Concept lattice orbifolds – first steps

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Abstract. Concept lattices with symmetries may be simplified by "folding" them along the orbits of their automorphism group. The resulting diagram is often more intuitive than the full lattice diagram, but well defined annotations are required to make the folded diagram as informative as the original one. The folding procedure can be extended to formal contexts.
A typical situation where such lattice foldings are useful is when hierarchies of structures are considered "up to isomorphisms".

1 Introduction

Much effort has been made to develop techniques for handling large and complex concept lattices. For lattices built from real-word data, methods allowing for aspects and views of the lattice are often a good choice. In more mathematical situations, a different strategy is promising: using symmetry. The lattice of, say, all quasi-orders on a fixed base set, or the lattice of all subgroups of a given group, are structures with rich automorphism groups, and it is advisable to use these for simplification.

The basic elements of theory for such investigations do already exist. They were invented in Monika Zickwolff's thesis of 1991 [8], and were tailored for applications in *rule exploration*, a generalisation of attribute exploration to first order predicate logic. First algorithmic results were also obtained at that time [5]. Since then, little progress has been made, presumably because the methods tend to be difficult.

In recent years we have often met situations in which the application of symmetry techniques would have been appropriate. We see a growing demand for a solid and intuitive theory. In the present paper, we give an introduction to the basic ideas, mainly by means of an example. Most of the results presented here are already contained in Zickwolff's work, but in a very condensed form. Our aim is to make them better accessible to the FCA community.

Any lattice or ordered set with automorphisms (in fact, any relational structure) can be "folded" in such a manner that the orbits of the group become the elements of a new structure. However, in order not to loose information, this "orbifold" needs to be carefully annotated, so that the original structure can be reconstructed.

Formal contexts can be folded as well, and it is possible to compute concept lattice orbifolds from context orbifolds. Such computations require a combination of lattice and group algorithms and are not easy to handle.

The present paper concentrates on folding orders and lattices. In Section 5 we sketch a first example of a context orbifold. The details are to be treated in a subsequent paper.

2 Group annotated ordered sets

Definition 1 (Group annotated ordered set): Let $\underline{P} := (P, \leq)$ be an ordered set and let $\underline{G} := (G, \circ)$ be some group. A mapping

$$\lambda : P \times P \to \mathcal{P}(G)$$

is called a $\underline{G}$**-annotation** of $\underline{P}$ iff

1. $\lambda(a,b) \neq \emptyset$ if and only if $a \leq b$ in P,
2. each set $\lambda(a,a)$, $a \in P$, is a subgroup G_a of $\underline{G}$, and
3. $\lambda(a,b) \circ \lambda(b,c) \subseteq \lambda(a,c)$ for all $a \leq b \leq c$ in P.

$(P, \leq, \lambda)$ is then called a $\underline{G}$**-annotated ordered set**. ◊

The following example, though small, seems complicated at first. In the sequel we shall introduce techniques easing readability. Moreover, it will be shown where the example comes from.

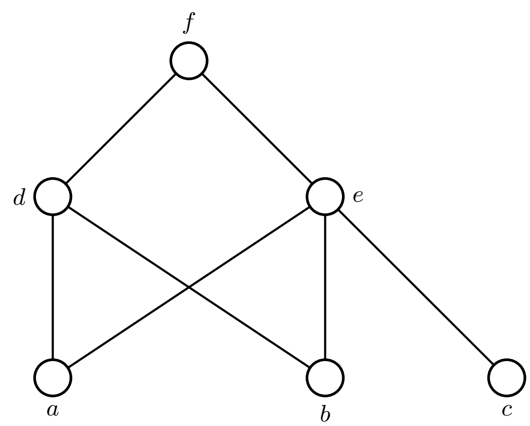


Fig. 1. A small ordered set

Example 1 Let $\underline{P} := (\{a,b,c,d,e,f\}, \leq)$ be the six-element ordered set depicted in Figure 1. Let G be the alternating group on the four element set $\{1,2,3,4\}$, i.e., the group of all even permutations of these elements. Table 1 gives an annotation map.

Giving an annotation for an ordered set by means of a full table, as it was done in Table 1, is informative but unpleasant to read. We therefore introduce a simplified notation based on double cosets of subgroups.

It is immediate from the definition that for each pair $a \leq b$ in an annotated ordered set the set $\lambda(a,b)$ is a union of double cosets of the "stabiliser" subgroups $G_a := \lambda(a,a)$ and $G_b := \lambda(b,b)$, i.e., that

$$\lambda(a,a) \circ \lambda(a,b) \circ \lambda(b,b) = \lambda(a,b).$$

Since the double cosets of any subgroup pair partition the group, it suffices to give a system of representatives of these double cosets. Moreover, since

$$\lambda(a,c) \circ \lambda(c,b)$$

also is a union of double cosets, we may simplify further and and define as follows:

