Technische Universität Dresden

International Center for Computational Logic

COMPLEXITY THEORY

Lecture 17: The Polynomial Hierarchy

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Knowledge-Based Systems

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Review: ATM vs. DTM

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How? Analyse the exponential ATM configuration graph deterministically.

$$\text{APSpace} \supseteq \text{ExpTime}$$

How? Re-trace exponential computation path by verifying local changes.

From Deterministic Time To Alternating Space

Let $h\colon \mathbb{N} \to \mathbb{R}$ be a function in $O(g)$ that defines the exact time bound for $\mathcal{M}$ (no O-notation) and that can be computed in space $O(\log g)$.

ATMSIMULATETM(TM $\mathcal{M}$, input word w, time bound h) :
  existentially guess $s \leq h(|w|)$ // halting step
  existentially guess $i \in \{0, \ldots, s\}$ // halting position
  existentially guess $\omega \in Q \times \Gamma$ // halting cell + state
  if $\mathcal{M}$ would not accept in ω:
    return false
  for $j = s, \ldots, 1$ **do**:
    existentially guess $\langle \omega_{-1}, \omega_0, \omega_1 \rangle \in \Omega^3$
    if $\mathcal{M}(\omega_{-1}, \omega_0, \omega_{+1}) \neq \omega$:
      return false
    universally choose $\ell \in \{-1, 0, 1\}$
    $\omega := \omega_\ell$
    $i := i + \ell$
  // after tracing back s steps, check input configuration:
  return "input configuration of $\mathcal{M}$ on w has ω at position i"

A Remark on (Non)determinism

For each cell that is to be verified:

- we guess three predecessor cells,
- which we then verify "recursively".

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how do we know that the guesses do not contradict each other?

Because of determinism:

- The simulated TM is deterministic.
- Hence, if the starting point is determined, every future cell in every position is determined too.
- Therefore, for every cell, there is only one possible guess that eventually leads to the right input tape.

$\leadsto$ Independent guesses, if correct, must generally be the same.

A Remark on Space-Constructibility

Our algorithm needs to compute h in logarithmic space w.r.t. its absolute value to implement the line

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However, we could also avoid this:

- The algorithm from line `03` on checks if the TM accepts after s steps.
- We can make algorithms that check if the TM does or does not halt after s steps.
- We can then use an algorithm that increments s one by one (starting from 1):
  - For each value of s, guess if the TM halts after this time or not;
  - Check the guess using the above procedures;
  - Stop when the halting configuration has been found.
- Because of the time bound on the simulated TM, s will not become larger than $2^{O(f)}$ here; so we can always store it in space $O(f)$.

Summary: Alternating vs. Deterministic Classes

We can sum up our findings as follows:

$$
\begin{array}{ccccccccc}
\mathsf{L} & \subseteq & \mathsf{PTime} & \subseteq & \mathsf{PSpace} & \subseteq & \mathsf{ExpTime} & \subseteq & \mathsf{ExpSpace} \\
 & & \parallel & & \parallel & & \parallel & & \parallel \\
 & & \mathsf{ALogSpace} & \subseteq & \mathsf{APTime} & \subseteq & \mathsf{APSpace} & \subseteq & \mathsf{AExpTime}
\end{array}
$$

The Polynomial Hierarchy

Bounding Alternation

For ATMs, alternation itself is a resource. We can distinguish problems by how much alternation they need to be solved.

We first classify computations by counting their quantifier alternations:

Definition 17.1: Let $\mathcal{P}$ be a computation path of an ATM on some input.

- $\mathcal{P}$ is of type Σ_1 if all of its non-halting configurations are existential.[a]
- $\mathcal{P}$ is of type Π_1 if all of its non-halting configurations are universal.[a]
- $\mathcal{P}$ is of type Σ_{i+1} if it starts with a sequence of existential configurations, followed by a path of type Π_i.
- $\mathcal{P}$ is of type Π_{i+1} if it starts with a sequence of universal configurations, followed by a path of type Σ_i.

[a] Recall that we used existential and universal halting configurations for rejecting and accepting, respectively. These are always allowed in all types of paths.

