TU Dresden, Knowledge-Based Systems Group
Dr. Stephan Mennicke Summer 2024

Concurrency Theory

Exercise Sheet 2: Semantics of Programming Languages

30th of April 2024

Exercise 2.1.

Suppose we represent nummerals **Num** in the binary format:

$$n \quad ::= \quad \texttt{0} \quad | \quad \texttt{1} \quad | \quad n\texttt{0} \quad | \quad n\texttt{1}$$

Define the total function $\mathcal{N} : \mathbf{Num} \to \mathbb{Z}$ mapping binary numerals to their positive integers.

Exercise 2.2.

Determine the functional F associated with the statement

$$\texttt{while}\ \neg(x \equiv \texttt{0})\ \texttt{do}\ x\texttt{:=}x \oplus \texttt{1}$$

Consider the following partial functions of **State** $\hookrightarrow$ **State**:

$$g_1\ s = \bot \text{ for all } s$$

$$g_2\ s = \begin{cases} s[x \mapsto 0] \text{ if } s\,x \geq 0 \\ \bot \text{ if } s\,x < 0 \end{cases}$$

$$g_3\ s = \begin{cases} s[x \mapsto 0] \text{ if } s\,x \geq 0 \\ s \text{ if } s\,x < 0 \end{cases}$$

$$g_4\ s = s[x \mapsto 0] \text{ for all } s$$

$$g_5\ s = s \text{ for all } s$$

Which of the function g_i ($i \in \{1, 2, 3, 4, 5\}$) are fixed points of F?

If there is more than one, which one is the *desired fixed point*?

Exercise 2.3.

Consider again the requirements on fixed points (from slide 20 on, lecture 4). We left option B (local looping) as an exercise. Study this option and carry out an example in the style of the ones given in options A and C. Do we learn anything about the fixed point requirements?

Exercise 2.4.

Let g_1, g_2, and g_3 be defined as follows:

$$g_1\ s = \begin{cases} s & \text{if } s\,x \text{ is even} \\ \texttt{undef} & \text{otherwise} \end{cases}$$

$$g_2\ s = \begin{cases} s & \text{if } s\,x \\ \texttt{undef} & \text{otherwise} \end{cases} \qquad a'$$

$$g_3\ s = s$$

(a) Determine the ordering between these partial functions.

(b) Determine a partial function g_4 such that $g_4 \sqsubseteq g_1$, $g_4 \sqsubseteq g_2$, and $g_4 \sqsubseteq g_3$ (i.e., g_4 is a lower bound of $\{g_1, g_2, g_3\}$).

(c) Determine a partial function g_5 such that $g_1 \sqsubseteq g_5$, $g_2 \sqsubseteq g_5$, and $g_5 \sqsubseteq g_3$ but g_5 is distinct from g_1, g_2, and g_3.

Exercise 2.5.

Let S be a nonempty set and define $\mathcal{P}_{\text{fin}}(S) = \{K \mid K \text{ if finite and } K \subseteq S\}$.

(a) Show that $(\mathcal{P}_{\text{fin}}(S), \subseteq)$ as well as $(\mathcal{P}_{\text{fin}}(S), \supseteq)$ are po-sets.
(b) Do both po-sets have a least element for all choices of S?
(c) Show that every subset of $\mathcal{P}_{\text{fin}}(S)$ has a least upper bound w.r.t. $\subseteq$.
(d) Provide a set S such that $(\mathcal{P}_{\text{fin}}(S), \subseteq)$ has a chain with no upper bound and, therefore, no least upper bound.
(e) Is any of the aforementioned po-sets a complete lattice? ccpo?
(f) Analyze $(\mathcal{P}(S), \subseteq)$ where $\mathcal{P}(S) = \{K \mid K \subseteq S\}$, whether it forms a complete lattice? How about ccpo?
(g) Construct a subset Y of **State** $\hookrightarrow$ **State** such that Y has no upper bound.

Exercise 2.6.

(a) Consider the ccpo $(\mathbb{N}, \subseteq)$. Determine which of the following functions in $\mathbb{N} \to \mathbb{N}$ are monotone:

$$\begin{aligned} f_1\, X &= \mathbb{N} \setminus X \\ f_2\, X &= X \cup \{27\} \\ f_3\, X &= X \cap \{7, 9, 13\} \\ f_4\, X &= \{n \in X \mid n \text{ is a prime}\} \\ f_5\, X &= \{2 \cdot n \mid n \in X\} \end{aligned}$$

(b) Which of the following functionals of

$$(\textbf{State} \hookrightarrow \textbf{State}) \to (\textbf{State} \hookrightarrow \textbf{State})$$

are monotone:

$$\begin{aligned} F_0\, g &= g \\ F_1\, g &= \begin{cases} g_1 & \text{if } g = g_2 \\ g_2 & \text{otherwise} \end{cases} \\ (F_2\, g)s &= \begin{cases} g\, s & \text{if } s\, x \neq 0 \\ s & \text{if } s\, x = 0 \end{cases} \end{aligned}$$

(c) Show that F_2 is even continuous.

Exercise 2.7.

Assume that $(D, \preccurlyeq)$ and $(D', \preccurlyeq')$ are ccpo's, and assume that function $f : D \to D'$ satisfies

$$\bigsqcup{}' \{f\, d \mid d \in Y\} = f\Big(\bigsqcup Y\Big)$$

for all non-empty chains Y of D. Show that f is monotone.