Definition 2 Let λ be a $\underline{G}$-annotation of an ordered set $\underline{P}$. A **simplified annotation** $\lambda_\bullet$ corresponding to λ gives for every pair $a \leq b$ in P a set of double coset representatives of

$$\lambda(a,b) \setminus \bigcup_{a<c<b} \lambda(a,c) \circ \lambda(c,b).$$

◊

$\lambda(a,a)=$	$\{id,(12)(34)\}$
$\lambda(b,b)=$	$\{id,(12)(34)\}$
$\lambda(c,c)=$	$\{id,(234),(243)\}$
$\lambda(d,d)=$	$\{id,(12)(34),(13)(24),(14)(23)\}$
$\lambda(e,e)=$	$\{id\}$
$\lambda(f,f)=$	$\{id,(13)(24)\}$
$\lambda(a,d)=$	$\{(124),(132),(143),(234)\}$
$\lambda(a,e)=$	$\{(132),(143)\}$
$\lambda(a,f)=$	$\{(234),(243),(123),(132),(124),(143)\}$
$\lambda(b,d)=$	$\{id,(12)(34),(13)(24),(14)(23)\}$
$\lambda(b,e)=$	$\{(134),(142)\}$
$\lambda(b,f)=$	$\{id,(12)(34),(13)(24),(14)(23),(123),(243)\}$
$\lambda(c,e)=$	$\{(12)(34),(132),(142)\}$
$\lambda(c,f)=$	$\{id,(243),(234),(123),(13)(24),(143)\}$
$\lambda(d,f)=$	$\{id,(12)(34),(13)(24),(14)(23)\}$
$\lambda(e,f)=$	$\{(12)(34),(14)(23),(124),(132)\}$

Table 1. An A_4-annotation of the ordered set in Figure 1.

Note that $\lambda_\bullet(a,b)$ may be empty. As a convention, such pairs will be omitted in our listings of $\lambda_\bullet$. Similarly, we shall not list neighbouring pairs $a \prec b$ for which $\lambda_\bullet(a,b)$ consists only of the neutral element of $\underline{G}$. Following this, Table 1 simplifies to, e.g., the data displayed in Table 2.

$\lambda_\bullet(a,a)=$	$\{id,(12)(34)\}$
$\lambda_\bullet(b,b)=$	$\{id,(12)(34)\}$
$\lambda_\bullet(c,c)=$	$\{id,(234),(243)\}$
$\lambda_\bullet(d,d)=$	$\{id,(12)(34),(13)(24),(14)(23)\}$
$\lambda_\bullet(e,e)=$	$\{id\}$
$\lambda_\bullet(f,f)=$	$\{id,(13)(24)\}$
$\lambda_\bullet(a,d)=$	$\{(234)\}$
$\lambda_\bullet(a,e)=$	$\{(132)\}$
$\lambda_\bullet(b,e)=$	$\{(134)\}$
$\lambda_\bullet(c,e)=$	$\{(12)(34)\}$
$\lambda_\bullet(e,f)=$	$\{(12)(34),(124)\}$

Table 2. The simplified annotation to Table 1.

The ordered set in Figure 1 can be interpreted as a set of graphs on four vertices $\{1,2,3,4\}$, ordered by embeddability, see Figure 2. Each connected graph with four vertices, with the exception of the complete graph, occurs exactly once up to *even* isomorphism, which means that each such graph occurs exactly once, except for the path, which occurs twice. The four-element path has only even automorphisms and is for this reason listed with two copies.

The small graphs in Figure 2 are all labelled in the manner indicated for the top element: counterclockwise, starting with the upper right vertex. The annotation listed in Table 1 can now be read off from this diagram. Then recall that the group under consideration is the alternating group, acting on these vertices. The annotation is obtained as follows:

1. For each $p \in P$, the annotation $\lambda(p,p)$ is simply the automorphism group of the labelling graph, allowing only permutations from the alternating group, i.e., only even permutations.

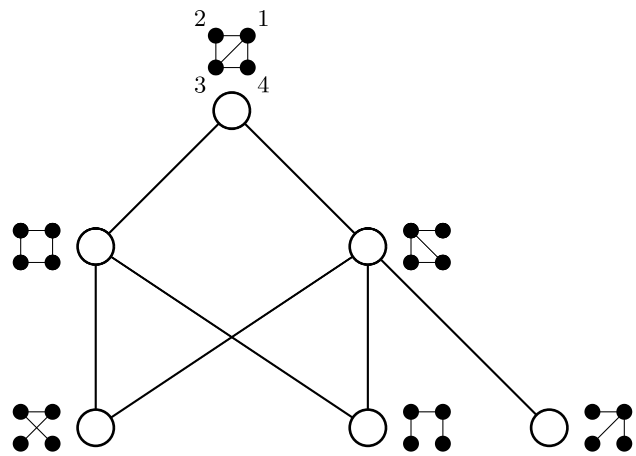


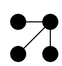
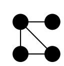
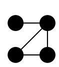
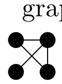
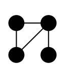
Fig. 2. The ordered set from Figure 1, labelled by graphs.

2. For $p < q$ in P, the set $\lambda(p, q)$ consists of all even permutations γ for which γ^{-1} is an embedding from p into q. (In other words: for which p is a subset of γq.)

Note that the second condition includes the first one if we allow $p = q$.

In small examples the simplified annotation can be written directly to the diagram, in particular when the stabiliser groups $\lambda(p, p)$ can be read off from the labelling. This is shown in Figure 3.

On the example of the pair $c \prec e$ we explain how to read the diagram in Figure 3:

- Point c is labelled by the graph , point e by .
- The graph at c is embeddable into the graph at e, but the given diagram is not a subdiagram. The graph at c is a subgraph of several isomorphic copies of the graph at e.
- There are precisely three isomorphic copies (all obtained by even permutations) of the graph at e that contain the diagram at point c, these are: , , and .
- These copies are obtained from the original label by the even permutations (12)(34), (132), and (142). These constitute the annotation $\lambda(c, e)$, cf. Table 1.
- The simplified annotation lists only (12)(34), because the other two permutations can be obtained from (12)(34) using automorphisms from the stabiliser groups. For example

$$(132) = (234) \circ (12)(34) \circ id,$$

where $(132) \in \lambda(c, c)$, $(12)(34) \in \lambda_\bullet(c, e)$, and $id \in \lambda(e, e)$.