Alternation-Bounded ATMs

We apply alternation bounds to every computation path:

Definition 17.2: A Σ_i Alternating Turing Machine is an ATM for which every computation path on every input is of type Σ_j for some $j \leq i$.
A Π_i Alternating Turing Machine is an ATM for which every computation path on every input is of type Π_j for some $j \leq i$.

Note that it's always OK to use fewer alternations ("$j \leq i$"), but computation has to start with the right kind of quantifier ($\exists$ for Σ_i and $\forall$ for Π_i).

Example 17.3: A Σ_1 ATM is simply an NTM.

Alternation-Bounded Complexity

We are interested in the power of ATMs that are both time/space-bounded and alternation-bounded:

Definition 17.4: Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. $\Sigma_i\text{Time}(f(n))$ is the class of all languages that are decided by some $O(f(n))$-time bounded Σ_i ATM. The classes $\Pi_i\text{Time}(f(n))$, $\Sigma_i\text{Space}(f(n))$ and $\Pi_i\text{Space}(f(n))$ are defined similarly.

The most popular classes of these problems are the alternation-bounded polynomial-time classes:

$$\Sigma_i\text{P} = \bigcup_{d \geq 1} \Sigma_i\text{Time}(n^d) \qquad \text{and} \qquad \Pi_i\text{P} = \bigcup_{d \geq 1} \Pi_i\text{Time}(n^d)$$

Hardness for these classes is defined by polynomial many-one reductions as usual.

Basic Observations

Theorem 17.5: $\Sigma_1\mathsf{P} = \mathsf{NP}$ and $\Pi_1\mathsf{P} = \mathsf{coNP}$.

Proof: Immediate from the definitions. □

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Theorem 17.6: $co\Sigma_i P = \Pi_i P$ and $co\Pi_i P = \Sigma_i P$.

Proof: We observed previously that ATMs can be complemented by simply exchanging their universal and existential states. This does not affect the amount of time or space needed. □

Example

MinFormula

Input:	A propositional formula φ.
Problem:	Is φ the shortest among formulas satisfied by the same assignments as φ?

One can show that **MinFormula** is Π_2P-complete. Inclusion is easy:

```
MinFormula(formula φ) :
  universally choose ψ := formula shorter than φ
  existentially guess I := assignment for variables in φ
  return φ^I ≠ ψ^I
```

The Polynomial Hierarchy

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What we do know, however, is this:

Theorem 17.7:

- $\Sigma_i\mathrm{P} \subseteq \Sigma_{i+1}\mathrm{P}$ and $\Sigma_i\mathrm{P} \subseteq \Pi_{i+1}\mathrm{P}$
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Proof: Immediate from the definitions. □

Thus, the classes Σ_iP and Π_iP form a kind of hierarchy:
the Polynomial (Time) Hierarchy. Its entirety is denoted PH:

$$\mathsf{PH} := \bigcup_{i\geq 1} \Sigma_i\mathsf{P} = \bigcup_{i\geq 1} \Pi_i\mathsf{P}$$

Problems in the Polynomial Hierarchy

The “typical” problems in the Polynomial Hierarchy are restricted forms of **True QBF**:

True Σ_kQBF

Input: A quantified Boolean formula φ with at most k quantifier alternations of the form $\exists X_1^1, X_2^1, \cdots \forall X_1^2, X_2^2, \cdots \mathsf{Q}_k X_1^k, X_2^k, \cdots .\psi$.

Problem: Is φ true?

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True Π_kQBF is defined analogously, using formulae with k quantifier alternations that start with $\forall$ rather than $\exists$.

Theorem 17.8: For every k, True Σ_kQBF is Σ_kP-complete and True Π_kQBF is Π_kP-complete.

Note: It is not known if there is any PH-complete problem.

Alternative Views on the Polynomial Hierarchy

Certificates

For NP, we gave an alternative definition based on polynomial-time verifiers that use a given polynomial certificate (witness) to check acceptance. Can we extend this idea to alternation-bounded ATMs?

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Notation: Given an input word w and a polynomial p, we write $\exists^p c$ as abbreviation for "there is a word c of length $|c| \leq p(|w|)$." Similarly for $\forall^p c$.

