TECHNISCHE UNIVERSITÄT DRESDEN

DATABASE THEORY

Lecture 9: First-Order Expressiveness / Introduction to Datalog

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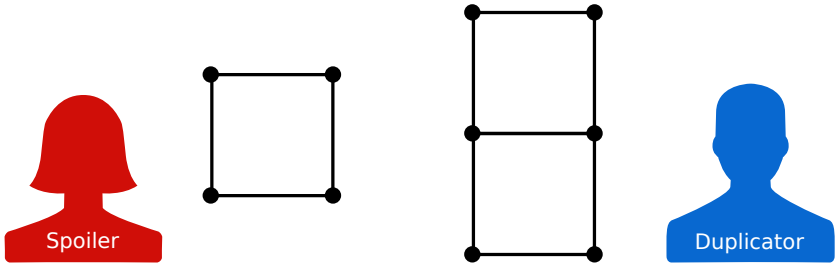
Overview

See course homepage [⇒ link] for more information and materials

Review: EF Games

Ehrenfeucht-Fraïssé games characterise expressivity of FO formulas:

- the quantifier rank needed to distinguish structure corresponds to
- the number of rounds needed by Spoiler to win the game



Using EF Games to Show FO-Undefinability

How to show that a query mapping M can not be FO-defined:

- Let C_M be the class of all databases recognised by M
- Find sequences of databases $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \ldots \in C_M$ and databases $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \ldots \notin C_M$, such that $\mathcal{I}_i \sim_i \mathcal{J}_i$

$\rightsquigarrow$ for any formula φ (however large its quantifier rank r), there is a counterexample $\mathcal{I}_r \in C_M$ and $\mathcal{J}_r \notin C_M$ that φ cannot distinguish

Problems:

- How to find such sequences of $\mathcal{I}_i$ and $\mathcal{J}_i$?
  $\rightsquigarrow$ No general strategy exists
- Given suitable sequences, how to show that $\mathcal{I}_i \sim_i \mathcal{J}_i$?
  $\rightsquigarrow$ Can be difficult, but doable for some special cases

Expressiveness on Linear Orders

Let's look at some very simple structures:

Definition

A structure $\mathcal{I}$ is a linear order if it has a single binary predicate $\leq$ interpreted as a total, transitive, reflexive and asymmetric relation.

Example:
$\mathcal{L}_6 : 1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6$
$\mathcal{L}_7 : 1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6 \leq 7$

Spoiler can win the 3-round EF game:

Spoiler plays 4 in $\mathcal{L}_7$
  Duplicator plays 4 in $\mathcal{L}_6$: Spoiler plays 6 in $\mathcal{L}_7$
    Duplicator plays 5 in $\mathcal{L}_6$: Spoiler plays 5 in $\mathcal{L}_7$ and wins
    Duplicator plays 6 in $\mathcal{L}_6$: Spoiler plays 7 in $\mathcal{L}_7$ and wins
  Duplicator plays 3 in $\mathcal{L}_6$: symmetric game (flipped horizontally)

Expressiveness on Linear Orders

Let's look at some very simple structures:

Definition

A structure $\mathcal{I}$ is a linear order if it has a single binary predicate $\leq$ interpreted as a total, transitive, reflexive and asymmetric relation.

Example:
$\mathcal{L}_7 : 1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6 \leq 7$
$\mathcal{L}_8 : 1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6 \leq 7 \leq 8$

Spoiler cannot win the 3-round EF game:

Spoiler plays 4 in $\mathcal{L}_8$: Duplicator plays 4 in $\mathcal{L}_7$
  Spoiler plays 6 in $\mathcal{L}_8$: Duplicator plays 6 in $\mathcal{L}_7$; spoiler cannot win
  Spoiler plays 7 in $\mathcal{L}_8$: Duplicator plays 6 in $\mathcal{L}_7$; spoiler cannot win
Other cases similar: Spoiler never wins

EF Games and Linear Orders

Theorem

The following are equivalent:

- $\mathcal{L}_m \sim_r \mathcal{L}_n$
- either (1) $m = n$, or (2) $m \geq 2^r - 1$ and $n \geq 2^r - 1$

Proof: see board

FO-Definability of PARITY

Theorem

PARITY is not FO-definable for linear orders, hence it is not FO-definable for arbitrary databases.