3 Folding orders and lattices

Figure 3 gives a clue what annotation maps are used for. The six-element ordered set shown there represents a much larger order, having 37 elements. These are the connected graphs on $\{1, 2, 3, 4\}$ (K_4 omitted). The smaller ordered set is obtained through folding the larger one: Isomorphic graphs are identified. The induced folding of the order relation is expressed by the annotation map. A general formulation is provided by the next definition.

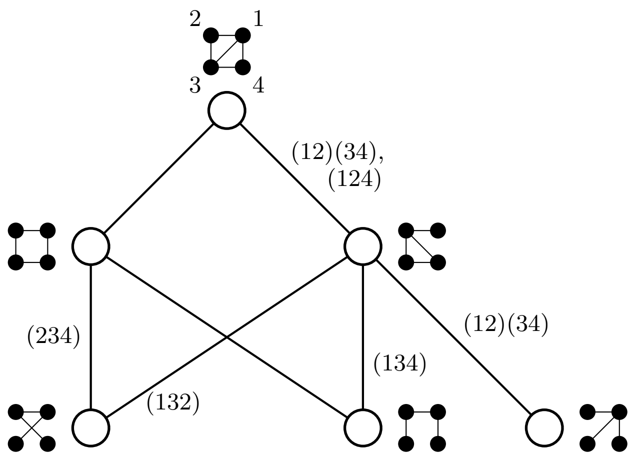


Fig. 3. Simplified annotation of the diagram. The labels are double coset representatives. The stabiliser $\lambda(p,p)$ is the automorphism group of the graph labelling p, restricted to the alternating group A_4. Non-neighbouring pairs are not drawn because their simplified labels are empty (in this example).

Definition 3 Let $\underline{P} := (P, \leq_P)$ be an ordered set and let $\Gamma \leq \mathrm{Aut}(\underline{P})$ be a subgroup of its automorphism group. A **Γ-orbifold** of $\underline{P}$ (also called an **order transversal**) is a triple

$$(P \setminus\!\setminus \Gamma, \leq, \lambda),$$

where

- $P \setminus\!\setminus \Gamma := \{p^\Gamma \mid p \in P\}$ is the set of orbits of Γ on P,
- $\leq$ is the order relation defined on $P \setminus\!\setminus \Gamma$ by

$$p^\Gamma \leq q^\Gamma :\Longleftrightarrow \exists_{\gamma\in\Gamma}\, p \leq_P \gamma q,$$

- and the mapping

$$\lambda : (P \setminus\!\setminus \Gamma) \times (P \setminus\!\setminus \Gamma) \to \mathfrak{P}(\Gamma)$$

  is defined using some fixed system Y of representatives of $P \setminus\!\setminus \Gamma$ by

$$\lambda(a^\Gamma, b^\Gamma) := \{\gamma \in \Gamma \mid a \leq_p \gamma b\},$$

  where $a, b \in Y$.

If $\underline{P}$ is a lattice, we speak of a **lattice orbifold**. ◊

Some details of this definition require justification. For example, it must be argued that $\leq$ is well defined and indeed an order. We include this in the proof of the following lemma.

Lemma 1 *Let $\underline{P} := (P, \leq_P)$ be an ordered set and let $\Gamma \leq \mathrm{Aut}(\underline{P})$ be a subgroup of its automorphism group. Then every orbifold of $\underline{P}$ is a Γ-annotated ordered set.*

Proof We first show that $\leq$, defined by

$$p^\Gamma \leq q^\Gamma :\Longleftrightarrow \exists_{\gamma\in\Gamma}\, p \leq_P \gamma q,$$

is well defined, i.e., independent of the choice of the representatives p, q. For representatives

$$p_1 \in p^\Gamma, q_1 \in q^\Gamma$$

of the same orbits we find automorphisms $\alpha, \beta \in \Gamma$ such that $p_1 = \alpha p$ and $q_1 = \beta q$. Then

$$\begin{aligned} p \leq_P \gamma q &\iff \alpha p \leq_P \alpha\gamma q \\ &\iff \alpha p \leq_P \alpha\gamma\beta^{-1}\beta q \\ &\iff p_1 \leq_P \gamma_1 q_1, \text{ where } \gamma_1 = \alpha\gamma\beta^{-1}. \end{aligned}$$

Thus $\exists_\gamma\ p \leq_P \gamma q \iff \exists\gamma_1\ p_1 \leq_P q_1$, as desired.

The rest of the proof is straightforward: $\leq$ obviously is an order on $P \setminus\!\setminus \Gamma$ and λ is an annotation map. That $\lambda(a, b)$ is nonempty for $a \leq b$ is immediate from the definition of $\leq$. Clearly

$$\lambda(a, a) = \{\gamma \mid a \leq \gamma a\} = \{\gamma \mid a = \gamma a\}$$

is a subgroup of Γ, it is the stabiliser Γ_a of a in Γ. For the third condition we obtain

$$\begin{aligned} \lambda(a, b) \circ \lambda(b, c) &= \{\alpha \mid a \leq_P \alpha b\} \circ \{\beta \mid b \leq_P \beta c\} \\ &= \{\alpha \circ \beta \mid a \leq_P \alpha b, b \leq_P \beta c\} \\ &\subseteq \{\gamma \mid a \leq_P \gamma c\} \\ &= \lambda(a, c). \end{aligned}$$

□

Now that we are able to fold ordered sets we also would like to unfold them in a way that reconstructs the original order. This is provided by the next definition.

