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Context Orbifolds

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Preface

From the very beginning one of the core interests of mathematics and mathematicians was to study regularity and symmetry. In modern mathematics the concept of symmetry has been condensed into the notion of *automorphisms*, bijective mappings which preserve the underlying structure, allowing a careful study of what symmetry is and can be. One of the views symmetry allows on itself is the one of redundancy: a huge structure with sufficiently enough symmetry can be composed from a small one together with the information *how* to create the huge from the small. This means that huge structures with symmetry can be *folded* to a small part of what they have been before together with a concise representation of their symmetry. This idea is by far not a new one and indeed the information how to construct the huge structure from the small one is very crucial, giving rise to one of the most fundamental concepts of modern mathematics, to that of a *group*.

The aim of this work is to study the possibilities of this idea applied to formal concept analysis and in particular to concept lattices and contexts. We want to examine whether it is possible and practicable to consider lattices and contexts with symmetry and fold them to a small representation. We then want to ask what properties these structures may have and which properties of the original structure they keep. And of course whether we are able to *unfold* the folded structures to give the original structures again.

The idea of folding is, at least from a mathematical point of view, very intuitive, but needs some basic ideas from group theory and in particular from the theory of permutation groups. After introducing the notions needed we shall firstly investigate on folding concept lattices or, more generally, preordered sets. As it turns out preordered sets form a suitable basis for studying the idea of folding and simultaneously allow an intuitive and concise graphical representation by means of a slight generalization of order diagrams. The abstract structures which arise here will be called *preorder orbifolds*, where the word "orbifold" is borrowed from algebraic topology, where it describes manifolds folded by orbits of certain functions.[1]

After we have considered preordered sets we shall try to transform the results we have achieved to formal contexts to get *context orbifolds*. They allow, as formal contexts do, a certain form of *derivation*, which will give us the possibility, at least in theory, to link together context orbifolds and concept lattice orbifolds. Thus we will be able to compute the one from the other and vice versa.

At the end we shall have a precise and formal understanding of what we mean when talking about folding structures by automorphisms. This knowledge might or might not help to understand certain concept lattices or contexts, provided that they yield enough symmetry.

[1] Note that however we shall use the term "orbifold" in a different manner and those two mathematical ideas are not directly linked to each other.

Contents

1 Preorder Orbifolds

We start by formalizing a very natural idea of "folding preorders by automorphism": Given a preordered set $\underline{P} = (P, \leq)$ and a group of automorphisms Γ of $\underline{P}$ we could fold $\underline{P}$ such that

- from every orbit of Γ on P we take one representative and
- for each two representatives a and b we memorize all automorphisms β such that $a \leq \beta(b)$. We shall denote the set of all these mappings with $\lambda(a, b)$.

We formalize all this in the notion of *preorder orbifolds*. Then we shall see that preorder orbifolds share a common structure called *group-annotated preordered sets* for which we can define the notion of *isomorphy*. With this we can prove that the arbitrary choice of representatives will not produce different preorder orbifolds up to isomorphy. Having this we shall define *unfolding* of group-annotated preordered sets in such a way that unfolding a folding of a preordered set delivers an isomorphic copy of the original and unfolding of isomorphic group-annotated preordered sets yields isomorphic preordered sets.

Note that this chapter is based on [Zw], but we restrict ourself to the case of preordered sets to have a formal basis for preorder orbifolds. This will be needed for concept lattice orbifolds. For context orbifolds the more general notion of binary relation structure orbifolds is then needed.

1.1 Basic Prerequisites

We start with some basic definitions and notational conventions. Let M be a set and $\underline{G}$ be a group acting on M, that is there exists a mapping

$$\begin{array}{rccc} \psi: & G \times M & \longrightarrow & M \\ & (g, m) & \longmapsto & gm \end{array}$$

with $\psi(gg', m) = \psi(g, \psi(g', m))$ and $\psi(e_G, m) = m$ for all $g, g' \in G$ and $m \in M$, where e_G is the neutral element of $\underline{G}$. Let $m \in M$. Then the *orbit of m under $\underline{G}$* is the set

$$G(m) := \{ gm \mid g \in G \}$$

and the *stabilizer of m under $\underline{G}$* is the set

$$G_m := \{ g \in G \mid gm = m \}.$$

The stabilizer of every element is the base set of a subgroup of $\underline{G}$. Also note that we write $\underline{G}$ for the group as structure, but simply G if we refer to the base set of $\underline{G}$, that is $\underline{G} = (G, \circ)$ where $\circ$ denotes the group operation of $\underline{G}$.

When we have a set M and a group $\underline{G}$ acting on M we can form the *set of all orbits of M under $\underline{G}$*

$$\underline{G} \setminus\!\setminus M = \{\, G(m) \mid m \in M \,\}.$$

We can also choose sets $Y \subseteq M$ such that for all $m \in M$ it is $|Y \cap G(m)| = 1$, i.e. Y is a set of *representatives of the orbits of $\underline{G}$ on M*. Those sets are also called *transversals* and the set of all transversals may be denoted by $\mathcal{T}(\underline{G} \setminus\!\setminus M)$.

Later we will need the notion of *preorder automorphisms*. This concept can be formulated in much more generality. For this let M be a set and $R \subseteq M \times M$.[1] Then the pair (M, R) is called a *(binary) relation structure*. Let (N, S) be another relation structure. A bijective mapping $\alpha : M \longrightarrow N$ is called a *relation isomorphism* if and only if

$$\forall x, y \in M : (x, y) \in R \iff (\alpha(x), \alpha(y)) \in S.$$

Then we also write $\alpha : (M, R) \longrightarrow (N, S)$. If (M, R) is a preordered set then we call α a preorder automorphism, likewise for ordered sets and lattices. Note that this means nothing more than α being a relation isomorphism but emphasizes the properties of the corresponding relation structures. If $(M, R) = (N, S)$ we call α a *relation automorphism* of (M, R). The set of all relation automorphisms of (M, R) is denoted by $\mathrm{Aut}(M, R)$ and forms a group under the composition

$$\alpha \circ \beta = (M, M, x \longmapsto \alpha(\beta(x))).$$

Note that the function application is from left and is denoted by $\circ$. Often $\circ$ will be the operation of a certain automorphism group and we may omit the explicit mentioning of the group operation if it is clear from the context which operation is meant. Furthermore functions $f : A \longrightarrow B : x \mapsto f(x)$ are denoted by the triple $f = (A, B, x \mapsto f(x))$ and the group of all relation automorphisms of (M, R) under function composition is denoted by $\underline{\mathrm{Aut}}(M, R)$.

1.2 Preorder Orbifolds and Group-annotated Preordered Sets

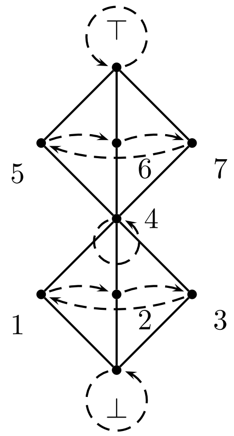
We first start by formalizing our idea of preorder orbifolds.

Definition 1.2.1 (Preorder Orbifolds) Let $\underline{P} = (P, \leq_P)$ be a preordered set and $\Gamma \leq \underline{\mathrm{Aut}}(\underline{P})$. Furthermore let Y be a transversal of the orbits of Γ on P. Then a *preorder orbifold (or representation) of $\underline{P}$ under* Γ is a quadruple

$$\mathrm{rep}_\Gamma(\underline{P}) := (Y, \leq_{\mathrm{rep}}, (\Gamma_y)_{y \in Y}, \lambda)$$

[1] It can be formulated even more general, that is for structures with more relations of arbitrary arity. The notion used here is then the one of *(bijective unary) polymorphisms*, i.e. permutations which *preserve* all relations on the set. The studying of polymorphisms and preserved relations is subject of the so called *clone theory*.

$\underline{L} =$



$\alpha = (\perp)(123)(4)(567)(\top)$, $\Gamma := \langle \alpha \rangle = (\{\,\mathrm{id}, \alpha, \alpha^2\,\}, \circ)$

Figure 1.1: Example lattice and an automorphism generating a subgroup of its automorphism group

where for $a, b \in Y$

$$a \leq_{\mathrm{rep}} b :\Longleftrightarrow \exists \beta \in \Gamma : a \leq_{\underline{P}} \beta(b)$$

and

$$\begin{array}{rccc} \lambda : & Y^2 & \longrightarrow & \mathfrak{P}(\Gamma) \\ & (a, b) & \longmapsto & \{\, \beta \in \Gamma \mid a \leq_{\underline{P}} \beta b \,\}. \end{array}$$

λ is then called a *(full) annotation function* and the relation structure $(Y, \leq_{\mathrm{rep}})$ is called the *base structure* of $\mathrm{rep}_\Gamma(\underline{P})$.

If $\underline{P}$ is a lattice (ordered set) we call $\mathrm{rep}_\Gamma(\underline{P})$ a *lattice (order) orbifold.* $\diamondsuit$

We may, if it is clear from the context which group Γ is meant, simply write the pair (Y, λ) for a preorder orbifold since the stabilizers and the relation $\leq_{\mathrm{rep}}$ can be reconstructed from this.

To convey a feeling for this notion we have a look at some simple examples.

Example 1.2.2 1) Let $\underline{L}$ be the lattice depicted in Figure 1.1. We want to compute a lattice orbifold of $\underline{L}$ under Γ. To do this, we choose the transversal $Y = \{\, \perp, 1, 4, 5, \top \,\}$

and get $\mathrm{rep}_\Gamma(\underline{L}) = (Y, \leq, (\Gamma_y)_{y\in Y}, \lambda)$ where

$$\begin{aligned}
&\lambda(\bot,\bot) = \Gamma_\bot = \Gamma && \lambda(\bot,1) = \Gamma \\
&\lambda(\bot,4) = \Gamma && \lambda(\bot,5) = \Gamma \\
&\lambda(\bot,\top) = \Gamma && \lambda(1,1) = \Gamma_1 = \{\,\mathrm{id}\,\} \\
&\lambda(1,4) = \Gamma && \lambda(1,5) = \Gamma \\
&\lambda(1,\top) = \Gamma && \lambda(4,4) = \Gamma_4 = \Gamma \\
&\lambda(4,5) = \Gamma && \lambda(4,\top) = \Gamma \\
&\lambda(5,5) = \Gamma_5 = \{\,\mathrm{id}\,\} && \lambda(5,\top) = \Gamma \\
&\lambda(\top,\top) = \Gamma_\top = \Gamma
\end{aligned}$$

and $\emptyset$ elsewhere. We see in this case that we actually do not need to carry along the stabilizers of the elements $y \in Y$ since we have

$$\lambda(y,y) = \Gamma_y$$

and we also observe that the relation $\leq$ is an order relation on Y.

