More about the Polynomial Hierarchy
The Polynomial Hierarchy Three Ways

We discovered a hierarchy of complexity classes between P and PSpace, with NP and coNP on the first level, and infinitely many further levels above:

**Definition by ATM:** Classes $\Sigma_i^P/\Pi_i^P$ are defined by polytime ATMs with bounded types of alternation, starting computation with existential/universal states.

**Definition by Verifier:** Classes $\Sigma_i^P/\Pi_i^P$ are given as projections of certain verifier languages in P, requiring existence/universality of polynomial witnesses.

**Definition by Oracle:** Classes $\Sigma_i^P/\Pi_i^P$ are defined as languages of NP/coNP oracle TMs with $\Sigma_{i-1}^P$ (or, equivalently, $\Pi_{i-1}^P$) oracle.

Using such oracles with deterministic TMs, we can also define classes $\Delta_i^P$. 
More Classes in PH

We defined $\Sigma^P_k$ and $\Pi^P_k$ by relativising NP and coNP with oracles.

What happens if we start from P instead?

**Definition 18.1:**

$\Delta^P_0 = P$

$\Delta^P_k + 1 = P^{\Sigma^P_k}$

Some immediate observations:

- $\Delta^P_1 = P^{P}$
- $\Delta^P_2 = P^{NP}$
- $\Delta^P_k \subseteq \Sigma^P_k$ (since $P \subseteq NP$) and $\Delta^P_k \subseteq \Pi^P_k$ (since $P \subseteq coNP$)
- $\Sigma^P_k \subseteq \Delta^P_k + 1$ and $\Pi^P_k \subseteq \Delta^P_k + 1$
More Classes in PH

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- $\Delta^P_k \subseteq \Sigma^P_k$ (since $P \subseteq NP$) and $\Delta^P_k \subseteq \Pi^P_k$ (since $P \subseteq coNP$)
- $\Sigma^P_k \subseteq \Delta^P_{k+1}$ and $\Pi^P_k \subseteq \Delta^P_{k+1}$
Problems for $\Delta^P_k$?

$\Delta^P_k$ seems to be less common in practice, but there are some known complete problems for $P^{NP} = \Delta^P_2$:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>UNIQUELY OPTIMAL TSP</strong> [Papadimitriou, JACM 1984]</td>
<td>Input: Undirected graph $G$ with edge weights (distances). Problem: Is there exactly one shortest travelling salesman tour on $G$?</td>
</tr>
<tr>
<td><strong>DIVISIBLE TSP</strong> [Krentel, JCSS 1988]</td>
<td>Input: Undirected graph $G$ with edge weights; number $k$. Problem: Is the shortest travelling salesman tour on $G$ divisible by $k$?</td>
</tr>
<tr>
<td><strong>ODD FINAL SAT</strong> [Krentel, JCSS 1988]</td>
<td>Input: Propositional formula $\varphi$ with $n$ variables. Problem: Is $X_n$ true in the lexicographically last assignment satisfying $\varphi$?</td>
</tr>
</tbody>
</table>
Is the Polynomial Hierarchy Real?

Questions:

...
Is the Polynomial Hierarchy Real?

Questions:

Are all of these classes really distinct?

Nobody knows.

Are any of these classes really distinct?

Nobody knows.

Are any of these classes distinct from $P$?

Nobody knows.

What do we know then?
Is the Polynomial Hierarchy Real?

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\[
\begin{align*}
\Sigma^P_k & = \text{NP}^\text{NP} \\
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\Delta^P_k & = \text{P}^\text{NP} \\
\Sigma^P_1 & = \text{NP} \\
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\Delta^P_0 & = \Sigma^P_0 = \Pi^P_0 = \Delta^P_1 = \text{P}
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What do we know then?

Markus Krötzsch, 17th Dec 2018
Complexity Theory slide 6 of 27
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\Sigma_0^P &= \Sigma_0^P = \Pi_0^P = \Delta_0^P = P \\
\Delta_1^P &= \Sigma_1^P = \Pi_1^P = \text{coNP} \\
\Pi_2^P &= \text{coNP}^{NP} \\
\Sigma_2^P &= \text{NP}^{NP} \\
\Delta_2^P &= \text{P}^{NP} \\
\Sigma_k^P &\quad \Pi_k^P \\
\Delta_k^P &\quad \text{PH}
\end{align*}
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What do we know then?
**Theorem 18.2:** If there is any $k$ such that $\Sigma^P_k = \Sigma^P_{k+1}$ then $\Sigma^P_j = \Pi^P_j = \Sigma^P_k$ for all $j > k$, and therefore $\text{PH} = \Sigma^P_k$.

In this case, we say that the polynomial hierarchy collapses at level $k$.