Definition 4 Let $(P, \leq, \lambda)$ be a G-annotated ordered set, and let $G_p := \lambda(p, p)$ for all $p \in P$. The **unfolding** (or **reconstruction**) of $(P, \leq, \lambda)$ is defined as

$$\operatorname{rec}(P, \leq, \lambda) := (\dot\bigcup_{p\in P} G/G_p, \leq_r),$$

with

$$gG_p \leq_r hG_q :\iff g^{-1}h \in \lambda(p, q).$$

♢

Proposition 1 *The unfolding* $\operatorname{rec}(P, \leq, \lambda)$ *of a G-annotated ordered set* $(P, \leq, \lambda)$ *is an ordered set having a group of automorphisms isomorphic to G.*

Proof Let

$$N := \dot\bigcup_{p\in P} G/G_p = \dot\bigcup\ \{gG_p \mid g \in G\}$$

be the set of all stabiliser cosets. Proving that $\leq_r$ is an order on N is easy: Clearly $\leq_r$ is reflexive, since $id \in \lambda(p, p)$. Antisymmetry follows from the fact that for $p \neq q$ at least one of the sets $\lambda(p, q)$ must be empty, and transitivity follows from the multiplicativity condition for annotation maps.

G operates on its power set as a permutation group Γ of left multiplications. Let $\phi : G \to \Gamma$ denote the canonical isomorphism. Each $\phi(h) \in \Gamma$ maps N to N by

$$gG_p \overset{\phi(h)}{\mapsto} hgG_p,$$

and

$$\begin{aligned} g_1G_p \le_r g_2G_q &\iff g_1^{-1}g_2 \in \lambda(p,q) \\ &\iff g_1^{-1}h^{-1}hg_2 \in \lambda(p,q) \\ &\iff (hg_1)^{-1}hg_2 \in \lambda(p,q) \\ &\iff hg_1G_p \le_r hg_2G_q. \end{aligned}$$

Therefore each $\phi(h) \in \Gamma$ acts as an automorphism on $(N, \le_r)$, and Γ is a subgroup of $\mathrm{Aut}(N, \le_r)$. □

4 Isomorphisms of annotated ordered sets

The annotation we have studied above depends on the choice of representatives for the isomorphism classes of graphs. If we choose other representatives, we obtain another annotation map. If we are lucky, the new annotation may be considerably simpler. An example is shown in Figure 4.

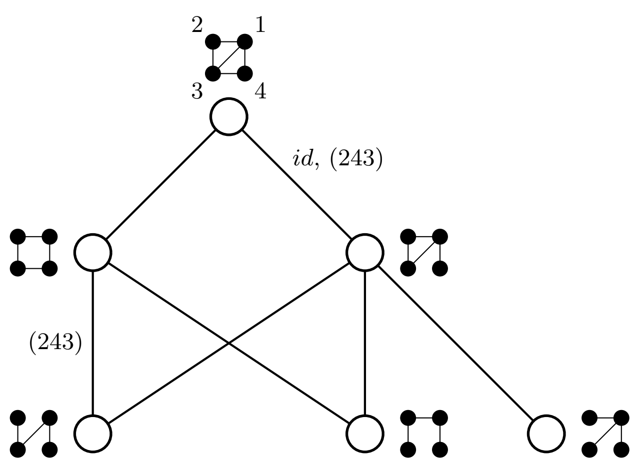


Fig. 4. An alternative simplified annotation obtained by using isomorphic graph diagrams.

The two diagrams in Figures 3 and 4 represent the same situation, and should be called isomorphic. They however differ considerably. It is not surprising that a rather complicated notion of isomorphy is needed.

Definition 5 [Isomorphy of group-annotated ordered sets]
Let Γ_1 and Γ_2 be groups, and let $\underline{P}_1 = (P_1, \le_1, \lambda_1)$ be a Γ_1-annotated ordered set and $\underline{P}_2 = (P_2, \le_2, \lambda_2)$ a Γ_2-annotated ordered set. Then $\underline{P}_1$ and $\underline{P}_2$ are said to be **isomorphic** if the following conditions hold:

- there exists an order isomorphism $\alpha : (P_1, \le_1) \longrightarrow (P_2, \le_2)$ and
- there exists a group isomorphism $\delta : \Gamma_1 \longrightarrow \Gamma_2$ and
- there exists a mapping $\phi : P_1 \longrightarrow \Gamma_2$

such that

$$\delta[\lambda_1(a,b)] = \phi(a)^{-1}\lambda_2(\alpha a, \alpha b)\phi(b)$$

holds for all $a \leq b$ in P_1. ◊

The two annotations in Figures 3 and 4 are indeed isomorphic according to this definition. The two groups are identical, so that we may choose δ to be the identity map. Two orders are canonically isomorphic and isomorphic to the order in Figure 1, so that we can omit α and simply use the letters from Figure 1 as element names for both. It remains to find a mapping $\phi : P \to A_4$ such that

$$\lambda_1(a,b) = \phi(a)^{-1}\lambda_2(a,b)\phi(b)$$

holds for all $a \leq b$ in P. For this, we may take

x	a	b	c	d	e	f
$\phi(x)$	(132)	id	id	id	(142)	id

For example, according to Table 1 we have $\lambda_1(c,e) = \{(12)(34), (132), (142)\}$, and from Figure 4 we read off that

$$\lambda_2(c,e) = \lambda_2(c,c) \circ \{id\} \circ \lambda_2(e,e)$$

(recall that a missing edge label stand for $\{id\}$). We conclude that

$$\begin{aligned} \lambda_2(c,e) &= \lambda_2(c,c) \circ \{id\} \circ \lambda_2(e,e) \\ &= \{id, (234), (243)\} \circ \{id\} \\ &= \{id, (234), (243)\}. \end{aligned}$$

Therefore

$$\begin{aligned} \phi(c)^{-1} \circ \lambda_2(c,e) \circ \phi(e) &= id \circ \lambda_2(c,e) \circ (142) \\ &= \{id, (234), (243)\} \circ (142) \\ &= \{(142), (12)(34), (132)\}, \end{aligned}$$

which is indeed $\lambda_1(c,e)$, as can be seen from Table 1.