2) We consider the ordered set $(\mathbb{Z}, \leq)$ and the automorphism $\alpha : \mathbb{Z} \longrightarrow \mathbb{Z} : x \longmapsto x + 2$. Then with $\Gamma = \langle\alpha\rangle$ we get

$$\Gamma \setminus\!\!\setminus \mathbb{Z} = \{\,\Gamma(0), \Gamma(1)\,\} = \{\,2\mathbb{Z}, 2\mathbb{Z}+1\,\} = \mathbb{Z}/2\mathbb{Z}.$$

Thus when choosing the transversal $Y = \{\,0,1\,\}$ we get for λ:

$$\begin{aligned}
\lambda(0,0) &= \{\,\alpha \in \Gamma \mid 0 \leq \alpha(0)\,\} = \{\,\alpha \in \Gamma \mid \alpha = (\mathbb{Z},\mathbb{Z}, x \longmapsto x+2k), k \geq 0\,\} \\
\lambda(0,1) &= \{\,\alpha \in \Gamma \mid 0 \leq \alpha(1)\,\} = \{\,\alpha \in \Gamma \mid \alpha = (\mathbb{Z},\mathbb{Z}, x \longmapsto x+2k), k \geq 0\,\} \\
\lambda(1,0) &= \{\,\alpha \in \Gamma \mid \alpha = (\mathbb{Z},\mathbb{Z}, x \longmapsto x+2k), k > 0\,\} \\
\lambda(1,1) &= \{\,\alpha \in \Gamma \mid \alpha = (\mathbb{Z},\mathbb{Z}, x \longmapsto x+2k), k \geq 0\,\}
\end{aligned}$$

and for the stabilizers Γ_0 and Γ_1

$$\begin{aligned}
\Gamma_0 &= \{\,\mathrm{id}\,\} \\
\Gamma_1 &= \{\,\mathrm{id}\,\}.
\end{aligned}$$

Here we see that stabilizers Γ_y are not redundant since in general they are different from $\lambda(y,y)$ and hence cannot be reconstructed from the annotation function λ. $\diamondsuit$

Preorder orbifolds have some properties which can be easily seen. The first one regards the map λ: given a preordered set $\underline{P} = (P, \leq)$, $\Gamma \leq \underline{\mathrm{Aut}}(\underline{P})$ and $a, b, c \in P$ such that $a \leq b \leq c$ we immediately have

$$\lambda(a,b) \circ \lambda(b,c) \subseteq \lambda(a,c)$$

because if we have $\alpha_1 \in \lambda(a,b)$ and $\alpha_2 \in \lambda(b,c)$ it is $a \leq \alpha_1(b)$ and $b \leq \alpha_2(c)$. Hence $a \leq \alpha_1(\alpha_2(c)) = \alpha_1 \circ \alpha_2(c)$ and $\alpha_1 \circ \alpha_2 \in \lambda(a,c)$.

A second property regards the relation $\leq_{\text{rep}}$. The name is not chosen arbitrarily since $\leq_{\text{rep}}$ will always be a preorder. If we further have a preordered set $\underline{P}$ and a group of automorphisms Γ where all orbits are antichains in $\underline{P}$ we can show that $\leq$ is indeed an order relation.

The third property we like to mention is that the intersection of all groups Γ_p for $p \in P$ is trivial, i.e.

$$\bigcap_{p\in P} \Gamma_p = \{\, \mathrm{id} \,\}$$

since the only automorphism having all points of P as fixpoints is the identity map. Choosing a transversal Y of $\Gamma \mathbin{\backslash\!\backslash} P$ we can write $P = \{\, \Gamma(y) \mid y \in Y \,\}$ and therefore for $p = \gamma(y) \in P$

$$\Gamma_p = \Gamma_{\gamma(y)} = \gamma \Gamma_y \gamma^{-1}$$

because $p = \delta(p)$ for some $\delta \in \Gamma$ implies $\gamma(y) = \delta(\gamma(y))$ and thus $y = \gamma^{-1}(\delta(\gamma(y)))$. Hence we have

$$\bigcap_{\gamma\in\Gamma, y\in Y} \gamma \Gamma_y \gamma^{-1} = \{\, \mathrm{id} \,\}.$$

To summarize all these observations we may formulate the following abstraction.

Definition 1.2.3 (Group-annotated Preordered Set) Let $\underline{G} = (G, \circ)$ be a group, $\underline{P} = (P, \leq)$ be a preordered set and

$$\lambda : P^2 \longrightarrow \mathfrak{P}(G)$$

such that

- $\lambda(a,b) = \emptyset$ if and only if $a \not\leq b$ and
- $\lambda(a,b) \circ \lambda(b,c) \subseteq \lambda(a,c)$ for all $a \leq b \leq c$ in P.

Furthermore let $\underline{G_p} \leq \underline{G}$ for every $p \in P$ such that $\underline{G_p} \subseteq \lambda(p,p)$ and

$$\bigcap_{p\in P, g\in G} g G_p g^{-1} = \{\, e_g \,\}$$

where e_G is the neutral element of G. Then the pair $((\underline{G_p})_{p\in P}, \lambda)$ is called a *$\underline{G}$-annotation of P* and the quadruple $(P, \leq, (\underline{G_p})_{p\in P}, \lambda)$ is called a *$\underline{G}$-annotated preordered set.* $\diamondsuit$

Of course we get the following result.

Proposition 1.2.4 *Let $(P, \leq, (\Gamma_p)_{p\in P}, \lambda)$ be a preorder orbifold under Γ. Then $(P, \leq, (\Gamma_p)_{p\in P}, \lambda)$ is a Γ-annotated preordered set.*

Proof Let $\underline{\bar{P}} = (\bar{P}, \leq_{\bar{P}})$ be a preordered set and $\Gamma \leq \underline{\mathrm{Aut}}(\underline{\bar{P}})$ such that $\mathrm{rep}(\underline{\bar{P}}) = (P, \leq, (\Gamma_p)_{p\in P}, \lambda)$. Then we have $P \subseteq \bar{P}$ and clearly $a \leq b \iff \exists \beta \in \Gamma : a \leq_{\bar{P}} \beta(b) \iff \lambda(a,b) \neq \emptyset$. For $a \in P$ it is $a \leq_{\bar{P}} a = \mathrm{id}(a)$ and therefore $a \leq a$. Furthermore for $a, b, c \in P$ with $a \leq b \leq c$ there exist $\beta_1, \beta_2 \in \Gamma$ such that $a \leq_{\bar{P}} \beta_1(b)$ and $b \leq_{\bar{P}} \beta_2(c)$. This gives $a \leq_{\bar{P}} \beta_1(\beta_2(c))$ and hence $a \leq c$. Therefore $(P, \leq)$ is a preordered set. Everything else has already been shown. □

One has to mention that group-annotated preordered sets are a special case of so called *relation transversals* as introduced in [Zw]. We are only interested in the case of binary relations here but the generalization to relations with arbitrary arity is straightforward. (See the footnote on page 2 for this).

Definition 1.2.5 ((Binary) Relation Transversal) Let $\underline{G}$ be a group, Y be a set, $R \subseteq Y \times Y$, $(\underline{G}_y \mid y \in Y)$ be a family of subgroups of $\underline{G}$ and $\beta : Y^2 \longrightarrow \mathfrak{P}(G)$ such that

i) $\beta(a,b) \neq \emptyset \iff (a,b) \in R$,

ii) $G_s\beta(s,t)G_t \subseteq \beta(s,t)$ and

iii) $\bigcap_{y\in Y, g\in G} g\underline{G}_y g^{-1} = \{\,\mathrm{id}\,\}$.

Then $(Y, R, \underline{G}, (\underline{G}_y)_{y\in Y}, \beta)$ is said to be a *(binary) relation transversal.* ◇

Proposition 1.2.6 *Let $(P, \leq, \lambda, (\underline{G}_p)_{p\in P})$ a $\underline{G}$-annotated preordered set. Then $(P, \leq, \underline{G}, (\underline{G}_p)_{p\in P}, \lambda)$ is a relation transversal.*

Proof The only thing we have to show is that

$$\underline{G}_s\lambda(s,t)\underline{G}_t \subseteq \lambda(s,t).$$

But this is immediately clear since $G_s \subseteq \lambda(s,s), G_t \subseteq \lambda(t,t)$ and therefore

$$\underline{G}_s\lambda(s,t)\underline{G}_t \subseteq \lambda(s,s)\lambda(s,t)\lambda(t,t) \subseteq \lambda(s,t).$$ □

As already mentioned in the above example we can omit the stabilizers under certain circumstances. Those cases are of particular interest for the implementation in computer programs since they allow a short and concise representation of preorder orbifolds.

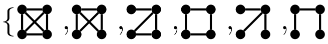
Proposition 1.2.7 *Let $\underline{P} = (P, \leq)$ a ordered set and $\Gamma \leq \underline{\mathrm{Aut}}(\underline{P})$ such that every orbit of an element is an antichain in $\underline{P}$. Let $\underline{P}_{\mathrm{rep}} = (P_{\mathrm{rep}}, \leq_{\mathrm{rep}}, (\Gamma_p)_{p\in P}, \lambda)$ a preorder orbifold of $\underline{P}$ under Γ. Then $\leq_{\mathrm{rep}}$ is an order relation on P_{rep} and $\lambda(p,p) = \Gamma_p$ for all $p \in P$.*

Proof We first show that $\leq_{\mathrm{rep}}$ is antisymmetric. Let $a, b \in P_{\mathrm{rep}}$ such that $a \leq_{\mathrm{rep}} b$ and $b \leq_{\mathrm{rep}} a$. Then there exist $\beta_1, \beta_2 \in \Gamma$ such that $a \leq \beta_1(b)$ and $b \leq \beta_2(a)$, hence $a \leq \beta_1(\beta_2(a))$ and due to $\beta_1(\beta_2(a)) \in \Gamma(a)$ and all orbits are antichains it is $a = \beta_1(\beta_2(a))$. Because $\leq$ is antisymmetric we therefore have $a = \beta_1(b) = \beta_1(\beta_2(a))$. Therefore it is $b \in \Gamma(a)$, $\Gamma(a) = \Gamma(b)$ and hence $a = b$ as required.

It remains to show that $\lambda(p,p) = \Gamma_p$. Let $p \in P_{\mathrm{rep}}$. We already have $\lambda(p,p) \supseteq \Gamma_p$. So let $\beta \in \lambda(p,p)$. Then it is $p \leq \beta(p)$ and since the orbit $\Gamma(p)$ is an antichain it must be $p = \beta(p)$ and therefore $\beta \in \Gamma_p$. □

To consolidate these ideas we want to consider the following example.

Example 1.2.8 As has been done in [GB], we consider all connected graphs on four vertices up to isomorphy. These are

$H = \{$  $\}$.