We can rephrase our earlier characterisation of polynomial-time verifiers:

L $\in$ NP iff there is a polynomial p and language **V** $\in$ P such that

$$\mathbf{L} = \{w \mid \exists^p c \text{ such that } (w\#c) \in \mathbf{V}\}$$

Certificates for bounded ATMs

Theorem 17.9: **L** $\in \Sigma_k$P iff there are a polynomial p and a language **V** $\in$ P such that

$$\mathbf{L} = \{w \mid \exists^p c_1.\forall^p c_2 \ldots \mathsf{Q}^p_k c_k \text{ such that } (w\#c_1\#c_2\#\ldots\#c_k) \in \mathbf{V}\},$$

where $\mathsf{Q}_k = \exists$ if k is odd and $\mathsf{Q}_k = \forall$ if k is even.

An analoguous result holds for **L** $\in \Pi_k$P.

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Proof sketch:
⇒: Similar as for NP. Use c_i to encode the non-deterministic choices of the ATM. With all choices given, the acceptance on the specified path can be checked in polynomial time.

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Proof sketch:

$\Rightarrow$: Similar as for NP. Use c_i to encode the non-deterministic choices of the ATM. With all choices given, the acceptance on the specified path can be checked in polynomial time.

$\Leftarrow$: Use an ATM to implement the certificate-based definition of **L** by using universal and existential choices to guess the certificate before running a polynomial time verifier. □

Oracles (Revision)

Recall how we defined oracle TMs:

Definition 3.15: An Oracle Turing Machine (OTM) is a Turing machine $\mathcal{M}$ with a special tape, called the oracle tape, and distinguished states $q_?$, q_{yes}, and q_{no}. For a language **O**, the oracle machine $\mathcal{M}^{\mathbf{O}}$ can, in addition to the normal TM operations, do the following:

Whenever $\mathcal{M}^{\mathbf{O}}$ reaches $q_?$, its next state is q_{yes} if the content of the oracle tape is in **O** and q_{no} otherwise.

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Let C be a complexity class:

- For a language **O**, we write $\mathrm{C}^{\mathbf{O}}$ for the class of all problems that can be solved by a C-TM with oracle **O**.
- For a complexity class O, we write C^{O} for the class of all problems that can be solved by a C-TM with an oracle from class O.

Note: this notation will only be used for complexity classes C where it is clear what a "C-TM with an oracle" is.

The Polynomial Hierarchy – Alternative Definition

We recursively define the following complexity classes:

Definition 17.10:

- $\Sigma_0^{\mathsf{P}} := \mathsf{P}$ and $\Sigma_{k+1}^{\mathsf{P}} := \mathsf{NP}^{\Sigma_k^{\mathsf{P}}}$
- $\Pi_0^{\mathsf{P}} := \mathsf{P}$ and $\Pi_{k+1}^{\mathsf{P}} := \mathsf{coNP}^{\Pi_k^{\mathsf{P}}}$

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Remark:

Complementing an oracle (language/class) does not change expressivity: we can just swap states q_{yes} and q_{no}. Therefore $\Sigma_{k+1}^{\mathsf{P}} = \mathsf{NP}^{\Pi_k^{\mathsf{P}}}$ and $\Pi_{k+1}^{\mathsf{P}} := \mathsf{coNP}^{\Sigma_k^{\mathsf{P}}}$.

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Question:
How do these relate to our earlier definitions of the PH classes?

Oracle TMs vs. ATMs

It turns out that this new definition leads to a familiar class of problems:[1]

Theorem 17.11: For all $k \geq 1$, we have $\Sigma_k^{\mathsf{P}} = \Sigma_k\mathsf{P}$ and $\Pi_k^{\mathsf{P}} = \Pi_k\mathsf{P}$.

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Proof: We only prove the case $\Sigma_k^{\mathsf{P}} = \Sigma_k\mathsf{P}$ – the other follows by complementation.
The proof is by induction on k.

Base case: $k = 1$.
The claim follows from $\Sigma_1^{\mathsf{P}} = \mathsf{NP}^{\mathsf{P}} = \mathsf{NP}$ and $\Sigma_1\mathsf{P} = \mathsf{NP}$ (as noted before).

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Oracle TMs vs. ATMs (2)

Induction step: Assume the claim holds for k. We show $\Sigma^{\mathsf{P}}_{k+1} = \Sigma_{k+1}\mathsf{P}$.

"$\supseteq$" Assume $\mathbf{L} \in \Sigma_{k+1}\mathsf{P}$.