Our first theorem states that the two structure necessarily are isomorphic.

Theorem 1 *Any two Γ-orbifolds of an ordered set $\underline{P}$ are isomorphic. More generally, if $\underline{P_1}$ and $\underline{P_2}$ are isomorphic ordered sets, $\alpha : P_1 \to P_2$ is an isomorphism and $\Gamma_1 \leq \mathrm{Aut}(\underline{P_1})$ and $\Gamma_2 \leq \mathrm{Aut}(\underline{P_2})$ are groups of automorphisms such that*

$$\Gamma_2 = \alpha \circ \Gamma_1 \circ \alpha^{-1},$$

then each Γ_1-orbifold of $\underline{P_1}$ is isomorphic to each Γ_2-orbifold of $\underline{P_2}$.

Proof We only prove the special case. A proof of the general statement can be found in [8]. Let

$$(P \setminus\!\setminus \Gamma, \leq, \lambda_1) \quad \text{and} \quad (P \setminus\!\setminus \Gamma, \leq, \lambda_2)$$

be two Γ-orbifolds of $\underline{P}$ and let Y_1 and Y_2 be the two orbit transversals used to define the annotation maps λ_1 and λ_2. For each $y \in Y_1$ there exists an automorphism $\phi_y \in \Gamma$ such that

$$\phi_y(y) \in Y_2.$$

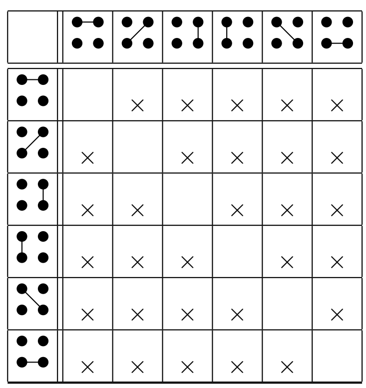
We get for $a, b \in Y_1$ that

$$\begin{aligned}\lambda_1(a,b) &= \{\gamma \mid a \leq \gamma b\} \\ &= \{\gamma \mid \phi_a^{-1}\phi_a a \leq \gamma \phi_b^{-1}\phi_b b\} \\ &= \{\gamma \mid \phi_a a \leq \phi_a \gamma \phi_b^{-1}\phi_b b\} \\ &= \{\gamma \mid \phi_a \gamma \phi_b^{-1} \in \lambda_2(a,b)\} \\ &= \phi_a^{-1}\lambda_2(a,b)\phi_b.\end{aligned}$$

The mapping $y \mapsto \phi_y$ therefore has the properties required by Definition 5. □

5 An example of a concept lattice orbifold

The ordered set in Figures 1–4 is part of a lattice orbifold. The lattice to be folded is the boolean lattice of all graphs with vertex set $V := \{1,2,3,4\}$. There are 64 such graphs, and 11 up to isomorphism. This lattice can naturally be written as the concept lattice of the ×6–formal context $(\binom{V}{2}, \binom{V}{2}, \neq)$, see Figure 5.



		×	×	×	×	×
	×		×	×	×	×
	×	×		×	×	×
	×	×	×		×	×
	×	×	×	×		×
	×	×	×	×	×	

Fig. 5. The standard context for the lattice of all graphs on four points.

(A, B) is a formal concept of this context iff A is (the edge set of) some graph on V and B is (the edge set of) its complement.

When folding this lattice, we have several groups to choose between. The full automorphism group is, of course, isomorphic to the symmetric group S_6. The S_6-orbifold of this lattice is simply a chain of length four, with trivial annotation. Two formal concepts are in the same orbit iff their extents have equal cardinality.

More interesting in the sense of graph theory is the subgroup Γ_4 isomorphic to S_4, that is induced by the action of the vertex permutation on the edges. Two concepts are in the same orbit of this group iff their extents are isomorphic as graphs.

In Figures 1–4 the group Γ of our choice was the alternating group A_4, consisting of the 12 even permutations of V, in its induced action on the two-element subsets. The Γ-orbifold of the lattice of all graphs on V is shown in Figure 6. Obviously, it is not a lattice. The orbifold diagram has a dual automorphism because the lattice it was generated from has one.

Since Γ is a subgroup of the group S_4, the Γ-orbifold in Figure 6 can itself be folded to obtain the diagram in Figure 7.