We order them by the relation "embeddable" to obtain the order diagram shown in Figure 1.2. We now interpret this ordered set as a preorder orbifold obtained by folding

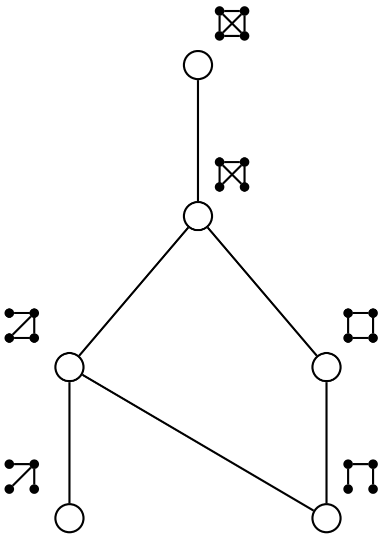
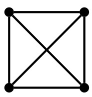


Figure 1.2: The embeddable-ordering of the connected graphs on four vertices up to isomorphy.

the set of all connected graphs with four vertices ordered by inclusion by the group Γ. Thereby $\Gamma \cong S_4$ is the group of permutations of the edges of every graph induced by the permutations on four elements, the graphs labeled as shown:



We then compute

$$\lambda(a, b) = \{\, \alpha \in \Gamma \mid a \subseteq \alpha(b)\,\}$$

and get the mapping shown in table 1.1. Now we have that $(H, \leq, (\Gamma_p)_{p \in H}, \lambda)$ is a Γ-annotated preordered set where $\leq$ denotes the "embeddable"-ordering. This is indeed the same Γ-annotated preordered set we would obtain when computing the preorder orbifold

$\lambda($, $) = \Gamma$
$\lambda($, $) = \{\, (1), (12), (34), (12)(34) \,\}$
$\lambda($, $) = \{\, (1), (34) \,\}$
$\lambda($, $) = \{\, (1), (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432) \,\}$
$\lambda($, $) = \{\, (1), (23), (24), (34), (234), (243) \,\}$
$\lambda($, $) = \{\, (1), (12)(34) \,\}$
$\lambda($, $) = \Gamma$
$\lambda($, $) = \Gamma$
$\lambda($, $) = \Gamma$
$\lambda($, $) = \Gamma$
$\lambda($, $) = \Gamma$
$\lambda($, $) = \{\, (23), (24), (132), (142), (234), (243), (1342), (1432) \,\}$
$\lambda($, $) = \{\, (14), (23), (132), (124), (143), (234), (1243), (1342) \,\}$
$\lambda($, $) = \{\, (1), (12), (23), (24), (34), (132), (142), (234), (243), (12)(34), (1342), (1432) \,\}$
$\lambda($, $) = \{\, (1), (12), (14), (23), (34), (132), (124), (143), (234), (12)(34), (1243), (1342) \,\}$
$\lambda($, $) = \{\, (1), (23), (24), (34), (234), (243) \,\}$
$\lambda($, $) = \{\, (24), (123), (243), (1234) \,\}$
$\lambda($, $) = \{\, (1), (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432) \,\}$

Table 1.1: Annotation of the ordered set of Figure 1.2 interpreted as preorder orbifold under the group Γ.

of the ordered set of all connected graphs on 4 vertices by Γ choosing H as transversal of the orbits of Γ. It is also obvious that

$$a \leq b \iff \lambda(a, b) \neq \emptyset \qquad \diamondsuit$$

where $a, b \in H$.

Two things are important to mention: First of all if we choose another transversal H we obviously get a different annotation map λ. But of course we then want to consider both preorder orbifolds as isomorphic. So we carefully have to develop a suitable understanding of isomorphy between group-annotated preordered sets.

Secondly we see that λ is not very easy to handle. Therefore we need a technique to simplify λ. We shall see that this is indeed possible by using so called *double cosets*.

1.3 Isomorphy between Group-annotated Preordered Sets

We now want to develop a precise understanding of what it means for two group-annotated preordered sets to be isomorphic. Although this definition can already be found in [Zw] for binary relation transversals we try to give a detailed description how this notion can be comprehended intuitively.

Let $\underline{P} = (P, \leq)$ be a preordered set and $\Gamma \leq \underline{\mathrm{Aut}}(\underline{P})$. Given two transversals Y, Z of the orbits of Γ we always want to consider the two preorder orbifolds $\underline{Y} = (Y, \leq_Y, (G_{Y,y})_{y \in Y}, \lambda_Y)$ and $\underline{Z} = (Z, \leq_Z, (G_{Z,z})_{z \in Z}, \lambda_Z)$ as isomorphic. For this let $\varphi : Y \longrightarrow \Gamma$ be a mapping such that

$$\varphi(y)(y) \in Z$$

for all $y \in Y$. The mapping φ then represents the difference between Y and Z by giving for every element $y \in Y$ a mapping $\varphi(y)$ that maps y to the element in Z that is in the same orbit as y. Having this mapping we are able to work out the necessary connections between λ_Y and λ_Z. To see this let $p, q \in Y$. Then it is

$$\begin{aligned}
\lambda_Y(p,q) &= \{\, \beta \in \Gamma \mid p \leq \beta(q) \,\} \\
&= \{\, \beta \in \Gamma \mid \varphi(p)(p) \leq (\varphi(p) \circ \beta \circ \varphi(q)^{-1} \circ \varphi(q))(q) \,\} \\
&= \varphi(p)^{-1} \circ \{\, \beta \in \Gamma \mid \varphi(p)(p) \leq (\beta \circ \varphi(q))(q) \,\} \circ \varphi(q) \\
&= \varphi(p)^{-1} \circ \lambda_Z(\varphi(p)(p), \varphi(q)(q)) \circ \varphi(q). \qquad (1.1)
\end{aligned}$$

With the help of φ we are also able to define a preorder automorphism α between $(Y, \leq_Y)$ and $(Z, \leq_Z)$ by simply setting

$$\alpha(y) := \varphi(y)(y).$$

Then the condition 1.1 simplifies to

$$\lambda_Y(p,q) = \varphi(p)^{-1} \circ \lambda_Z(\alpha(p), \alpha(q)) \circ \varphi(q)$$

and for the stabilizers $\Gamma_{Y,y}$ and $\Gamma_{Z,z}$ where $y \in Y$ and $z \in Z$ we get

$$\begin{aligned}
\Gamma_{Y,y} &= \{\, \beta \in \Gamma \mid y = \beta(y) \,\} \\
&= \{\, \beta \in \Gamma \mid \varphi(y)(y) = (\varphi(y) \circ \beta \circ \varphi(y)^{-1} \circ \varphi(y))(y) \,\} \\
&= \varphi(y)^{-1} \circ \{\, \beta \in \Gamma \mid \varphi(y)(y) = (\beta \circ \varphi(y))(y) \,\} \circ \varphi(y) \\
&= \varphi(y)^{-1} \circ \Gamma_{Z,\alpha(y)} \circ \varphi(y).
\end{aligned}$$

Let us now examine the general case. For this let $\underline{P} = (P, \leq_P)$ and $\underline{Q} = (Q, \leq_Q)$ be two isomorphic, preordered sets, $\Gamma_P \leq \underline{\mathrm{Aut}}(\underline{P})$, $\Gamma_Q \leq \underline{\mathrm{Aut}}(\underline{Q})$ and $\alpha : \underline{P} \longrightarrow \underline{Q}$ be a preorder automorphism. Let $\underline{Y} = (Y, \leq_Y, (G_{Y,y})_{y \in Y}, \lambda_Y) = \mathrm{rep}(\underline{P})$ and $\underline{Z} = (Z, \leq_Z, (G_{Z,z})_{z \in Z}, \lambda_Z) = \mathrm{rep}(\underline{Q})$. Additionally α has to have the property that

$$\begin{array}{rccc}
\delta : & \Gamma_P & \longrightarrow & \Gamma_Q \\
& \beta & \longmapsto & \alpha \circ \beta \circ \alpha^{-1}
\end{array}$$

is a group isomorphism. Furthermore we again need a function $\varphi : P \longrightarrow \Gamma_Q$ such that

$$\varphi(x)(\alpha(x)) \in Z.$$

With this we can find that the mapping

$$\begin{array}{rccc} \bar{\alpha}: & (Y, \leq_Y) & \longrightarrow & (Z, \leq_Z) \\ & y & \longmapsto & \varphi(y)(\alpha(y)) \end{array}$$

is a preorder automorphism and

$$\begin{aligned} \delta[\lambda_Y(a,b)] &= \delta[\{\, \beta \in \Gamma_Y \mid a \leq_P \beta(b) \,\}] \\ &= \{\, \alpha\beta\alpha^{-1} \in \Gamma_Z \mid a \leq_Q \beta(b) \,\} \\ &= \{\, \beta \in \Gamma_Z \mid \alpha(a) \leq_Q (\beta\alpha)(b) \,\} \\ &= \{\, \beta \in \Gamma_Z \mid \varphi(a)(\alpha(a)) \leq_Q (\varphi(a)\beta\varphi(b)^{-1}\varphi(b))(\alpha(b)) \,\} \\ &= \varphi(a)^{-1}\{\, \beta \in \Gamma_Z \mid \bar{\alpha}(a) \leq_Q (\beta\bar{\alpha})(b) \,\}\varphi(b) \\ &= \varphi(a)^{-1}\lambda_Z(\bar{\alpha}(a), \bar{\alpha}(b))\varphi(b) \end{aligned}$$

for $a, b \in Y$. Analogously we find

$$\delta[\Gamma_{Y,a}] = \varphi(a)^{-1}\Gamma_{Z,\bar{\alpha}(a)}\varphi(a).$$

With all these preliminary remarks we can now define what is meant for two relation transversals to be isomorphic. We shall formulate the definition for the general case of relation transversals because this notion will be needed again in later sections for context orbifolds.

Definition 1.3.1 (Isomorphy of Relation Transversals) Let $\underline{Y} = (Y, R_Y, \Gamma_Y, (\Gamma_{Y,y})_{y\in Y}, \lambda_Y)$ and $\underline{Z} = (Z, R_Z, \Gamma_Z, (\Gamma_{Z,z})_{z\in Z}, \lambda_Z)$ be two relation transversals. $\underline{Y}$ and $\underline{Z}$ are said to be *isomorphic*, written as $\underline{Y} \cong \underline{Z}$, if the following conditions hold:

- There exists a bijective mapping $\alpha : (Y, R_Y) \longrightarrow (Z, R_Z)$ such that

$$(x, y) \in R_Y \iff (\alpha(x), \alpha(y)) \in R_Z,$$

- there exists a group isomorphism $\delta : \Gamma_Y \longrightarrow \Gamma_Z$ and
- there exists a mapping $\varphi : Y \longrightarrow \Gamma_Z$

such that

$$\delta[\lambda_Y(a,b)] = \varphi(a)^{-1}\lambda_Z(\alpha(a), \alpha(b))\varphi(b)$$

and

$$\delta[\Gamma_{Y,a}] = \varphi(a)^{-1}\Gamma_{Z,\alpha(a)}\varphi(a)$$

hold for all $a, b \in Y$. $\diamondsuit$

Theorem 1.3.2 *Let $\underline{P}_1 = (P_1, \leq_1)$ and $\underline{P}_2 = (P_2, \leq_2)$ be two preordered sets and let $\Gamma_1 \leq \underline{\mathrm{Aut}}(\underline{P}_1)$, $\Gamma_2 \leq \underline{\mathrm{Aut}}(\underline{P}_2)$. Let $\alpha : \underline{P}_1 \longrightarrow \underline{P}_2$ a preorder isomorphism such that*

$$\begin{array}{rccc} \delta: & \Gamma_1 & \longrightarrow & \Gamma_2 \\ & \beta & \longmapsto & \alpha \circ \beta \circ \alpha^{-1} \end{array}$$

is a group isomorphism. Then $\mathrm{rep}_{\Gamma_1}(\underline{P}_1) \cong \mathrm{rep}_{\Gamma_2}(\underline{P}_2)$.