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- By Theorem 17.9, for some language $\mathbf{V} \in \mathsf{P}$ and polynomial p:
  $\mathbf{L} = \{w \mid \exists^p c_1.\forall^p c_2 \ldots \mathsf{Q}^p_{k+1} c_{k+1}$ such that $(w\#c_1\#c_2\#\ldots\#c_{k+1}) \in \mathbf{V}\}$

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- The following algorithm decides $\mathbf{L}$, thus showing $\mathbf{L} \in \mathsf{NP}^{\mathbf{L}'}$:
  on input w, non-deterministically guess c_1;
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- By induction, $\mathbf{L}' \in \Pi^{\mathrm{P}}_k$. Hence, $\mathbf{L} \in \mathrm{NP}^{\Pi^{\mathrm{P}}_k} = \mathrm{NP}^{\Sigma^{\mathrm{P}}_k} = \Sigma^{\mathrm{P}}_{k+1}$.

Oracle TMs vs. ATMs (3)

Induction step: Assume the claim holds for k. We show $\Sigma^{\mathsf{P}}_{k+1} = \Sigma_{k+1}\mathsf{P}$.

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- There is a polynomial-time nondeterministic TM $\mathcal{M}$ that decides $\mathbf{L}$ using an oracle $\mathbf{O} \in \Sigma^{\mathsf{P}}_{k}$.

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- For an $\Sigma_{k+1}\mathsf{P}$ algorithm, first guess an accepting path of $\mathcal{M}$ including results of all oracle queries.
- Then universally branch to verify all guessed oracle replies:

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Oracle TMs vs. ATMs (3)

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  - For queries $w \in \mathbf{O}$ with guessed answer "yes", use $\Pi_{k-1}\mathsf{P}$ check for $(w\#c_1) \in \mathbf{O}'$, where $\mathbf{O}'$ is constructed as in the $\supseteq$-case and c_1 is guessed in the first $\exists$-phase. □

More about the Polynomial Hierarchy

The Polynomial Hierarchy Three Ways

We discovered a hierarchy of complexity classes between P and PSpace, with NP and coNP on the first level and infinitely many further levels above:

Definition by ATM: Classes $\Sigma_i^{\mathsf{P}}/\Pi_i^{\mathsf{P}}$ are defined by polytime ATMs with bounded types of alternation, starting computation with existential/universal states.

Definition by Verifier: Classes $\Sigma_i^{\mathsf{P}}/\Pi_i^{\mathsf{P}}$ are given as projections of certain verifier languages in P, requiring existence/universality of polynomial witnesses.

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Using such oracles with deterministic TMs, we can also define classes Δ_i^{P}.

More Classes in PH

We defined Σ_k^{P} and Π_k^{P} by relativising NP and coNP with oracles.

What happens if we start from P instead?

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Problems for Δ_k^{P}?

Δ_k^{P} seems to be less common in practice, but there are some known complete problems for $\mathsf{P}^{\mathsf{NP}} = \Delta_2^{\mathsf{P}}$:

Uniquely Optimal TSP [Papadimitriou, JACM 1984]

Input: Undirected graph G with edge weights (distances).

Problem: Is there exactly one shortest travelling salesman tour on G?

Divisible TSP [Krentel, JCSS 1988]

Input: Undirected graph G with edge weights; number k.

Problem: Is the shortest travelling salesman tour on G divisible by k?

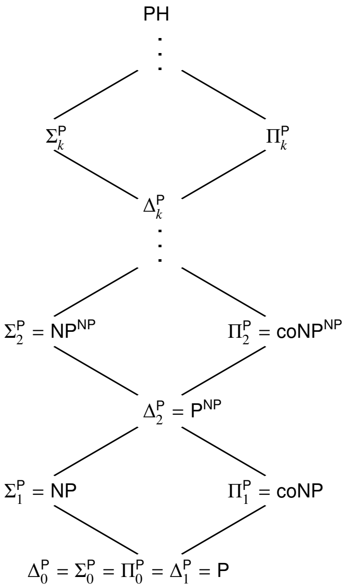
Odd Final SAT [Krentel, JCSS 1988]

Input: Propositional formula φ with n variables.

Problem: Is X_n true in the lexicographically last assignment satisfying φ?

Is the Polynomial Hierarchy Real?