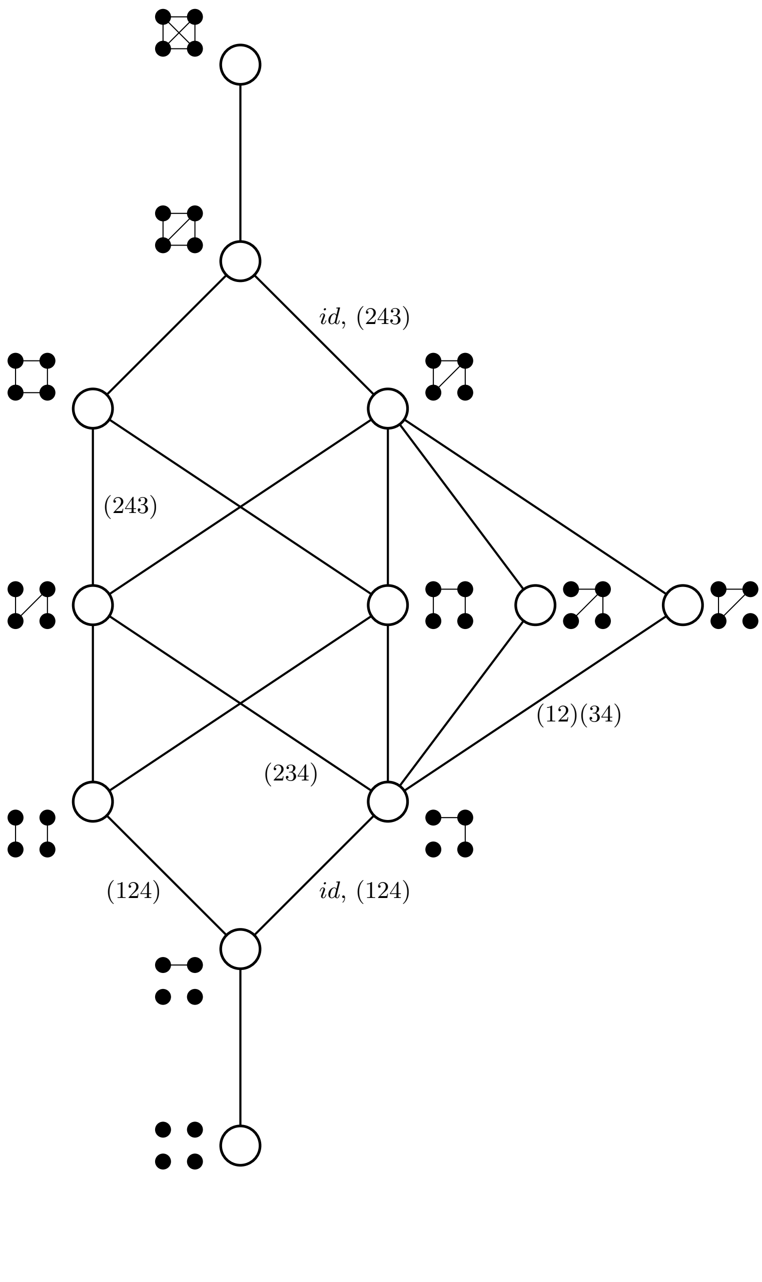


Fig. 6. An A_4– orbifold of the lattice of all graphs on four points.

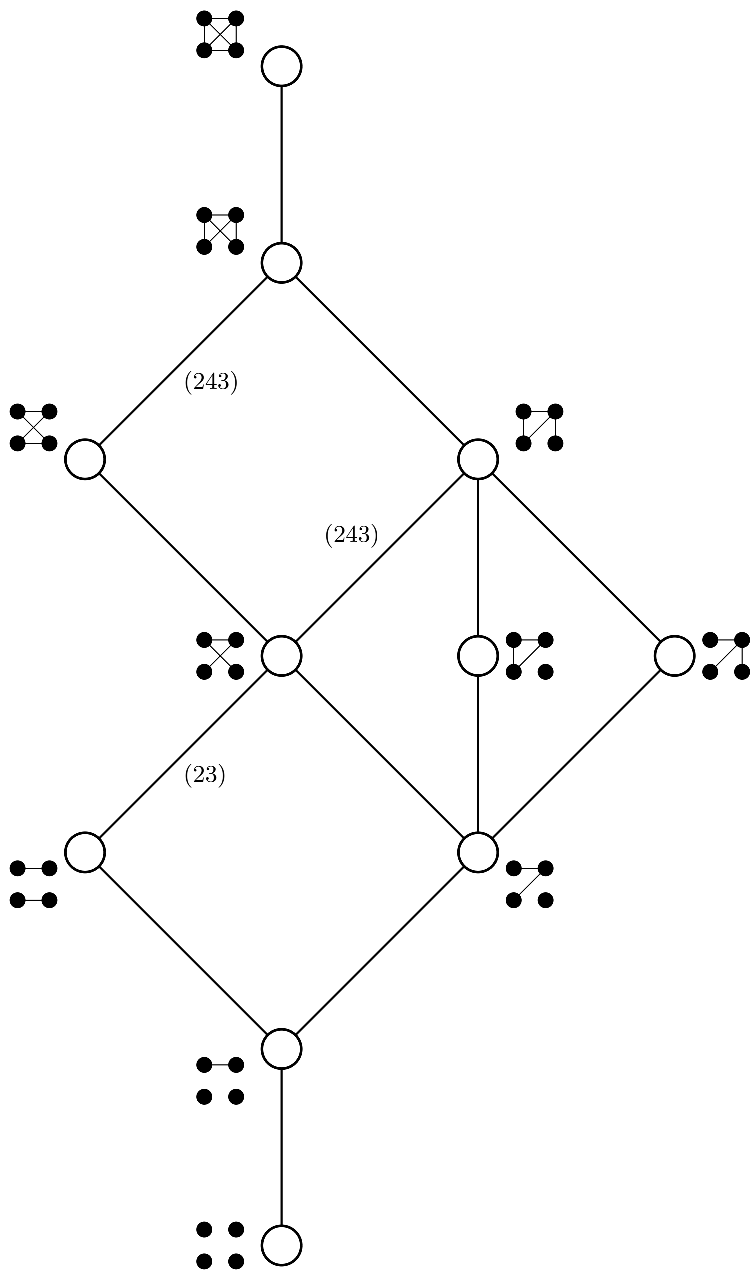


Fig. 7. An S_4- orbifold of the lattice of all graphs on four points.