Proof Let $\mathrm{rep}_{\Gamma_1}(\underline{P}_1) = (Y_1, \leq_{Y_1}, (\Gamma_{1,y})_{y\in Y_1}, \lambda_{Y_1})$ and $\mathrm{rep}_{\Gamma_2}(\underline{P}_2) = (Y_2, \leq_{Y_2}, (\Gamma_{2,y})_{y\in Y_2}, \lambda_{Y_2})$. By definition of δ it holds

$$a \leq_1 \beta(b) \iff \alpha(a) \leq_2 \delta(\beta)(\alpha(b))$$

for $a, b \in P_1$ and $\beta \in \Gamma_1$. Now for every $x \in Y_1$ there exists a $\varphi_x \in \Gamma_2$ such that

$$\varphi_x(\alpha(x)) \in Y_2.$$

We then define

$$\begin{array}{rrcl} \bar{\alpha}: & Y_1 & \longrightarrow & Y_2 \\ & x & \longmapsto & \varphi_x(\alpha(x)). \end{array}$$

Then $\bar{\alpha}$ is bijective and for $x, y \in Y_1$ it is

$$\begin{aligned} x \leq_{Y_1} y &\iff \exists\beta \in \Gamma_1 : x \leq_1 \beta(y) \\ &\iff \exists\beta \in \Gamma_1 : \alpha(x) \leq_2 (\delta(\beta)\alpha)(y) \\ &\iff \exists\beta \in \Gamma_1 : \varphi_x\alpha(x) \leq_2 (\varphi_x\delta(\beta)\varphi_y^{-1}\varphi_y)(\alpha(y)) \\ &\iff \exists\bar{\beta} \in \Gamma_2 : \bar{\alpha}(x) \leq_2 \bar{\beta}(\bar{\alpha}(y)) \\ &\iff \bar{\alpha}(x) \leq_{Y_2} \bar{\alpha}(y), \end{aligned}$$

hence $\bar{\alpha}$ is a preorder automorphism. This also shows

$$\beta \in \lambda_{Y_1}(x, y) \iff \varphi_x\delta(\beta)\varphi_y^{-1} \in \lambda_{Y_2}(\bar{\alpha}(x), \bar{\alpha}(y))$$

and thus $\delta[\lambda_{Y_1}(x, y)] = \varphi_x^{-1}\lambda_{Y_2}(\bar{\alpha}(x), \bar{\alpha}(y))\varphi_y$. We also have

$$\begin{aligned} \beta \in \Gamma_{1,y} &\iff y = \beta(y) \\ &\iff \alpha(y) = \delta(\beta)\alpha(y) \\ &\iff \varphi_y\alpha(y) = \varphi_y\delta(\beta)\varphi_y^{-1}\varphi_y\alpha(y) \\ &\iff \bar{\alpha}(y) = \varphi_y\delta(\beta)\varphi_y^{-1}\bar{\alpha}(y) \\ &\iff \varphi_y\delta(\beta)\varphi_y^{-1} \in \Gamma_{2,y} \end{aligned}$$

hence $\delta[\Gamma_{1,y}] = \varphi_y^{-1}\Gamma_{2,y}\varphi_y$. So if we define

$$\begin{array}{rrcl} \varphi: & Y_1 & \longrightarrow & \Gamma_2 \\ & y & \longmapsto & \varphi_y \end{array}$$

we see that $\mathrm{rep}_{\Gamma_1}(\underline{P}_1) \cong \mathrm{rep}_{\Gamma_2}(\underline{P}_2)$ as required. □

This immediately proves the following important result:

Corollary 1.3.3 *Let $\underline{P}_1 \cong \underline{P}_2$ be isomorphic preordered sets. Then* $\mathrm{rep}_{\underline{\mathrm{Aut}(P_1)}}(\underline{P}_1) \cong \mathrm{rep}_{\underline{\mathrm{Aut}(P_2)}}(\underline{P}_2)$.

Proof Let $\alpha : \underline{P}_1 \longrightarrow \underline{P}_2$ be a preorder isomorphism. Then the mapping

$$\begin{array}{rccc} \delta : & \underline{\mathrm{Aut}}(\underline{P}_1) & \longrightarrow & \underline{\mathrm{Aut}}(\underline{P}_2) \\ & \beta & \longmapsto & \alpha \circ \beta \circ \alpha^{-1} \end{array}$$

is a group isomorphism and the corollary follows from Theorem 1.3.2. □

One might ask whether the restriction on δ being a group isomorphism which is somehow induced by an automorphism is really necessary. The following example shows that at least the condition on Γ_1 and Γ_2 being two isomorphic groups does not suffice.

Example 1.3.4 We consider the lattice shown in Figure 1.3. This lattice has the auto-

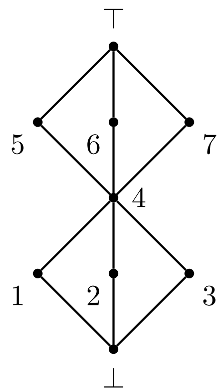


Figure 1.3: Example lattice.

morphisms $\alpha = (123)$ and $\beta = (567)$ and $\langle\alpha\rangle \cong \mathbb{Z}_3 \cong \langle\beta\rangle$. But the base structures of the preorder orbifolds obtained when folding the given lattice by $\langle\alpha\rangle$ and $\langle\beta\rangle$ respectively yield the lattices shown in Figure 1.4. But these preorder orbifolds are not isomorphic as ordered sets and can therefore not be isomorphic as preorder orbifolds. ♢

We are even able to show that the groups Γ_1 and Γ_2 have to be more than just isomorphic in the case of isomorphic binary relation transversals.

Lemma 1.3.5 *Let $\underline{P}_1 = (P_1, \leq_{\underline{P}_1})$ and $\underline{P}_2 = (P_2, \leq_{\underline{P}_2})$ be two preordered sets and $\Gamma_1 \leq \underline{\mathrm{Aut}}(\underline{P}_1)$, $\Gamma_2 \leq \underline{\mathrm{Aut}}(\underline{P}_2)$. If $\mathrm{rep}_{\Gamma_1}(\underline{P}_1) \cong \mathrm{rep}_{\Gamma_2}(\underline{P}_2)$ then there exists a preorder isomorphism $\psi : \underline{P}_1 \longrightarrow \underline{P}_2$ such that $\Gamma_2 = \psi\Gamma_1\psi^{-1}$.*

Proof Let $\mathrm{rep}_{\Gamma_i}(\underline{P}_i) = \underline{Y}_i = (Y_i, \leq_{\underline{Y}_i}, (\Gamma_{i,y})_{y \in Y_i}, \lambda_i)$ for $i \in \{1, 2\}$ and

- $\alpha : (Y_1, \leq_{\underline{Y}_1}) \longrightarrow (Y_2, \leq_{\underline{Y}_2})$ be a preorder isomorphism,
- $\delta : \Gamma_1 \longrightarrow \Gamma_2$ be a group isomorphism and

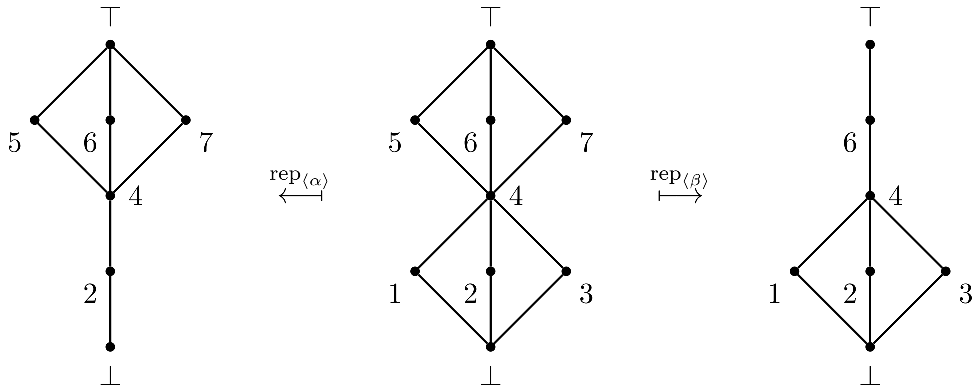


Figure 1.4: Base structures of the preorder orbifolds obtained when folding by $\langle\alpha\rangle$ and $\langle\beta\rangle$ respectively.

- $\varphi : Y_1 \longrightarrow \Gamma_2$

such that

$$\delta[\lambda_1(a,b)] = \varphi(a)^{-1}\lambda_2(\alpha(a),\alpha(b))\varphi(b)$$

and

$$\delta[\Gamma_{1,a}] = \varphi(a)^{-1}\Gamma_{2,\alpha(a)}\varphi(a).$$

Then we define

$$\begin{array}{cccc} \psi : & P_1 & \longrightarrow & P_2 \\ & \beta(x) & \longmapsto & \delta(\beta)\varphi(x)^{-1}\alpha(x) \end{array}$$

for $x \in Y_1$ and $\beta \in \Gamma_1$.

Then ψ is well defined since for $\beta_1(x_1) = \beta_2(x_2)$ we have $x_1 \in \Gamma_1(x_2)$ and hence $x_1 = x_2$ (because $x_1, x_2 \in Y_1$) and $\beta_1^{-1}\beta_2 \in \Gamma_{1,x_1}$. It follows that

$$\delta(\beta_1)^{-1}\delta(\beta_2) \in \varphi(x_1)^{-1}\Gamma_{2,\alpha(x_1)}\varphi(x_1)$$

and therefore

$$\varphi(x_1)\delta(\beta_1)^{-1}\delta(\beta_2)\varphi(x_1)^{-1} \in \Gamma_{2,\alpha(x_1)}$$

which means nothing else but

$$\delta(\beta_1)(\varphi(x_1)^{-1}\alpha(x_1)) = \delta(\beta_2)(\varphi(x_1)^{-1}\alpha(x_1))$$

and thus $\psi(\beta_1(x_1)) = \psi(\beta_2(x_2))$ as required.

We also have for $x_1, x_2 \in Y_1$ and $\beta_1, \beta_2 \in \Gamma$

$$\begin{aligned}
\beta_1(x_1) \leq_{\underline{P}_1} \beta_2(x_2) &\iff x \leq_{\underline{P}_1} (\beta_1^{-1}\beta_2)(x_2) \\
&\iff \beta_1^{-1}\beta_2 \in \lambda_1(x_1, x_2) \\
&\iff \delta(\beta_1)^{-1}\delta(\beta_2) \in \delta[\lambda_1(x_1, x_2)] \\
&\iff \delta(\beta_1)^{-1}\delta(\beta_2) \in \varphi(x_1)^{-1}\lambda_2(\alpha(x_1), \alpha(x_2))\varphi(x_2) \\
&\iff \varphi(x_1)\delta(\beta_1)^{-1}\delta(\beta_2)\varphi(x_2)^{-1} \in \lambda_2(\alpha(x_1), \alpha(x_2)) \\
&\iff \alpha(x_1) \leq_{\underline{P}_2} (\varphi(x_1)\delta(\beta_1)^{-1}\delta(\beta_2)\varphi(x_2)^{-1})(\alpha(x_2)) \\
&\iff \delta(\beta_1)(\varphi(x_1)^{-1}(\alpha(x_1))) \leq_{\underline{P}_2} \delta(\beta_2)(\varphi(x_2)^{-1}(\alpha(x_2))) \\
&\iff \psi(\beta_1(x_1)) \leq_{\underline{P}_2} \psi(\beta_2(x_2))
\end{aligned}$$

and hence ψ is preorder-reflecting and preorder-preserving. Now let

$$\begin{array}{rccc}
\bar{\psi}: & P_2 & \longrightarrow & P_1 \\
& \beta(y) & \longmapsto & \delta^{-1}(\beta\varphi(\alpha^{-1}(y)))(\alpha^{-1}(y))
\end{array}$$

for $y \in Y_2, \beta \in \Gamma_2$.