Questions:



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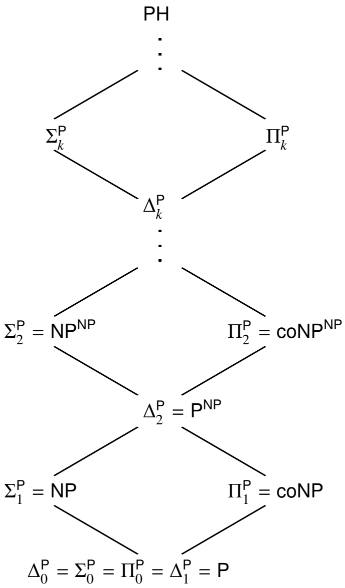

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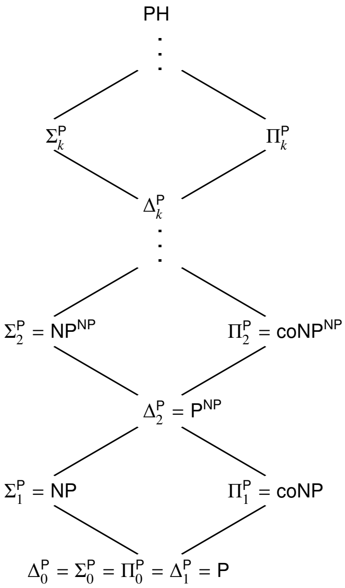

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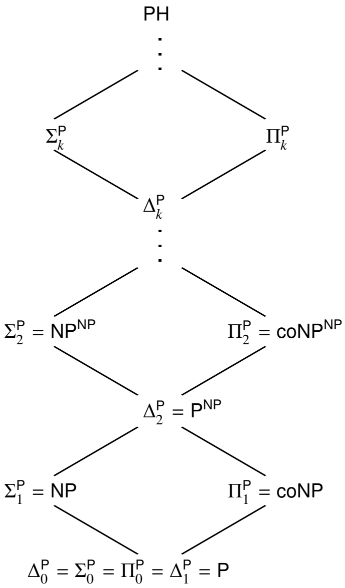

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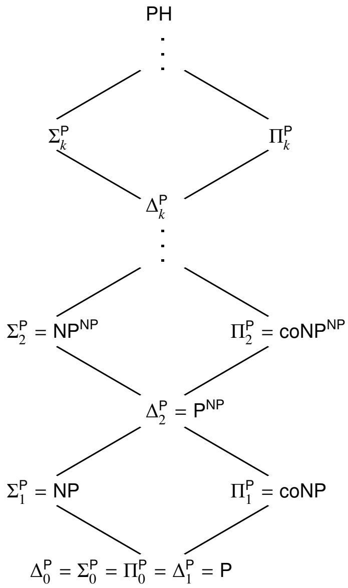

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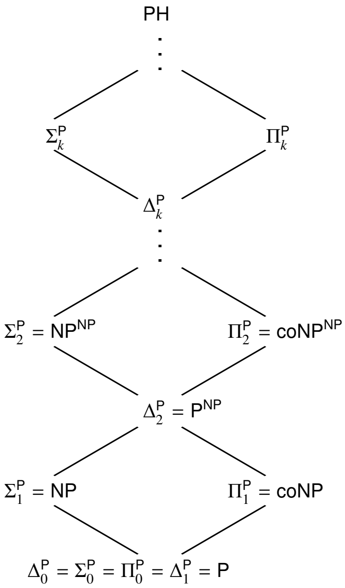

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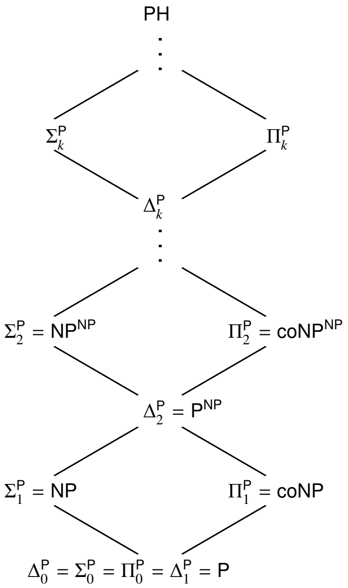