**Proof:** Left as exercise (not too hard to get from definitions). $\square$
**Theorem 18.2:** If there is any $k$ such that $\Sigma^P_k = \Sigma^P_{k+1}$ then $\Sigma^P_j = \Pi^P_j = \Sigma^P_k$ for all $j > k$, and therefore $\text{PH} = \Sigma^P_k$.

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**Corollary 18.3:** If $\text{PH} \neq P$ then $\text{NP} \neq P$.

Intuitively speaking: “The polynomial hierarchy is built upon the assumption that NP has some additional power over P. If this is not the case, the whole hierarchy collapses.”
**Theorem 18.4:** $\text{PH} \subseteq \text{PSpace}$. 

**Proof:** Left as exercise (induction over PH levels, using that $\text{PSpace}^{\text{PSpace}} = \text{PSpace}$). □
Theorem 18.4: \( \text{PH} \subseteq \text{PSpace} \).

Proof: Left as exercise (induction over PH levels, using that \( \text{PSpace}^\text{PSpace} = \text{PSpace} \)). □

Theorem 18.5: If \( \text{PH} = \text{PSpace} \) then there is some \( k \) with \( \text{PH} = \Sigma^P_k \).

Proof: If \( \text{PH} = \text{PSpace} \) then \( \text{TRUE QBF} \in \text{PH} \). Hence \( \text{TRUE QBF} \in \Sigma^P_k \) for some \( k \). Since \( \text{TRUE QBF} \) is PSpace-hard, this implies \( \Sigma^P_k = \text{PSpace} \). □
“Most experts” think that:

- The polynomial hierarchy does not collapse completely (same as $P \neq NP$)
- The polynomial hierarchy does not collapse on any level
  (in particular $PH \neq PSpace$ and there is no PH-complete problem)

But there can always be surprises . . .
Computing with Circuits
Motivation

One might imagine that \( P \neq NP \), but \( \text{Sat} \) is tractable in the following sense: for every \( \ell \) there is a very short program that runs in time \( \ell^2 \) and correctly treats all instances of size \( \ell \). – Karp and Lipton, 1982
Motivation

One might imagine that P ≠ NP, but **SAT** is tractable in the following sense: for every \( \ell \) there is a very short program that runs in time \( \ell^2 \) and correctly treats all instances of size \( \ell \). – Karp and Lipton, 1982

**Some questions:**

- Even if it is hard to find a universal algorithm for solving all instances of a problem, couldn’t it still be that there is a simple algorithm for every fixed problem size?
- What can complexity theory tell us about parallel computation?
- Are there any meaningful complexity classes below LogSpace? Do they contain relevant problems?
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$\sim$ circuit complexity provides some answers

Intuition: use circuits with logical gates to model computation
A Boolean circuit is a finite, directed, acyclic graph where

- each node that has no predecessor is an input node
- each node that is not an input node is one of the following types of logical gate:
  - AND with two input wires
  - OR with two input wires
  - NOT with one input wire
- one or more nodes are designated output nodes

The outputs of a Boolean circuit are computed in the obvious way from the inputs.

\[ \text{circuits with } k \text{ inputs and } \ell \text{ outputs represent functions } \{0, 1\}^k \rightarrow \{0, 1\}^{\ell} \]

We often consider circuits with only one output.
Example 1
Example 1

XOR function:
Example 2

Parity function with four inputs:

(true for odd number of 1s)
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Propositional formulae

- propositional formulae are special circuits:
  each non-input node has only one outgoing wire
- each variable corresponds to one input node
- each logical operator corresponds to a gate
- each sub-formula corresponds to a wire

\[((\neg x_1 \land x_2) \lor (x_1 \land \neg x_2))\]
Alternative Ways of Viewing Circuits (2)

Straight-line programs

- are programs without loops and branching (if, goto, for, while, etc.)
- that only have Boolean variables
- and where each line can only be an assignment with a single Boolean operator

\[ \sim n \]-line programs correspond to \( n \)-gate circuits

\[
\begin{align*}
01 & \quad z_1 := \neg x_1 \\
02 & \quad z_2 := \neg x_2 \\
03 & \quad z_3 := z_1 \land x_2 \\
04 & \quad z_4 := z_2 \land x_1 \\
05 & \quad \text{return} \quad z_3 \lor z_4
\end{align*}
\]
Example: Generalised AND

The function that tests if all inputs are 1 can be encoded by combining binary AND gates:

- works similarly for OR gates
- number of gates: $n - 1$
- we can use $n$-way AND and OR (keeping the real size in mind)
Solving Problems with Circuits

Circuits are not universal: they have a fixed number of inputs!
How can they solve arbitrary problems?

**Definition 18.7:**
A circuit family is an infinite list \( C = C_1, C_2, C_3, \ldots \) where each \( C_i \) is a Boolean circuit with \( i \) inputs and one output.

We say that \( C \) decides a language \( L \) (over \( \{0, 1\} \)) if \( w \in L \) if and only if \( C_n(w) = 1 \) for \( n = |w| \).