Proof:

- Suppose for a contradiction that PARITY is FO-definable by some query φ.
- Let r be the quantifier rank of φ.
- Consider databases $\mathcal{L}_m$ and $\mathcal{L}_n$ with $m = 2^r$ and $n = 2^r + 1$.
- We know that $\mathcal{L}_m \sim_r \mathcal{L}_n$, and therefore $\mathcal{L}_m \equiv_r \mathcal{L}_n$.
- Hence, $\mathcal{L}_m \models \varphi$ if and only if $\mathcal{L}_n \models \varphi$.
- But $\mathcal{L}_m \in$ PARITY while $\mathcal{L}_n \notin$ PARITY.
- Therefore, φ does not FO-define PARITY. Contradiction.

FO-Definability of Connectivity

The Connectivity problem over finite graphs is as follows:

- Input: A finite graph (relational structure with one binary relation “edge”)
- Output: “true” if there is an (undirected) path between any pair of vertices

Theorem

Connectivity is not FO-definable.

Proof:

- Suppose for a contradiction that Connectivity is FO-definable using a query φ.
- We show that this would make Parity FO-definable on linear orders.
- For a linear order $\mathcal{L}$ with order predicate $\leq$, we define a finite graph $\mathcal{G}(\mathcal{L})$ over a binary predicate “edge” such that $\mathcal{G}(\mathcal{L})$ is connected if and only if $\mathcal{L}$ has an even number of elements.

Defining a Graph From a Linear Order

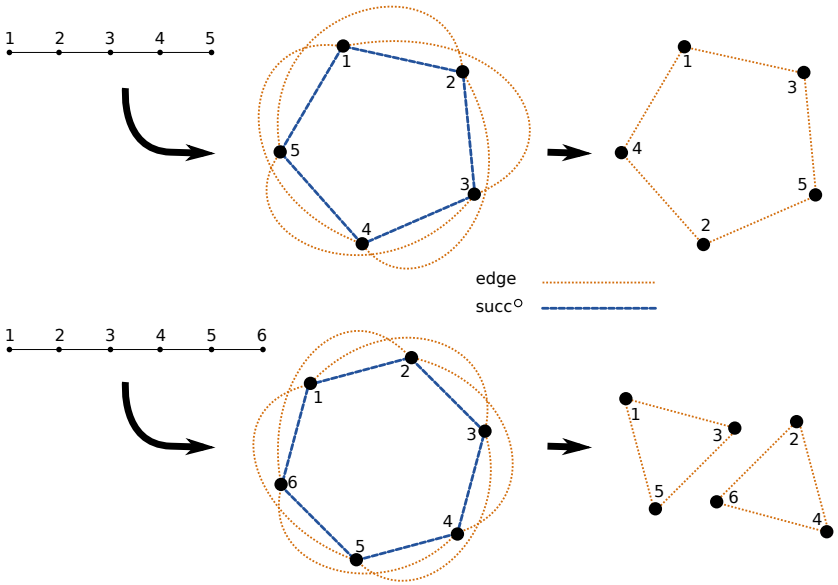
We use abbreviations for the following FO formulas:

$$\begin{aligned}
\text{succ}[x,y] &= (x \leq y) \wedge \neg(y \leq x) \wedge \forall z.(z \leq x \vee y \leq z) && \text{y is the successor of x}\\
\min[x] &= \forall z.x \leq z && \text{x is the first element}\\
\max[x] &= \forall z.z \leq x && \text{x is the last element}\\
\text{succ}^{\circ}[x,y] &= \text{succ}[x,y] \vee (\max[x] \wedge \min[y]) && \text{circular version of succ}
\end{aligned}$$

We now define the formula ψ that derives edges from a linear order:

$$\forall x, y.\text{edge}(x,y) \leftrightarrow \exists z.\text{succ}^{\circ}[x,z] \wedge \text{succ}^{\circ}[z,y]$$

Illustration: Graphs From Linear Orders



Completing the Proof

Observation:
The graph $\mathcal{G}(\mathcal{L})$ is connected if and only if $\mathcal{L}$ has odd parity.

Therefore, if φ FO-defines Connectivity on graphs with predicate edge, then $\neg(\varphi \wedge \psi)$ FO-defines Parity on linear orders.

Since Parity is not FO-definable, no such φ can exist.