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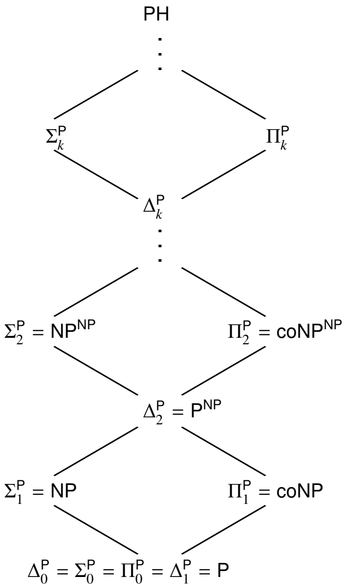

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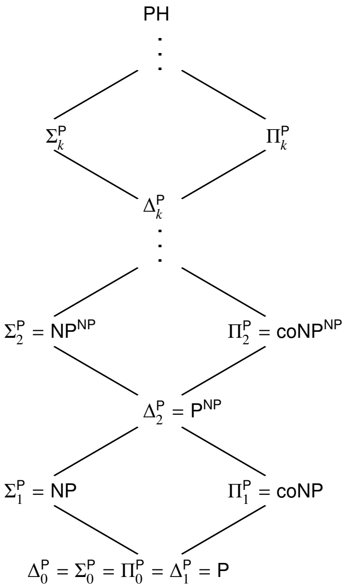

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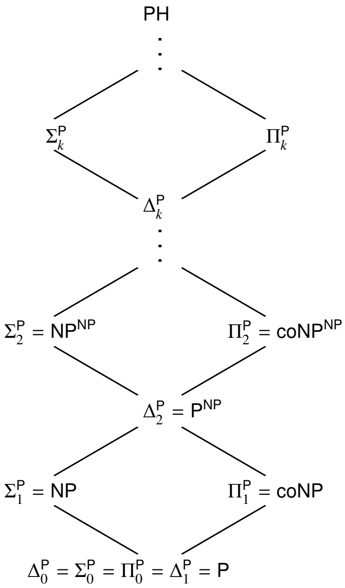
What do we know then?



What We Know (Excerpt)

Theorem 17.13: If there is any k such that $\Sigma_k^{\mathsf{P}} = \Sigma_{k+1}^{\mathsf{P}}$, then $\Sigma_j^{\mathsf{P}} = \Pi_j^{\mathsf{P}} = \Sigma_k^{\mathsf{P}}$ for all $j > k$ and, therefore, $\mathsf{PH} = \Sigma_k^{\mathsf{P}}$.
In this case, we say that the polynomial hierarchy collapses at level k.

Proof: Left as an exercise (not too hard to get from definitions). □

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Proof: Left as an exercise (not too hard to get from definitions). □

Corollary 17.14: If $\mathsf{PH} \neq \mathsf{P}$ then $\mathsf{NP} \neq \mathsf{P}$.

Intuitively speaking: "The polynomial hierarchy is built upon the assumption that NP has some additional power over P. If this is not the case, the whole hierarchy collapses."

What We Know (Excerpt)

Theorem 17.15: PH $\subseteq$ PSpace.

Proof: Left as an exercise (induction over PH levels, using $\text{PSpace}^{\text{PSpace}} = \text{PSpace}$). □

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Theorem 17.16: If PH = PSpace then there is some k with PH = Σ_k^{P}.

Proof: If PH = PSpace, then **True QBF** $\in$ PH. Hence **True QBF** $\in \Sigma_k^{\text{P}}$ for some k. Since **True QBF** is PSpace-hard, this implies $\Sigma_k^{\text{P}} = \text{PSpace}$. □

What We Believe (Excerpt)

"Most experts" think that

- The polynomial hierarchy does not collapse completely (same as P $\neq$ NP);
- The polynomial hierarchy does not collapse on any level
  (in particular, PH $\neq$ PSpace and there is no PH-complete problem).

But there can always be surprises...

Summary and Outlook

The Polynomial Hierarchy is a hierarchy of complexity classes between P and PSpace.

It can be defined by stacking NP-oracles on top of P/NP/coNP or, equivalently, by bounding alternation in polytime ATMs.

The typical complete problems for the classes in the polynomial hierarchy are QBF with bounded forms of quantifier alternation.

What's next?

- Computing with circuits
- End-of-year consultation