**Example 18.8:**
The circuits we gave for generalised AND are a circuit family that decides the language \( \{1^n | n \geq 1\} \).
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**Example 18.8:** The circuits we gave for generalised AND are a circuit family that decides the language $\{1^n \mid n \geq 1\}$. 
Circuit Complexity

To measure difficulty of problems solved by circuits, we can count the number of gates needed:

**Definition 18.9:** The *size* of a circuit is its number of gates.

Let $f : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function. A circuit family $C$ is *$f$-size bounded* if each of its circuits $C_n$ is of size at most $f(n)$.

$\text{Size}(f(n))$ is the class of all languages that can be decided by an $O(f(n))$-size bounded circuit family.

**Example 18.10:** Our circuits for generalised AND show that $\{1^n \mid n \geq 1\} \in \text{Size}(n)$. 
Many simple operations can be performed by circuits of polynomial size:

- Boolean functions such as parity (=sum modulo 2), sum modulo $n$, or majority
- Arithmetic operations such as addition, subtraction, multiplication, division (taking two fixed-arity binary numbers as inputs)
- Many matrix operations

See exercise for some more examples
Polynomial Circuits
A natural class of problems to consider are those that have polynomial circuit families:

**Definition 18.11:** $P_{\text{poly}} = \bigcup_{d \geq 1} \text{Size}(n^d)$.

**Note:** A language is in $P_{\text{poly}}$ if it is solved by some polynomial-sized circuit family. There may not be a way to compute (or even finitely represent) this family.

How does $P_{\text{poly}}$ relate to other classes?
Theorem 18.12: For $f(n) \geq n$, we have $\text{DTIME}(f) \subseteq \text{SIZE}(f^2)$. 

Proof sketch (see also Sipser, Theorem 9.30)

• We can represent the DTIME computation as in the proof of Theorem 16.10: as a list of configurations encoded as words $\sigma_1 \cdots \sigma_i - 1 \langle q, \sigma_i \rangle \sigma_i + 1 \cdots \sigma_m \ast$ of symbols from the set $\Omega = \{\ast\} \cup \Gamma \cup (Q \times \Gamma)$.

Tableau (i.e., grid) with $O(f^2)$ cells.

• We can describe each cell with a list of bits (wires in a circuit).

• We can compute one configuration from its predecessor by $O(f)$ circuits (idea: compute the value of each cell from its three upper neighbours as in Theorem 16.10).

• Acceptance can be checked by assuming that the TM returns to a unique configuration position/state when accepting $\square$. 

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From $\text{DTime}(f) \subseteq \text{Size}(f^2)$ we get:

**Corollary 18.13:** $P \subseteq \text{P/poly}$. 
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**Corollary 18.13:** $P \subseteq P/\text{poly}$.

This suggests another way of approaching the $P$ vs. $NP$ question:

If any language in $NP$ is not in $P/\text{poly}$, then $P \neq NP$.

(but nobody has found any such language yet)
**Circuit-Sat**

**Input:** A Boolean Circuit $C$ with one output.

**Problem:** Is there any input for which $C$ returns 1?

Theorem 18.14: Circuit-Sat is NP-complete.

Proof: Inclusion in NP is easy (just guess the input).

For NP-hardness, we use that NP problems are those with a P-verifier:

- The DTM simulation of Theorem 18.12 can be used to implement a verifier (input: $(w\#c)$ in binary)
- We can hard-wire the $w$-inputs to use a fixed word instead (remaining inputs: $c$)
- The circuit is satisfiable iff there is a certificate for which the verifier accepts $w$

Note: It would also be easy to reduce Sat to Circuit-Sat, but the above yields a proof from first principles.
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A New Proof for Cook-Levin

**Theorem 18.15**: \(3\text{Sat}\) is NP-complete.

Proof:

Membership in NP is again easy (as before).

For NP-hardness, we express the circuit that was used to implement the verifier in Theorem 18.14 as propositional logic formula in 3-CNF:

- Create a propositional variable \(X_i\) for every wire in the circuit
- Add clauses to relate input wires to output wires, e.g., for AND gate with inputs \(X_1\) and \(X_2\) and output \(X_3\), we encode \((X_1 \land X_2) \iff X_3\) as:
  \[
  (\neg X_1 \lor \neg X_2 \lor X_3) \land (X_1 \lor \neg X_3) \land (X_2 \lor \neg X_3)
  \]
- Fixed number of clauses per gate = constant factor size increase
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  \]

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- Add a clause \((X)\) for the output wire \(X\)
Summary and Outlook

We do not know if the **Polynomial Hierarchy** is real or collapses

Circuits provide an alternative model of computation

\[ P \subseteq P_{/\text{poly}} \]

**Circuit-Sat** is NP-complete.

**What's next?**

- Circuits for parallelism
- Complexity classes (strictly!) below P
- Randomness