Then given $\beta \in \Gamma_2$ and $y \in Y_2$ it is

$$\begin{aligned}
\psi(\bar{\psi}(\beta(y))) &= \psi(\delta^{-1}(\beta\varphi(\alpha^{-1}(y)))(\alpha^{-1}(y))) \\
&= \delta(\delta^{-1}(\beta\varphi(\alpha^{-1}(y))))(\varphi(\alpha^{-1}(y))^{-1}\alpha(\alpha^{-1}(y))) \\
&= \beta\varphi(\alpha^{-1}(y))\varphi(\alpha^{-1}(y))^{-1}y \\
&= \beta(y)
\end{aligned}$$

and for $\gamma \in \Gamma_1$ and $x \in Y_1$

$$\begin{aligned}
\bar{\psi}(\psi(\gamma(x))) &= \bar{\psi}(\delta(\gamma)\varphi(x)^{-1}\alpha(x)) \\
&= \delta^{-1}(\delta(\gamma)\varphi(x)^{-1}\varphi(\alpha^{-1}\alpha(x)))(\alpha^{-1}\alpha(x)) \\
&= \delta^{-1}\delta(\gamma)(x) \\
&= \gamma(x),
\end{aligned}$$

therefore the mapping $\bar{\psi}$ is inverse to ψ and thus ψ is also bijective.

If we now define for $\omega \in \Gamma_1$

$$\bar{\delta}(\omega) := \psi\omega\psi^{-1}$$

we have for $\omega \in \Gamma_1$, $\beta \in \Gamma_2$ and $y \in Y_2$ that

$$\begin{aligned}
\psi\omega\psi^{-1}(\beta y) &= \psi(\underbrace{\omega(\delta^{-1}(\beta\varphi(\alpha^{-1}(y))))}_{=:\bar{\omega}\in\Gamma_1}(\alpha^{-1}(y)))) \\
&= \psi(\bar{\omega}(\alpha^{-1}(y))) \\
&= \delta(\bar{\omega})\varphi(\alpha^{-1}(y))^{-1}(y) \\
&= \underbrace{\delta(\bar{\omega})\varphi(\alpha^{-1}(y))^{-1}\beta^{-1}}_{\in\Gamma_2}(\beta y)
\end{aligned}$$

and therefore $\bar{\delta} : \Gamma_1 \longrightarrow \Gamma_2$ and $\psi\Gamma_1\psi^{-1} \subseteq \Gamma_2$. On the other hand, given $x \in Y_1$, we have

$$\begin{aligned}\psi^{-1}\beta\psi(\omega x) &= \psi^{-1}(\underbrace{\beta\delta(\omega)\varphi(x)^{-1}}_{=:\bar{\beta}\in\Gamma_2}(\alpha(x))) \\ &= \psi^{-1}(\bar{\beta}(\alpha(x))) \\ &= \delta^{-1}(\bar{\beta}\varphi(x))(x) \\ &= \underbrace{\delta^{-1}(\bar{\beta}\varphi(x))\omega^{-1}}_{\in\Gamma_1}(\omega x)\end{aligned}$$

and hence $\psi^{-1}\Gamma_2\psi \subseteq \Gamma_1$. In sum we get $\Gamma_2 = \psi\Gamma_1\psi^{-1}$ as required. □

So putting together what we have proven about isomorphy of group-annotated preordered sets we get:

Corollary 1.3.6 *Let $\underline{P}_1$ and $\underline{P}_2$ be two preordered sets and $\Gamma_1 \leq \underline{\mathrm{Aut}}(\underline{P}_1)$, $\Gamma_2 \leq \underline{\mathrm{Aut}}(\underline{P}_2)$. Then the following conditions are equivalent*

1. $\mathrm{rep}_{\Gamma_1}(\underline{P}_1) \cong \mathrm{rep}_{\Gamma_2}(\underline{P}_2)$ *and*
2. *there exists a preorder isomorphism $\alpha : \underline{P}_1 \longrightarrow \underline{P}_2$ such that $\Gamma_2 = \alpha\Gamma_1\alpha^{-1}$.*

Proof This is Theorem 1.3.2 together with Lemma 1.3.5. □

1.4 Unfolding Group-annotated Preordered Sets

Now that we have a precise notion of folding preordered sets we also desire the possibility to "reverse the folding", i.e. to unfold preorder orbifolds or, more general, to unfold group annotated preordered sets.

The idea is fairly simple: Given a group of automorphisms Γ and a transversal Y we get the original base set by

$$P := \{\, \gamma(y) \mid \gamma \in \Gamma, y \in Y \,\}.$$

To recover the preorder relation on P we observe that for every $b \in P$ there exist $\bar{b} \in Y$ and $\gamma \in \Gamma$ such that $b = \gamma(\bar{b})$ and hence $a \leq b \iff a \leq \gamma(\bar{b}) \iff \gamma \in \lambda(a, \bar{b})$.

But we can go further and define unfolding for every group-annotated preordered set (indeed, we can do so for every relation transversal) by considering the following proposition.

Proposition 1.4.1 *Let $\underline{P} = (P, \leq_{\underline{P}})$ be a preordered set and $\Gamma \leq \underline{\mathrm{Aut}}(\underline{P})$. Furthermore let Y be a transversal of the orbits of Γ on P. Then the mapping*

$$\begin{array}{rccc}\Psi : & \dot{\bigcup}_{y\in Y} \Gamma/\Gamma_y & \longrightarrow & \underline{P} \\ & \gamma\Gamma_y & \longmapsto & \gamma(y)\end{array}$$

is a preorder isomorphism, where

$$\alpha\Gamma_y \leq \beta\Gamma_z :\iff \alpha(y) \leq_{\underline{P}} \beta(z).$$

Proof First of all we see that the assignment $\Psi(\gamma\Gamma_y) = \gamma(y)$ indeed describes a function because if we have $\gamma_1\Gamma_{y_1} = \gamma_2\Gamma_{y_2}$ (as an equality of elements in the disjoint union) it is $y_1 = y_2$ and $\gamma_1^{-1}\gamma_2\Gamma_{y_1} = \Gamma_{y_1}$. Therefore it is $\gamma_1^{-1}\gamma_2 \in \Gamma_{y_1}$ and hence $\gamma_1(y_1) = \gamma_2(y_2)$ as required. It is also easy to see that Ψ is surjective. Finally, if we have $\gamma_1(y_1) = \gamma_2(y_2)$ we get $y_1 = y_2$ since $y_1, y_2 \in Y$, thus $\gamma_1^{-1}\gamma_2 \in \Gamma_{y_1}$ and therefore $\gamma_1\Gamma_{y_1} = \gamma_2\Gamma_{y_1} = \gamma_2\Gamma_{y_2}$. Hence Ψ is injective. It is clear that Ψ is preorder-preserving and preorder-reflecting since $\leq$ is induced by $\leq_{\underline{P}}$. □

Remark 1.4.2 Note that by the previous proposition we are now allowed to restrict ourselves to group annotated preordered sets where the group acts on the base set. For a group annotated preordered set $(Y, \leq_y, (\underline{G}_y)_{y\in Y}, \lambda)$ we are then able to expose a group action by
$$gy := \Psi(g\Psi^{-1}(y)),$$
that is we set $gy := x$ if and only if $g\Gamma_y = \Gamma_x$. Therefore we may omit the explicit notion of cosets $g\Gamma_y$ and can simply write gy.

So we can now identify every element $\gamma(y)$ with the set $\gamma\Gamma_y$ and formulate the following definition.

Definition 1.4.3 (Unfolding Group-anntotated Preordered Sets) Let $(Y, \leq, (\underline{G}_y)_{y\in Y}, \lambda)$ be a $\underline{G}$-annotated preordered set. Then the *unfolding (or reconstruction) of* $(Y, \leq, (\underline{G}_y)_{y\in Y}, \lambda)$ *under* $\underline{G}$ is defined as
$$\mathrm{rec}_{\underline{G}}(Y, \leq, \lambda) := (\dot{\bigcup_{y\in Y}} \underline{G}/\underline{G}_y, \leq_r)$$
where
$$gG_y \leq_r hG_z :\iff g^{-1}h \in \lambda(y, z). \qquad \diamondsuit$$

Remark 1.4.4 The relation $\leq_r$ is well defined since $g_1G_y = g_2G_y$, $h_1G_z = h_2G_z$ and $g_1^{-1}h_1 \in \lambda(y, z)$ implies
$$g_2^{-1}h_2 = \underbrace{g_2^{-1}g_1}_{\in G_y}\underbrace{g_1^{-1}h_1}_{\in\lambda(y,z)}\underbrace{h_1^{-1}h_2}_{\in G_z} \in G_y\lambda(y, z)G_z \subseteq \lambda(y, z).$$

We may remark that this definition can be generalized easily to binary relation transversals.

Definition 1.4.5 (Unfolding Binary Relation Transversals) Let $\underline{Y} = (Y, R, \underline{G}, (\underline{G}_y)_{y\in Y}, \lambda)$ a binary relation transversal. Then the *unfolding (or reconstruction) of* $\underline{Y}$ is given by
$$\mathrm{rec}(\underline{Y}) = (\dot{\bigcup_{y\in Y}} \underline{G}/\underline{G}_y, R_{\mathrm{rec}})$$
where
$$y_1G_{z_1}\ R_{\mathrm{rec}}\ y_2G_{z_2} :\iff y_1^{-1}y_2 \in \lambda(z_1, z_2). \qquad \diamondsuit$$

We are now going to show what can be expected: that unfolding of a folding of a preordered set yields an isomorphic copy of the original preordered set and that unfolded isomorphic group-annotated preordered sets are again isomorphic. But we also want to prove that folding an unfolding of a group-annotated preordered set is isomorphic to the original group-annotated preordered set. To do this we need the following observation which can again be found in [Zw].

Proposition 1.4.6 *Let $\underline{G}$ be a group, Y be a set and $(\underline{G}_y \mid y \in Y)$ be a family of subgroups such that*

$$\bigcap_{g\in G, y\in Y} gG_yg^{-1} = \{ e_G \}$$

where e_G is the neutral element of $\underline{G}$. Then with $N := \dot{\bigcup}_{y\in Y} G/G_y$ the mapping

$$\begin{array}{rccc} \iota: & G & \longrightarrow & S_N \\ & g & \longmapsto & (N, N, hG_y \longmapsto ghG_y) \end{array}$$

is an injective group homomorphism.

Proof Let $g \in G$. Then $(hG_y \longmapsto ghG_y) \in S_N$ since $(hG_y \longmapsto g^{-1}hG_y)$ is the inverse mapping. Clearly ι is a group homomorphism since

$$\begin{aligned} \iota(gh) &= (N, N, lG_y \longmapsto ghlG_y) \\ &= (N, N, lG_y \longmapsto glG_y) \circ (N, N, lG_y \longmapsto hlG_y) \\ &= \iota(g) \circ \iota(h). \end{aligned}$$

To show that ι is injective let $g \in G$ such that $\iota(g) = \mathrm{id}$. Then we have $hG_y = ghG_y$ for every $hG_y \in N$ and therefore

$$g \in \bigcap_{h\in G, y\in Y} hG_yh^{-1} = \{ e_G \}$$

thus $g = e_G$ and ι is injective. □

We shall call $\iota[G]$ the *(faithful) permutation representation of $\underline{G}$*. With this we are now able to prove the following result.