Beyond Linear Orders: Locality

Intuition: Duplicator can win an EF game if selected nodes have the same "neighbourhood"
$\rightsquigarrow$ let's define this for graphs (structures with binary predicates)

Definition

Consider a graph $\mathcal{G}$. For a natural number $d \geq 0$ and a vertex v, the d-neighbourhood of v, $N(v, d)$, is defined inductively:

- $N(v, 0) = \{v\}$
- $N(v, d+1) = N(v, d) \cup$
  $\{w \mid w \text{ is a direct neighbour of some } w' \in N(v, d)\}$

Two vertices v and w have the same d-type if the subgraphs $\mathcal{G}|_{N(v,d)}$ and $\mathcal{G}|_{N(w,d)}$ are isomorphic.
Two graphs are d-equivalent if, for every d-type, they have the same number of d-neighbourhoods of this type.

Locality and FO-definability

A special case of Gaifman's Locality Theorem of first-order logic:

Theorem

For every integer $r \geq 1$:

- if $\mathcal{G}_1$ is 3^{r-1}-equivalent to $\mathcal{G}_2$
- then $\mathcal{G}_1 \sim_r \mathcal{G}_2$, and thus $\mathcal{G}_1 \equiv_r \mathcal{G}_2$

$\rightsquigarrow$ Intuition: FO can only express local properties

How to show that a query mapping M can not be FO-defined:

- Let C_M be the class of all databases recognised by M
- Find sequences of graphs $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \ldots \in C_M$ and graphs $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \ldots \notin C_M$, such that $\mathcal{I}_i$ is i-equivalent to $\mathcal{J}_i$

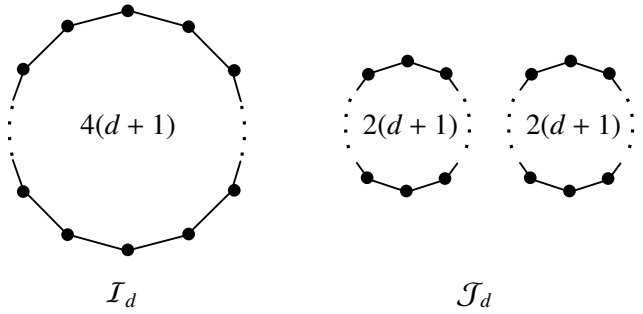
$\rightsquigarrow$ for any formula φ (however large its quantifier rank r), there is a counterexample $\mathcal{I}_{3^{r-1}} \in C_M$ and $\mathcal{J}_{3^{r-1}} \notin C_M$ that φ cannot distinguish

Connectivity is not FO-definable (Proof 2)

Theorem

Connectivity is not FO-definable.

Proof: counterexample for quantifier rank r: set $d = 3^r$



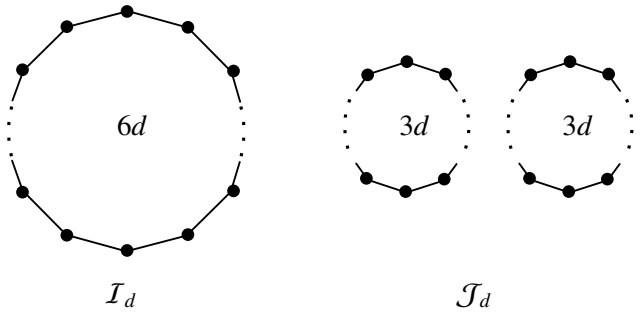
- the only d-type is a path of $2d + 1$ nodes
- $\mathcal{I}_d$ and $\mathcal{J}_d$ are d-equivalent

2-Colourability

Theorem

2-Colourability is not FO-definable.

Proof: counterexample for quantifier rank r: set $d = 3^r$ (odd number)

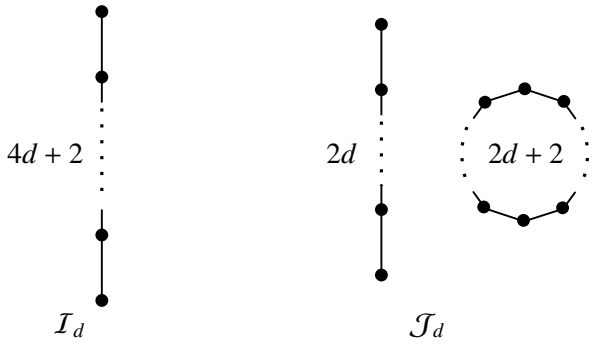


- the only d-type is a path of $2d + 1$ nodes
- $\mathcal{I}_d$ and $\mathcal{J}_d$ are d-equivalent

Theorem

ACYCLICITY is not FO-definable.

