(s(0)), \ldots\} \\ &\quad \cup \{positive(t) \mid t \in HU_{\{s,0\}} \setminus \{0\}\} & &\quad \cup \{positive(s(0)), positive(s(s(0))), \ldots\} \end{aligned}$$

Hence $M_P = M_3$ is the standard model of P.

Properties of Standard Models

Theorem 6.7

Consider a stratified program P. Then:

- M_P does not depend on the chosen stratification of P,
- M_P is a minimal model of P,
- M_P is a supported model of P.

Corollary

For a stratified program P, $comp(P)$ admits a Herbrand model.

Perfect Models and Local Stratification

Stratification may be too demanding

Consider the first-order program P_1 over $\Pi_1 = \{even/1\}$ and $F_1 = \{s/1, 0/0\}$:

- P_1 is not stratified, since $even/1$ depends negatively on itself.

$$
\begin{aligned}
even(0) &\leftarrow \\
even(s(x)) &\leftarrow {\sim} even(x)
\end{aligned}
$$

Observation

P_1 has a clear intended model: $\{even(0), even(s(s(0))), even(s(s(s(s(0))))), \ldots\}$.

Consider, in contrast, the propositional program P_0 over $\Pi_0 = HB_{\{even\},\{s,0\}} = \{even(0)/0, even(s(0))/0, \ldots\}$ and $F_0 = \emptyset$:

- P_0 is stratified.
- The standard model of P_0 is the intended model of P_1.

$$
\begin{aligned}
even(0) &\leftarrow \\
even(s(0)) &\leftarrow {\sim} even(0) \\
even(s(s(0))) &\leftarrow {\sim} even(s(0)) \\
&\vdots
\end{aligned}
$$

Perfect Models

Definition

Let P be a normal program over Π and F, and let $\prec$ be a well-founded order on $HB_{\Pi,F}$. Further, let M and N be Herbrand interpretations of P.

- N is **preferable** to M (written $N \lhd M$)
  $:\iff$ for every $B \in N \setminus M$ there exists an $A \in M \setminus N$ such that $A \prec B$.
- A Herbrand model M of P is **perfect** (w.r.t. $\prec$)
  $:\iff$ there is no Herbrand model of P that is preferable to M.

Well-founded orders admit no infinite descending chains $\ldots \prec c_2 \prec c_1 \prec c_0$.

Example

$$\begin{array}{lcl} p & \leftarrow & \sim q \\ q & \leftarrow & q \end{array}$$

For the well-founded order $q \prec p$, we obtain $\{p\} \lhd \{q\}$.

The Standard Model is Perfect

Lemma 6.9

Let P be a normal program and $\prec$ be a well-founded order on $HB_{\Pi,F}$.

- If $N \subsetneq M$ then $N \lhd M$.
- Every perfect model of P is minimal.
- The relation $\lhd$ is a partial order on Herbrand interpretations.

Theorem 6.10

Let P be a stratified normal program over Π and F and for $A, B \in HB_{\Pi,F}$ define $A \prec B$ $:\Longleftrightarrow$ the predicate symbol of B depends negatively on the predicate symbol of A.

Then M_P is a unique perfect model of P (w.r.t. $\prec$).

The standard model M_P is thus the $\lhd$-least Herbrand model of P.

But how to come up with an order $\prec$ for non-stratified programs?

Local Stratification

Definition

Let P be a normal program over Π and F.

- A **local stratification** for P is a function *strat* from $HB_{\Pi,F}$ to the countable ordinals.
- For a given local stratification *strat* and $A \in HB_{\Pi,F}$, we define $\mathit{strat}(\sim A) := \mathit{strat}(A) + 1$.
- A clause $c \in P$ is **locally stratified w.r.t.** *strat*
  $:\Longleftrightarrow$ for every $A \leftarrow \vec{K}, L, \vec{M} \in \mathit{ground}(c)$, we have $\mathit{strat}(A) \geq \mathit{strat}(L)$.
- P is **locally stratified w.r.t.** *strat*
  $:\Longleftrightarrow$ all $c \in P$ are locally stratified w.r.t. *strat*.
- P is **locally stratified**
  $:\Longleftrightarrow$ it is locally stratified w.r.t. to some local stratification.

$\rightsquigarrow$ A first-order program is locally stratified iff its ground version is stratified.

Locally Stratified Programs & Perfect Models

Lemma 6.12

Every stratified program is locally stratified.

Example

The program

$$even(0) \leftarrow$$
$$even(s(x)) \leftarrow \sim even(x)$$

is locally stratified (via $\{even(s^n(0)) \mapsto n\}$), but not stratified.

Theorem 6.13

Let P be a normal logic program (over Π and F) that is locally stratified (w.r.t. $strat$), and for $A, B \in HB_{\Pi,F}$ define $A \prec B \;:\Longleftrightarrow\; strat(A) < strat(B)$.

Then P has a unique perfect model (w.r.t. $\prec$).

Well-Supported Models

From Supported to Well-Supported Models

$$p \leftarrow q$$
$$q \leftarrow p$$

has two supported models, $\emptyset$ and $\{p, q\}$.

Only the minimal supported model is intended.

$$p \leftarrow \sim q$$
$$q \leftarrow q$$

has two minimal supported models, $\{p\}$ and $\{q\}$.

Only $\{p\}$ is intended: the support of q ("q because q") is unfounded.

$$p \leftarrow \sim q$$
$$q \leftarrow \sim p$$

has two minimal supported models, $\{p\}$ and $\{q\}$.

- The program is not (locally) stratified.
- The situation is symmetric, so why should we prefer one model over the other?

Well-Supported Models

Definition

Let P be a normal logic program over vocabulary Π, F.

A Herbrand interpretation $I \subseteq HB_{\Pi,F}$ is **well-supported**

$:\Longleftrightarrow$

there is a well-founded order $\prec$ on $HB_{\Pi,F}$ such that:

for each $A \in I$ there is a clause $A \leftarrow \vec{B} \in ground(P)$ with:

- $I \Vdash \vec{B}$
- for every positive atom $C \in \vec{B}$, we have $C \prec A$.

If additionally $I \Vdash P$, then I is a **well-supported model** of P.

Intuitively: Well-supported models disallow circular justifications.

Theorem 6.20

Any locally stratified normal logic program P has a unique well-supported model that coincides with its perfect model.

Well-Supported Models: Examples

$p \leftarrow \sim q$ $q \leftarrow q$	has $\{p\}$ as only well-supported model.
$p \leftarrow \sim q$ $q \leftarrow \sim p$	has two well-supported models, $\{p\}$ and $\{q\}$.
$p \leftarrow q$ $p \leftarrow \sim q$ $q \leftarrow p$ $q \leftarrow \sim p$	has no well-supported model.

Preview: Well-supported models are also known as stable models.

Conclusion

Summary

- The immediate consequence operator T_P for a normal logic program P characterizes the **supported models** of P ($=$ the models of $comp(P)$).
- The **stratification** of a program P partitions the program in layers (strata) such that predicates in one layer only negatively/positively depend on predicates in strictly lower/lower or equal layers.
- Every **stratified** logic program P has an intended **standard model** M_P.
- A program is **locally stratified** iff its ground instantiation is stratified.
- Locally stratified programs allow for a unique **perfect model**.
- A normal program P may have zero or more **well-supported models**.

Suggested action points:

- Prove Lemma 6.5; show that every well-supported model is supported.