Proposition 1.4.7 *Let $(Y, \leq_Y, (\underline{G}_y)_{y\in Y}, \lambda)$ be a $\underline{G}$-annotated preordered set. Then the unfolding $\mathrm{rec}_{\underline{G}}(Y, \leq_Y, (\underline{G}_y)_{y\in Y}, \lambda)$ is a preordered set such that $\iota[\underline{G}]$ is a subgroup of its automorphism group.*

Proof Let $(P, \leq) = \mathrm{rec}_{\underline{G}}(Y, \leq_Y, (\underline{G}_y)_{y\in Y}, \lambda)$. Then $gG_y \leq gG_y$ since $g^{-1}g = \mathrm{id} \in \lambda(y, y)$ and if we have $gG_y \leq hG_z \leq lG_u$ it is $g^{-1}h \in \lambda(y, z)$ and $h^{-1}l \in \lambda(z, u)$, thus $g^{-1}l = g^{-1}hh^{-1}l \in \lambda(y, z)\lambda(z, u) \subseteq \lambda(y, u)$ and therefore $\leq$ is transitive. Hence $(P, \leq)$ is a preordered set. Now by Proposition 1.4.6 the mapping

$$\begin{array}{rccc} \iota: & G & \longrightarrow & S_P \\ & g & \longmapsto & (P, P, hG_y \longmapsto ghG_y) \end{array}$$

is a group monomorphism and for $g \in G$ we have

$$\begin{aligned} hG_z \leq lG_u &\iff h^{-1}l \in \lambda(z,u) \\ &\iff (gh)^{-1}(gl) \in \lambda(z,u) \\ &\iff ghG_z \leq glG_u \\ &\iff \iota(g)(hG_y) \leq \iota(g)(lG_u) \end{aligned}$$

where $hG_z, lG_u \in P$. Therefore $\iota[\underline{G}] \leq \underline{\mathrm{Aut}}(P, \leq)$ as required. □

Corollary 1.4.8 *Let $(Y, \leq_Y, (\underline{G}_y)_{y\in Y}, \lambda)$ be a $\underline{G}$-annotated preordered set where $\leq_Y$ is an order relation on Y and $\lambda(y,y) = G_y$ for all $y \in Y$. Then $\mathrm{rec}_{\underline{G}}(Y, \leq_Y, (\underline{G}_y)_{y\in Y}, \lambda)$ is an ordered set such that $\iota[\underline{G}]$ is a subgroup of its automorphism group.*

Proof Let $(P, \leq) = \mathrm{rec}_{\underline{G}}(Y, \leq_Y, (\underline{G}_y)_{y\in Y}, \lambda)$. Let $gG_y, hG_z \in P$ with $gG_y \leq hG_z$ and $hG_z \leq gG_y$. Then $h^{-1}g \in \lambda(y,z)$ and $g^{-1}h \in \lambda(z,y)$ hence $y \leq_Y z$ and $z \leq_Y y$. Since $\leq_Y$ is antisymmetric we get $y = z$ and $g^{-1}h \in \lambda(y,y) = G_y$. This yields $gG_y = hG_y = hG_z$ as required. The rest follows from Proposition 1.4.7. □

Now we can apply our intuitive idea of unfolding group-annotated preordered sets if we already have a preorder orbifold. This might in some cases simplify necessary calculations.

Proposition 1.4.9 *Let $(Y, \leq_Y, (\Gamma_y)_{y\in Y}, \lambda)$ be a preorder orbifold under Γ and let $(Q, \leq_Q)$ be the pair obtained by*

$$Q := \{\, \gamma(x) \mid \gamma \in \Gamma, x \in P \,\}$$

and

$$\gamma_1(x) \leq_Q \gamma_2(y) \iff \gamma_1^{-1}\gamma_2 \in \lambda(x,y).$$

Then $(Q, \leq_Q)$ is a well-defined preordered set and is isomorphic to $\mathrm{rec}_\Gamma(Y, \leq_Y, (\Gamma_y)_{y\in Y}, \lambda)$.

Proof It has already been shown in Remark 1.4.4 that $\leq_Q$ is well defined. Clearly $\leq_Q$ is reflexive since $\mathrm{id} \in \lambda(x,x)$, so $\gamma(x) \leq_Q \gamma(x)$ for each $\gamma \in \Gamma$ and $x \in Y$. Furthermore $\leq_Q$ is transitive since $\lambda(x,y)\lambda(y,z) \subseteq \lambda(x,z)$. Hence $(Q, \leq_Q)$ is a well-defined preordered set.

Now let $\beta \in \Gamma$, $\gamma_1(x_1), \gamma_2(x_2) \in Q$. Then

$$\begin{aligned} \gamma_1(x_1) \leq_Q \gamma_2(x_2) &\iff \gamma_1^{-1}\gamma_2 \in \lambda(x_1,x_2) \\ &\iff (\beta\gamma_1)^{-1}\beta\gamma_2 \in \lambda(x_1,x_2) \\ &\iff \beta\gamma_1(x_1) \leq_Q \beta\gamma_2(x_2) \end{aligned}$$

thus $\beta \in \mathrm{Aut}(Q, \leq_Q)$ and hence $\Gamma \leq \underline{\mathrm{Aut}}(Q, \leq_Q)$. It follows that Y is a transversal of the orbits of Γ on Q.

Let $(P, \leq) = \mathrm{rec}_\Gamma(Y, \leq_Y, (\Gamma_y)_{y\in Y}, \lambda)$. Now because of

$$\gamma_1(x_1) \leq_Q \gamma_2(x_2) \iff \gamma_1^{-1}\gamma_2 \in \lambda(x,y) \iff \gamma_1 G_{x_1} \leq \gamma_2 G_{x_2}$$

we can apply Proposition 1.4.1 and get $(P, \leq) \cong (Q, \leq_Q)$. □

Example 1.4.10 We like to compute the unfolding of the preorder orbifold computed in Example 1.2.2, 2). By Proposition 1.4.9 we can do this without considering disjoint unions of cosets. Let $(Z, \leq_Z, (G_z)_{z \in Z}, \lambda_Z) = \mathrm{rep}_{\langle x \longmapsto x+2 \rangle}(\mathbb{Z}, \leq_{\mathbb{Z}})$. Then we find as a base set for the unfolding

$$\begin{aligned} Z_{\mathrm{rec}} &= \{ \gamma(z) \mid \gamma \in \langle x \longmapsto x+2 \rangle , z \in Z \} \\ &= \{ 2k + z \mid z \in \{ 0, 1 \}, k \in \mathbb{Z} \} \\ &= \mathbb{Z}. \end{aligned}$$

For $2k_1 + z_1, 2k_2 + z_2 \in Z_{\mathrm{rec}}$ we have

$$\begin{aligned} 2k_1 + z_1 \leq_{\mathrm{rec}} 2k_2 + z_2 &\iff (x \longmapsto x + 2(k_2 - k_1)) \in \lambda_Z(z_1, z_2) \\ &\iff z_1 \leq_{\mathbb{Z}} 2(k_2 - k_1) + z_2 \\ &\iff 2k_1 + z_1 \leq_{\mathbb{Z}} 2k_2 + z_2 \end{aligned}$$

hence $\leq_{\mathrm{rec}} = \leq_{\mathbb{Z}}$ and therefore $\mathrm{rec}_{\langle x \longmapsto x+2 \rangle}(\mathrm{rep}_{\langle x \longmapsto x+2 \rangle}(\mathbb{Z}, \leq_{\mathbb{Z}})) \cong (Z_{\mathrm{rec}}, \leq_{\mathrm{rec}}) = (\mathbb{Z}, \leq_{\mathbb{Z}})$. $\diamondsuit$

The next theorem covers a general property of unfolding isomorphic, group-annotated preordered sets.

Theorem 1.4.11 *Let* $\underline{Y}_1 = (Y_1, \leq_{Y_1}, (G_{1,y})_{y \in Y_1}, \lambda_{Y_1})$ *be a* $\underline{G}_1$*-annotated preordered set and* $\underline{Y}_2 = (Y_2, \leq_{Y_2}, (G_{2,y})_{y \in Y_2}, \lambda_{Y_2})$ *be a* $\underline{G}_2$*-annotated preordered set with* $\underline{Y}_1 \cong \underline{Y}_2$*. Then* $\mathrm{rec}_{\underline{G}_1}(\underline{Y}_1) \cong \mathrm{rec}_{\underline{G}_2}(\underline{Y}_2)$.

Proof Let $\underline{P}_1 = (P_1, \leq_1) = \mathrm{rec}_{\underline{G}_1}(\underline{Y}_1)$ and $\underline{P}_2 = (P_2, \leq_2) = \mathrm{rec}_{\underline{G}_2}(\underline{Y}_2)$. Let α, δ and φ as in Definition 1.3.1. Then we define

$$\begin{array}{rccc} \psi : & P_1 & \longrightarrow & P_2 \\ & g_1 G_{1,y} & \longmapsto & \delta(g_1)\varphi(y)^{-1} G_{2,\alpha(y)}. \end{array}$$

One might be tempted to compare this definition to the one found in Lemma 1.3.5 and indeed this theorem together with the following one yields a generalization of this statement. The proof now is very similar to the one of Lemma 1.3.5.

First of all ψ is well-defined. To see this let $gG_{1,y}, hG_{1,z} \in P_1$ with $gG_{1,y} = hG_{1,z}$. Then $y = z$ and thus $g^{-1}hG_{1,y} = G_{1,y}$. Therefore $g^{-1}h \in G_{1,y}$. It follows that $\delta(g^{-1}h) \in \delta[G_{1,y}] = \varphi(y)^{-1}G_{2,\alpha(y)}\varphi(y)$ and hence

$$(\delta(g)\varphi(y)^{-1})^{-1}(\delta(h)\varphi(y)^{-1}) \in G_{2,\alpha(y)}$$

which is equivalent to

$$\delta(g)\varphi(y)^{-1}G_{2,\alpha(y)} = \delta(h)\varphi(y)^{-1}G_{2,\alpha(y)}$$

as required.

One can verify that the mapping

$$\begin{array}{rccc} \bar{\psi}: & P_2 & \longrightarrow & P_1 \\ & \beta G_{2,y} & \longmapsto & \delta^{-1}(\beta\varphi(\alpha^{-1}(y)))G_{1,\alpha^{-1}(y)} \end{array}$$

is inverse to ψ, hence ψ is bijective.