Proof: counterexample for quantifier rank r: set $d = 3^r$



- d-types are paths of $\leq 2d + 1$ nodes
- $\mathcal{I}_d$ and $\mathcal{J}_d$ are d-equivalent

Summary: Limits of FO-Queries

FO queries (and hence Relational Calculus) cannot express properties that require a "global" view:

- properties where one needs to follow paths
- properties where one needs to count elements

Remember Lecture 1?

"Stops at distance 2 from Helmholtzstr."

$$R_2 = \delta_{\mathsf{To}\to\mathsf{From}}(\pi_{\mathsf{To}}(\mathsf{Connect} \bowtie R_1))$$

What about all stops reachable from Helmholtzstr.?

$\rightsquigarrow$ Not expressible in Relational Calculus

Yet, all examples we saw are in P

$\rightsquigarrow$ Is there another query language that could help us?

Introduction to Datalog

Introduction to Datalog

Datalog introduces recursion into database queries

- Use deterministic rules to derive new information from given facts
- Inspired by logic programming (Prolog)
- However, no function symbols and no negation
- Studied in AI (knowledge representation) and in databases (query language)

Example: transitive closure C of a binary relation r

$$C(x, y) \leftarrow r(x, y)$$
$$C(x, z) \leftarrow C(x, y) \wedge r(y, z)$$

Intuition:

- some facts of the form $r(x, y)$ are given as input, and the rules derive new conclusions $C(x, y)$
- variables range over all possible values (implicit universal quantifier)

Syntax of Datalog

Recall: A term is a constant or a variable. An atom is a formula of the form $R(t_1, \ldots, t_n)$ with R a predicate symbol (or relation) of arity n, and $t_1, \ldots, t_n$ terms.

Definition

A Datalog rule is an expression of the form:

$$H \leftarrow B_1 \wedge \ldots \wedge B_m$$

where H and $B_1, \ldots, B_m$ are atoms. H is called the head or conclusion; $B_1 \wedge \ldots \wedge B_m$ is called the body or premise. A rule with empty body ($m = 0$) is called a fact. A ground rule is one without variables (i.e., all terms are constants).

A set of Datalog rules is a Datalog program.

Datalog: Example

$$\begin{aligned}
&\text{father(alice, bob)}\\
&\text{mother(alice, carla)}\\
&\text{mother(evan, carla)}\\
&\text{father(carla, david)}\\
\text{Parent}(x, y) &\leftarrow \text{father}(x, y)\\
\text{Parent}(x, y) &\leftarrow \text{mother}(x, y)\\
\text{Ancestor}(x, y) &\leftarrow \text{Parent}(x, y)\\
\text{Ancestor}(x, z) &\leftarrow \text{Parent}(x, y) \wedge \text{Ancestor}(y, z)\\
\text{SameGeneration}(x, x)&\\
\text{SameGeneration}(x, y) &\leftarrow \text{Parent}(x, v) \wedge \text{Parent}(y, w) \wedge \text{SameGeneration}(v, w)
\end{aligned}$$

Datalog Semantics by Deduction

What does a Datalog program express?
Usually we are interested in entailed ground atoms

What can be entailed? Informally:

- Restrict to set of constants that occur in program (finite) $\rightsquigarrow$ universe $\mathcal{U}$
- Variables can represent arbitrary constants from this set $\rightsquigarrow$ ground substitutions map variables to constants
- A rule can be applied if its body is satisfied for some ground substitution
  Example: rule $\text{Parent}(x, y) \leftarrow \text{mother}(x, y)$ can be applied to mother(alice, carla) under substitution $\{x \mapsto \text{alice}, y \mapsto \text{carla}\}$
- If a rule is applicable under some ground substitution, then the according instance of the rule head is entailed.

Datalog Semantics by Deduction (2)

An inductive definition of what can be derived:

Definition

Consider a Datalog program P. The set of ground atoms that can be derived from P is the smallest set of atoms A for which there is a rule $H \leftarrow B_1 \wedge \ldots \wedge B_n$ and a ground substitution θ such that

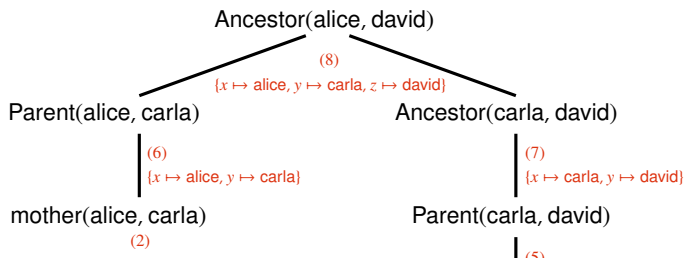
- $A = H\theta$, and
- for each $i \in \{1, \ldots, n\}$, $B_i\theta$ can be derived from P.