6 Context orbifolds and a lattice of lattices

The symmetry group Γ may also be used to fold the formal context. The resulting **context orbifold** is

$$(G \setminus\!\setminus \Gamma, M \setminus\!\setminus \Gamma, \lambda),$$

where

$$\lambda : (G \setminus\!\setminus \Gamma) \times (M \setminus\!\setminus \Gamma) \to \mathcal{P}(\Gamma)$$

is the mapping defined by

$$\lambda(g^\Gamma, m^\Gamma) := \{\gamma \in \Gamma \mid g\ I\ \gamma m\}.$$

In practical computations we use the group structure for simplification. The orbits are replaced by orbit representatives, and since the values of the λ-mapping are unions of double cosets of the respective stabiliser groups (of g and m), they may be given by double coset representatives. However, a context orbifold may have many different such representations, and a theorem similar to Theorem 1 is required (and can be given) to guarantee representation invariance.

In the case of our example (in Figure 5) the context orbifold is a 1×1–table, since Γ is transitive both on objects and on attributes. The annotation gives the set $A_4 \setminus \{id, (12)(34)\}$.

As a more instructive example we give an orbifold representation of the "lattice of all concept lattices" with attribute set $\{a, b, c\}$. Recall that a **closure system** on a set M is a set $\mathcal{C} \subseteq \mathfrak{P}(M)$ of subsets of M which contains M and is closed under arbitrary intersections. The family of concept intents of any formal context is a closure system (as well as the family of concept extents). Any closure system is the system of intents of some formal context, and this context is determined by its intents up to clarifying, reducing and renaming objects.

The intersection of closure systems on M yields a closure system. The family of all closure systems on M therefore is itself a closure system, on the power set $\mathfrak{P}(M)$ of M, and therefore forms a complete lattice. The mathematical properties of these lattices have been studied by Caspard and Monjardet [2]. It is well known that this is encoded by the formal context

$$(\mathfrak{P}(M), \mathrm{Imp}(M), \models),$$

where $\mathfrak{P}(M)$ is the set of all subsets of M, $\mathrm{Imp}(M)$ is the set of all implications on M, and the relation $\models$ is defined as

$$S \models A \to B \quad :\Longleftrightarrow \quad A \not\subseteq S \text{ or } B \subseteq S.$$

The extents of this formal context are precisely the closure systems on M, and the intents are the corresponding implicational theories. The extent lattice therefore is indeed the lattice of all closure systems on M.

The formal context given above is not reduced, and for computations it is convenient to use the standard context

$$(\mathfrak{P}(M) \setminus \{M\}, \mathrm{Imp}_r(M), \models),$$

where

$$\mathrm{Imp}_r(M) := \{A \to \{b\} \mid A \subseteq M, b \notin A\}.$$

For $M := \{a, b, c\}$ this yields the formal context in Figure 8. This formal context has 61 concepts, corresponding to the 61 closure systems on $\{a, b, c\}$. The cardinalities of these lattices are known up to $|M| = 6$ (see Habib and Nourine[3]). The values can be verified using the standard algorithm for generating concept lattices. Note,

	$\emptyset \to a$	$\emptyset \to b$	$\emptyset \to c$	$a \to b$	$a \to c$	$b \to a$	$b \to c$	$c \to a$	$c \to b$	$a, b \to c$	$a, c \to b$	$b, c \to a$
$\emptyset$				×	×	×	×	×	×	×	×	×
$\{a\}$	×					×	×	×	×	×	×	×
$\{b\}$		×		×	×			×	×	×	×	×
$\{a, b\}$	×	×		×		×		×	×		×	×
$\{c\}$			×	×	×	×	×			×	×	×
$\{a, c\}$	×		×		×	×	×	×		×		×
$\{b, c\}$		×	×	×	×		×		×	×	×	

Fig. 8. The reduced formal context for the lattice of closure systems on $\{a, b, c\}$. Each permutation of $\{a, b, c, \}$ induces an automorphism.

however, that the numbers grow rapidly. For $n = 1, \ldots, 6$ the numbers of closure systems on an n-element set are 2, 7, 61, 2480, 1385552, 75973751474 [7]. Up to isomorphism, there are 1, 2, 5, 19, 184, 14664, 108295846 closure systems. Note that there is a misprint in the sixth term of Sloane's sequence A108799 [7], as was noticed by Mike Behrisch [1].

The formal context in Figure 8 obviously has six automorphisms induced by the permutations of $\{a, b, c\}$. Folding the context by the induced action Γ of this symmetric group yield the context orbifold displayed in Figure 9. The concept lattice of the formal context in Figure 8 has 61 elements. Its lattice orbifold has 19 elements. It is displayed in Figure 10. Note that the diagram in Figure 10 is very intuitive, because it represents the closure systems "up to isomorphism". However, the containment order "up to isomorphism" does not give a lattice, it is actually not a mathematically precise notion right away. The annotated diagram, together with the definitions on which the annotation is built, make the idea of a hierarchy of structures "up to isomorphism" precise and mathematically accessible.

	$\emptyset \to c$	$a \to c$	$a, b \to c$
$\emptyset$	$\emptyset$	id	id
$\{a\}$	id	$(ab), (abc)$	$id, (abc)$
$\{a, b\}$	id	$id, (acb)$	(bc)

Fig. 9. An orbifold of the formal context in Figure 8. Objects and attributes are given by coset representatives. The cells of the table contain sets of double coset representatives, in analogy to Definition 2.