Now let $gG_{1,y}, hG_{1,z} \in P_1$. Then we have

$$\begin{aligned} gG_{1,y} \leq_1 hG_{1,z} &\iff g^{-1}h \in \lambda_{Y_1}(y,z) \\ &\iff \delta(g)^{-1}\delta(h) \in \varphi(y)^{-1}\lambda_{Y_2}(\alpha(y),\alpha(z))\varphi(z) \\ &\iff (\delta(g)\varphi(y)^{-1})^{-1}(\delta(h)\varphi(z)^{-1}) \in \lambda_{Y_2}(\alpha(y),\alpha(z)) \\ &\iff \psi(gG_{1,y}) \leq_2 \psi(hG_{1,z}) \end{aligned}$$

so ψ is preorder-preserving and preorder-reflecting and thus $\underline{P}_1 \cong \underline{P}_2$ as required. □

Corollary 1.4.12 *Let $\underline{Y}_1 = (Y_1, \leq_{Y_1}, (G_{1,y})_{y\in Y_1}, \lambda_{Y_1})$ be a $\underline{G}_1$-annotated preordered set and $\underline{Y}_2 = (Y_2, \leq_{Y_2}, (G_{2,y})_{y\in Y_2}, \lambda_{Y_2})$ be a $\underline{G}_2$-annotated preordered set with $\underline{Y}_1 \cong \underline{Y}_2$. Then there exists a preorder automorphism $\psi : \mathrm{rec}_{\underline{G}_1}(\underline{Y}_1) \longmapsto \mathrm{rec}_{\underline{G}_2}(\underline{Y}_2)$ such that $\iota[G_1] = \psi^{-1}\iota[G_2]\psi$.*

Proof Let ψ as in the proof of Theorem 1.4.11. Then one can see that $\psi\iota[G_1]\psi^{-1} \subseteq \iota[G_2]$ and $\psi^{-1}\iota[G_2]\psi \subseteq \iota[G_1]$ similar to the proof of Lemma 1.3.5. This shows $\iota[G_1] = \psi^{-1}\iota[G_2]\psi$. □

Finally we see that our idea of unfolding preorder orbifolds is indeed the inversion of folding preordered sets.

Theorem 1.4.13 *Let $\underline{P} = (P, \leq_P)$ be a preordered set, $\Gamma \leq \underline{\mathrm{Aut}}(\underline{P})$ and $(Q, \leq_Q, (\underline{G}_q)_{q\in Q}, \lambda_Q)$ a $\underline{G}$-annotated preordered set. Then*

1) $\mathrm{rec}_\Gamma(\mathrm{rep}_\Gamma(P, \leq_P)) \cong (P, \leq_P)$ *and*

2) $\mathrm{rep}_{\iota[\underline{G}]}(\mathrm{rec}_{\underline{G}}(Q, \leq_Q, (\underline{G}_q)_{q\in Q}, \lambda_Q)) \cong (Q, \leq_Q, (\underline{G}_q)_{q\in Q}, \lambda_Q)$.

Proof The claim 1) has already been proven in Proposition 1.4.1. For 2) to see let $\mathrm{rep}_{\iota[\underline{G}]}(\mathrm{rec}_{\underline{G}}(Q, \leq_Q, (\underline{G}_q)_{q\in Q}, \lambda_Q)) = (S, \leq_S, (\bar{\underline{G}}_s)_{s\in S}, \lambda_S)$ where $\bar{\underline{G}}_s \leq \iota[\underline{G}]$ for each $s \in S$. Let us choose for every $q \in Q$ a $g_q \in G$ such that $g_qG_q \in S$. We then define

$$\begin{array}{rccc} \alpha: & Q & \longrightarrow & S \\ & q & \longmapsto & g_qG_q. \end{array}$$

It is obvious that α is bijective since the mapping $g_qG_q \longmapsto q$ describes the inverse mapping of α. Furthermore by Proposition 1.4.6 the mapping $\iota : \underline{G} \longmapsto \iota[\underline{G}]$ is a group isomorphism. Finally we define

$$\begin{array}{rccc} \varphi: & Q & \longrightarrow & \iota[G] \\ & q & \longmapsto & \iota(g_q^{-1}). \end{array}$$

We then observe that $\alpha(q) = g_q G_q = \varphi(q)^{-1} G_q$ for $q \in Q$. With this we obtain

$$\begin{aligned}
\varphi(p)^{-1}\lambda_S(\alpha(p),\alpha(q))\varphi(q) &= \varphi(p)^{-1}\{\, \iota(g) \in \iota[G] \mid \alpha(p) \leq_{\mathrm{rec}_{\underline{G}}} \iota(g)\alpha(q) \,\}\varphi(q) \\
&= \varphi(p)^{-1}\{\, \iota(g) \in \iota[G] \mid \varphi(p)^{-1}G_p \leq_{\mathrm{rec}_{\underline{G}}} g\varphi(q)^{-1}G_q \,\}\varphi(q) \\
&= \{\, \iota(g) \in \iota[G] \mid G_p \leq_{\mathrm{rec}_{\underline{G}}} gG_q \,\} \\
&= \{\, \iota(g) \in \iota[G] \mid g \in \lambda_Q(p,q) \,\} \\
&= \iota[\lambda_Q(p,q)]
\end{aligned}$$

and

$$\begin{aligned}
\varphi(p)^{-1}\bar{G}_{\alpha(p)}\varphi(p) &= \varphi(p)^{-1}\{\, \iota(g) \in \iota[G] \mid \iota(g)(\alpha(p)) = \alpha(p) \,\}\varphi(p) \\
&= \varphi(p)^{-1}\{\, \iota(g) \in \iota[G] \mid g\varphi(p)^{-1}G_q = \varphi(p)^{-1}G_q \,\}\varphi(p) \\
&= \{\, \iota(g) \in \iota[G] \mid gG_q = G_q \,\} \\
&= \iota[G_q]
\end{aligned}$$

where $p, q \in Q$ and $\leq_{\mathrm{rec}_{\underline{G}}}$ is the preorder relation of $\mathrm{rec}_{\underline{G}}(Q, \leq_Q, (G_q)_{q\in Q}, \lambda_Q)$. Thus we get $(Q, \leq_Q, (G_q)_{q\in Q}, \lambda_Q) \cong \mathrm{rep}_{\iota[\underline{G}]}(\mathrm{rec}_{\underline{G}}(Q, \leq_Q, (G_q)_{q\in Q}, \lambda_Q))$ as required. □

1.5 Visualization of Group-annotated Preordered Sets

Now that we have a precise understanding of how to fold and unfold preordered sets it would be nice to have a way to visualize preorder orbifolds in a similar way as can be done with ordered or preordered sets. We shall see that the notion of order diagrams can be generalized to perform this task. The generalization will lead to the concept of *group-annotated preorder diagrams* in the same way as folding preordered sets leads to group-annotated preordered sets.

Firstly we start with a simple yet important observation.

Proposition 1.5.1 *Let $\underline{G} = (G, \circ)$ be a group, $\underline{P} = (P, \leq_P)$ be a preordered set and $((G_p)_{p\in P}, \lambda)$ be a $\underline{G}$-annotation of $\underline{P}$. Then for all $a, b \in P$ it is*

$$G_a \circ \lambda(a,b) \circ G_b = \lambda(a,b).$$

Proof Since $G_a \subseteq \lambda(a,a)$ and $\lambda(a,a) \circ \lambda(a,b) \circ \lambda(b,b) \subseteq \lambda(a,b)$ we only have to show that $\lambda(a,b) \subseteq G_a \circ \lambda(a,b) \circ G_b$. But this is clear since $e_{\underline{G}} \in G_a, e_{\underline{G}} \in G_b$ where $e_{\underline{G}}$ is the neutral element of $\underline{G}$. □

What this proposition gives us is that every annotation of a pair (a,b) where $a \neq b$ can be written as a union of *double cosets*

$$\lambda(a,b) = \bigcup_{c\in\lambda(a,b)} G_a c G_b.$$

Since double cosets yield, in the same way as cosets do, a partition of $\lambda(a,b)$ we can choose representatives of the double cosets instead of remembering the whole set $\lambda(a,b)$. We can further observe that the sets

$$\lambda(a,b) \setminus \bigcup_{a<c<b} \lambda(a,c) \circ \lambda(c,b) = \bigcup_{\substack{d \in \lambda(a,b),\\ d \notin \lambda(a,c)\circ\lambda(c,b),\\ a<c<b}} G_a d G_b$$

are also disjoint unions of double cosets. Therefore we can abridge the annotation function λ in a considerable way.

Definition 1.5.2 (Abridged Annotation Function) Let $\underline{Y} = (Y, \leq_Y, (G_y)_{y\in Y}, \lambda_Y)$ be a $\underline{G}$-annotated preordered set. Let for each $a, b \in Y$ with $a \neq b$ denote with $\lambda_{\text{abr}}(a,b)$ a set of representatives of

$$\lambda_Y(a,b) \setminus \bigcup_{a<_Y c<_Y b} \lambda_Y(a,c) \circ \lambda_Y(c,b).$$

Then the mapping λ_{abr} is called an *abridged annotation function* of $\underline{Y}$.

Let $e_{\underline{G}}$ denote the neutral element of $\underline{G}$. If $e_{\underline{G}} \in \lambda_{\text{abr}}(a,b)$ whenever $e_{\underline{G}}$ can be chosen as a representative of a double coset of $\lambda_Y(a,b) \setminus \bigcup_{a<_Y c<_Y b} \lambda_Y(a,c) \circ \lambda_Y(c,b)$ then λ_{abr} is said to be a *normalized abridged annotation function* of $\underline{Y}$. $\diamondsuit$

Remark 1.5.3 Double cosets are an abstract notion from group theory but they allow an interpretation in the case of folding a preordered set. For this let $\underline{P} = (P, \leq)$ be such a preordered set and let $\Gamma \leq \underline{\text{Aut}}(\underline{P})$. Then we want to consider the folding $\text{rep}_\Gamma(\underline{P})$ and in particular the interpretation of double cosets of annotations $\lambda(a,b)$ in the original structure $\underline{P}$.

So we fix an abridged annotation function λ_{abr}, $a, b \in P$ and choose an automorphism $\beta \in \lambda_{\text{abr}}(a,b)$. Then we know that

$$a \leq \beta(b)$$

by definition. But we know for all $\gamma \in \Gamma_a \circ \beta \circ \Gamma_b$ that

$$a \leq \gamma(b) = \gamma_a \beta \gamma_b(b) = \gamma_a \beta(b)$$

with $\gamma = \gamma_a \circ \beta \circ \gamma_b$ and $\gamma_a \in \Gamma_a, \gamma_b \in \Gamma_b$. On the other hand we know with $\gamma_a \in \Gamma_a$ that $a = \gamma_a(a) \leq \gamma_a\beta(b)$. Therefore the double coset $\Gamma_a \circ \beta \circ \Gamma_b$ corresponds to the collection of all elements $\{\, \gamma_a\beta(b) \mid \gamma_a \in \Gamma_a \,\}$ in a bijective way. But this allows us to directly see the number of cosets and some representatives from a preorder diagram if $a \lessdot b$: For the element a compute its stabilizer Γ_a. Then compute the orbits of Γ_a on $\Gamma(b)$ and for every orbit containing an element above a choose an automorphism $\beta \in \Gamma$ which maps b into this orbit. Then the collection of all those automorphisms will be a set of representatives for the double cosets of λ. Note that for the general case $a \leq b$ one has to take into account that certain double cosets might not occur because of the set difference we used to define abridged annotation functions.