Notes:

- $n = 0$ for ground facts, so they can always be derived (induction base)
- if variables in the head do not occur in the body, they can be any constant from the universe

Datalog Deductions as Proof Trees

We can think of deductions as tree structures:



(1) father(alice, bob)
(2) mother(alice, carla)
(3) mother(evan, carla)
(4) father(carla, david)
(5) $\text{Parent}(x, y) \leftarrow \text{father}(x, y)$
(6) $\text{Parent}(x, y) \leftarrow \text{mother}(x, y)$
(7) $\text{Ancestor}(x, y) \leftarrow \text{Parent}(x, y)$
(8) $\text{Ancestor}(x, z) \leftarrow \text{Parent}(x, y) \wedge \text{Ancestor}(y, z)$

Datalog Semantics by Least Fixed Point

Instead of using substitutions, we can also ground programs:

Definition

The grounding ground(P) of a Datalog program P is the set of all ground rules that can be obtained from rules in P by uniformly replacing variables with constants from the universe.

Derivations are described by the immediate consequence operator T_P that maps sets of ground facts I to sets of ground facts $T_P(I)$:

- $T_P(I) = \{H \mid H \leftarrow B_1 \wedge \ldots \wedge B_n \in \text{ground}(P) \text{ and } B_1, \ldots, B_n \in I\}$
- Least fixed point of T_P: smallest set L such that $T_P(L) = L$
- Bottom-up computation: $T_P^0 = \emptyset$ and $T_P^{i+1} = T_P(T_P^i)$
- The least fixed point of T_P is $T_P^\infty = \bigcup_{i \geq 0} T_P^i$ (exercise)

Observation: Ground atom A is derived from P if and only if $A \in T_P^\infty$

Datalog Semantics by Least Model

We can also read Datalog rules as universally quantified implications

Example: $\text{Ancestor}(x, z) \leftarrow \text{Parent}(x, y) \wedge \text{Ancestor}(y, z)$ corresponds to implication

$$\forall x, y, z. \text{Parent}(x, y) \wedge \text{Ancestor}(y, z) \rightarrow \text{Ancestor}(x, z).$$

A set of FO implications may have many models
$\rightsquigarrow$ consider least model over the domain defined by the universe

Theorem

A fact is entailed by the least model of a Datalog program if and only if it can be derived from the Datalog program.

Datalog Semantics: Overview

There three equivalent ways of defining Datalog semantics:

- Proof-theoretic: What can be proven deductively?
- Operational: What can be computed bottom up?
- Model-theoretic: What is true in the least model?

In each case, we restrict to the universe of given constants.
$\rightsquigarrow$ similar to active domain semantics in databases

Datalog as a Query Language

How can we use Datalog to query databases?
$\rightsquigarrow$ View database as set of ground facts
$\rightsquigarrow$ Specify which predicate yields the query result

Definition

A Datalog query is a pair $\langle R, P \rangle$, where P is a Datalog program and R is the answer predicate.
Results of the query: R-facts entailed by P

Datalog queries distinguish "given" relations from "derived" ones:

- predicates that occur in a head of P are intensional database (IDB) predicates
- predicates that only occur in bodies are extensional database (EDB) predicates

Requirement: database relations used as EDB predicates only

Datalog as a Generalisation of CQs

A conjunctive query $\exists y_1, \ldots, y_m . A_1 \wedge \ldots \wedge A_\ell$ with answer variables $x_1, \ldots, x_n$ can be expressed as a Datalog query $\langle \text{Ans}, P \rangle$ where P has the single rule:

$$\text{Ans}(x_1, \ldots, x_n) \leftarrow A_1 \wedge \ldots \wedge A_\ell$$

Unions of CQs can also be expressed (exercise)

Intuition: Datalog generalises UCQs with recursion

Open questions:

- How hard is it to answer Datalog queries?
- Can Datalog express all queries in P?
- What about query containment and equivalence?

Summary and Outlook

FO-queries can only express "local" properties

Possible proof techniques:

- Ehrenfeucht-Fraïssé Games
- Locality Theorems
- For more approaches see Chapter 17 of [Abiteboul, Hull, Vianu 1994]

Datalog can overcome some of these limitations

Next topics:

- Complexity and expressive power of Datalog
- Implementation techniques for Datalog