7 Outlook

A detailed theoretical framework and a good algorithmic basis are needed to make context and lattice orbifolds applicable. Algorithms must be given to compute the lattice orbifold directly from the context orbifold, and, even more interestingly, to compute the folded stem base. A package based on the GAP system [4] has been implemented and is available upon request. Some of these questions will be treated in a subsequent paper, but many are still open.

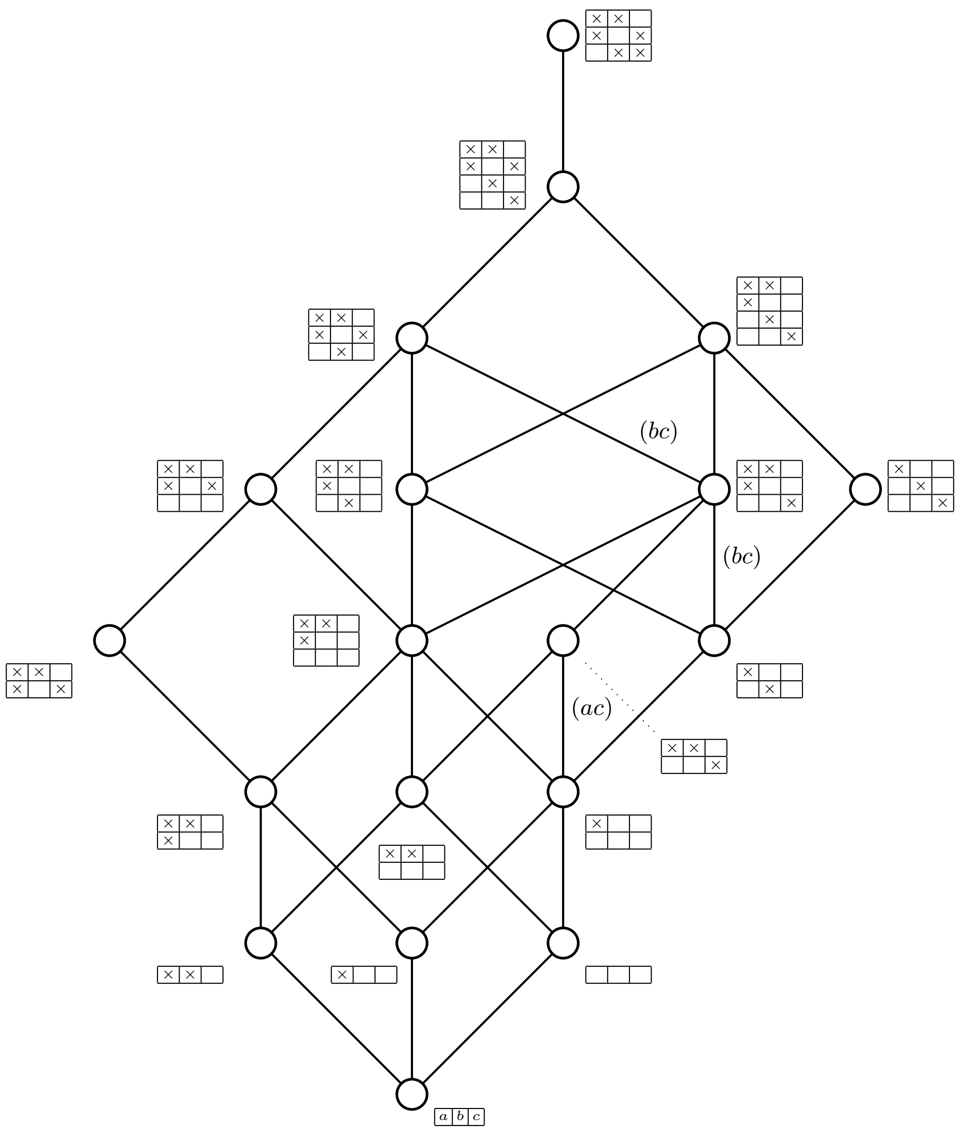


Fig. 10. The lattice orbifold for the lattice of the 61 closure systems on the set $\{a, b, c\}$. Each closure system is the system of intents of a unique row-reduced formal context with attribute set $\{a, b, c\}$. These contexts are given up to permutations of $\{a, b, c\}$. The context at the least element has empty object set.

8 Conclusion

The interplay between concept lattice orbifolds and context orbifolds offers a powerful technique for the investigation of lattices with symmetries. Although the necessary foundations were provided by Zickwolff [8] in a very general setting, it requires some effort to adapt them to the case of contexts and lattices. We have shown here how this can be done and that interesting results can be obtained.

References

1. Mike Behrisch, personal communication 2006.
2. N. Caspard, B. Monjardet. *The lattices of closure systems, closure operators, and implicational systems on a finite set: a survey.* Discrete Applied Mathematics 127, 2 (2003), pp 241–269.
3. M. Habib, L. Nourine. *The number of Moore families on* $n = 6$. Discrete Mathematics 294 (2005), pp 291–296.
4. The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. (http://www.gap-system.org)
5. B. Ganter and K. Reuter. *Finding closed sets: a general approach.* Order 8 (1991), 283–290.
6. B. Ganter and R. Wille. *Formal Concept Analysis – Mathematical Foundations.* Springer 1999.
7. N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences.* Sequences A102896 and A108799. `http://www.research.att.com/~njas/sequences/A108799`.
8. M. Zickwolff. *Rule Exploration: First Order Logic in Formal Concept Analysis.* Dissertation, Darmstadt 1991.