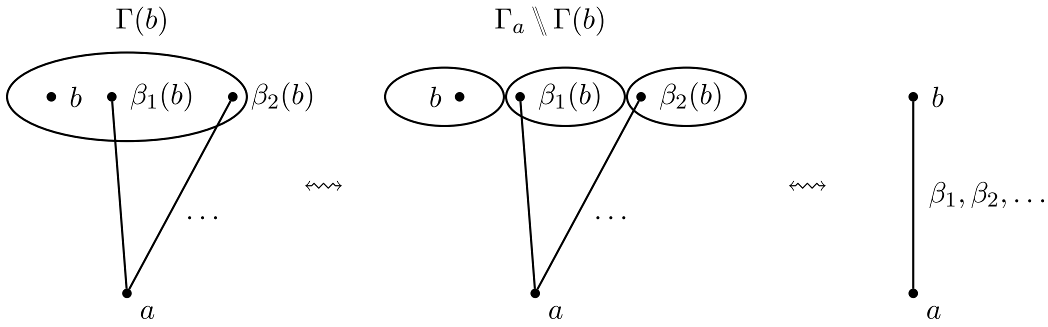


Figure 1.5: Interpretation of double cosets when folding preordered sets.

If we now want to draw a group-annotated preordered set we simply draw a preorder diagram with each edge attached to its abridged annotation given by an abridged annotation function. We might omit the annotation if it is trivial, i.e. the neutral element of the group is the only representative. In the special case of Proposition 1.2.7 we can simplify the annotation even more because $\underline{G}_a = \lambda_Y(a, a)$ for all $a \in Y$, therefore we do not have to mention $\underline{G}_a$ explicitly since it can be computed easily from the element a itself.

Example 1.5.4 Let us consider Example 1.2.8 again. The annotation function shown in table 1.1 can be abridged to the annotation function

$\lambda_{\text{abr}}($$,$$) = \{\,(1)\,\}$

$\lambda_{\text{abr}}($$,$$) = \emptyset$

$\lambda_{\text{abr}}($$,$$) = \emptyset$

$\lambda_{\text{abr}}($$,$$) = \{\,(23)\,\}$

$\lambda_{\text{abr}}($$,$$) = \emptyset$

$\lambda_{\text{abr}}($$,$$) = \{\,(24)\,\}$

$\lambda_{\text{abr}}($$,$$) = \emptyset$

$\lambda_{\text{abr}}($$,$$) = \emptyset$

$\lambda_{\text{abr}}($$,$$) = \{\,(23)\,\}$

$\lambda_{\text{abr}}($$,$$) = \emptyset$

$\lambda_{\text{abr}}($$,$$) = \{\,(1)\,\}$

$\lambda_{\text{abr}}($$,$$) = \{\,(1)\,\}$

$\diamondsuit$

With the help of this abridged annotation function we can now draw the preorder orbifold of the connected graphs on 4 vertices by the full symmetric group on 4 elements very concisely as can be seen in Figure 1.6. Note that this diagram contains all information of the original ordered set.

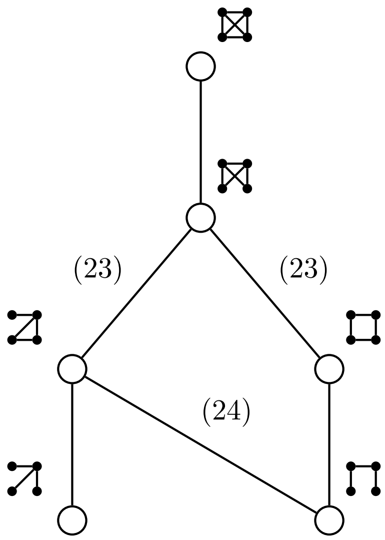


Figure 1.6: Preorder Orbifold of the connected graphs on 4 vertices by the full symmetric group on 4 elements.

In some cases it might be possible that we have to draw edges in the preorder diagram we normally do not have to draw. These edges then have to be drawn if they have a nonempty abridged annotation since otherwise we would lose information in the graphical visualization.

Example 1.5.5 Let us consider the lattice $\underline{L}$ shown in Figure 1.7 together with the automorphism

$$\alpha = (\bot)(123)(456)(789)(\top).$$

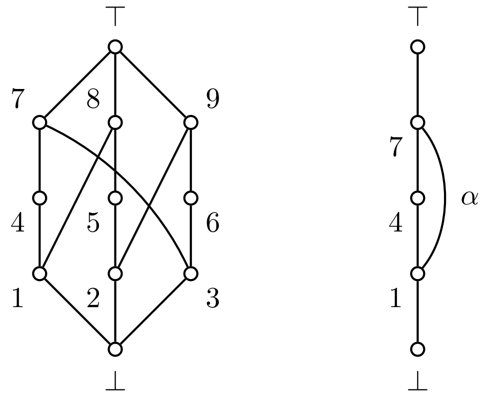
We would like to compute the preorder orbifold $\mathrm{rep}_\Gamma(\underline{L})$ where $\Gamma = \langle\alpha\rangle = (\{\,(1), \alpha, \alpha^2\,\}, \circ)$. The result is also shown in Figure 1.7 and we see that we had to draw an additional edge not to lose information given by the annotation function.

This special case motivates the following definition.

Definition 1.5.6 (Long Edges) Let $\underline{Y} = (Y, \leq_Y, (G_y)_{y\in Y}, \lambda_Y)$ be a $\underline{G}$-annotated preordered set and let λ_{abr} be an abridged annotation function of $\underline{Y}$. A pair (a, b) where $a, b \in Y$ is called a *long edge* of $\underline{Y}$ if and only if

- $\lambda_{\mathrm{abr}}(a, b) \neq \emptyset$ and
- there exists $c \in Y$ with $a <_Y c <_Y b$. $\diamondsuit$

One might ask whether the occurrence of long edges depends on the choice of the abridged annotation function. The following proposition answers this question.



$\diamondsuit$

Figure 1.7: An example lattice $\underline{L}$ and one lattice orbifold of $\underline{L}$ by Γ, taken from [Zw].

Proposition 1.5.7 *Let* $\underline{Y}_1 = (Y_1, \leq_{Y_1}, (\underline{G}_{1,y})_{y \in Y_1}, \lambda_{Y_1})$ *be a* $\underline{G}_1$*-annotated preordered set and* $\underline{Y}_2 = (Y_2, \leq_{Y_2}, (\underline{G}_{2,y})_{y \in Y_2}, \lambda_{Y_2})$ *be a* $\underline{G}_2$*-annotated preordered set with* $\underline{Y}_1 \cong \underline{Y}_2$ *and* α *a corresponding preorder automorphism. Let* $\lambda_{\mathrm{abr},1}$ *be an abridged annotation function of* $\underline{Y}_1$ *and* $\lambda_{\mathrm{abr},2}$ *be an abridged annotation function of* $\underline{Y}_2$*. Then* $|\lambda_{\mathrm{abr},1}(a,b)| = |\lambda_{\mathrm{abr},2}(\alpha(a), \alpha(b))|$ *for all* $a, b \in Y$*.*

Proof Let $a, b \in Y$. Let $\delta : \underline{G}_1 \longrightarrow \underline{G}_2$ and $\varphi : Y_1 \longrightarrow G_2$ such that

$$\delta[\lambda_{Y_1}(a,b)] = \varphi(a)^{-1} \lambda_{Y_2}(\alpha(a), \alpha(b)) \varphi(b) \quad \text{and} \quad \delta[G_{1,y}] = \varphi(y)^{-1} G_{2,\alpha(y)} \varphi(y).$$

We define the bijective mapping

$$\begin{array}{rccl} \zeta : & \lambda_{Y_1}(a,b) & \longrightarrow & \lambda_{Y_2}(\alpha(a), \alpha(b)) \\ & x & \longmapsto & \varphi(a)\delta(x)\varphi(b)^{-1}. \end{array}$$

Then ζ maps double cosets of $G_{1,a}$ and $G_{1,b}$ to double cosets of $G_{2,\alpha(a)}$ and $G_{2,\alpha(b)}$ because

$$\begin{aligned} \zeta[G_{1,a} x G_{1,b}] &= \varphi(a)\delta[G_{1,a}]\delta(x)\delta[G_{1,b}]\varphi(b)^{-1} \\ &= G_{2,\alpha(a)} \varphi(a)\delta(x)\varphi(b)^{-1} G_{2,\alpha(b)} \\ &= G_{2,\alpha(a)} \zeta(x) G_{2,\alpha(b)}. \end{aligned}$$

This shows that $G_{1,a} x G_{1,b} \longmapsto G_{2,\alpha(a)} \zeta(x) G_{2,\alpha(b)}$ is well defined and bijective and hence

$$\begin{aligned} |\lambda_{\mathrm{abr},1}(a,b)| = |\{ G_{1,a} x G_{1,b} \mid x \in \lambda_{Y_1}(a,b) \}| &= |\{ G_{2,\alpha(a)} y G_{2,\alpha(b)} \mid y \in \lambda_{Y_2}(a,b) \}| \\ &= |\lambda_{\mathrm{abr},2}(\alpha(a), \alpha(b))|. \quad \square \end{aligned}$$

With this we get the independence of the existence of long edges of the choice of the abridged annotation function.

Corollary 1.5.8 *Let $\underline{Y} = (Y, \leq_Y, (G_y)_{y\in Y}, \lambda_Y)$ be a $\underline{G}$-annotated preordered set and λ_{abr} an abridged annotation function of $\underline{Y}$. Then for every abridged annotation function λ'_{abr} of $\underline{Y}$ it holds*

$$\lambda_{\text{abr}}(a,b) = \emptyset \iff \lambda'_{\text{abr}}(a,b) = \emptyset.$$

Proof This follows from Proposition 1.5.7, the fact that $\underline{Y} \cong \underline{Y}$ and $\lambda_{\text{abr}}(a,b) = \emptyset \iff |\lambda_{\text{abr}}(a,b)| = 0$. □

Now the occurrence of long edges when folding preordered sets allows an easy characterization by means of the original preordered set and the chosen automorphism group.

Proposition 1.5.9 *Let $\underline{P} = (P, \leq_{\underline{P}})$ be a preordered set, $\underline{Y} = (Y, \leq_Y, (G_y)_{y\in Y}, \lambda_Y)$ a preorder orbifold of $\underline{P}$ under some group $\Gamma \leq \underline{\text{Aut}}(\underline{P})$ of automorphisms. Then for $p, q \in Y$ the following conditions are equivalent:*

1. *(p,q) is a long edge of $\underline{Y}$ and*
2. *there exist $\alpha, \beta \in \Gamma$ and $x \in P$ such that p is lower neighbor of $\beta(q)$ in $\underline{P}$, $p <_{\underline{P}} x$ and $\alpha(x) <_{\underline{P}} \beta(q)$.*

Proof The situation is shown in Figure 1.8.

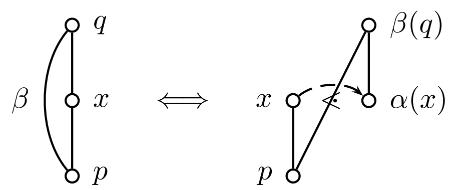


Figure 1.8: Situation where long edges appear in a preorder orbifold.

($\Longrightarrow$) Let (p,q) be a long edge in $\underline{Y}$ and let λ_{abr} be an abridged annotation function of $\underline{Y}$. Then $\lambda_{\text{abr}}(p,q) \neq \emptyset$ and there exists $r \in Y$ with $p <_Y r <_Y q$. Hence there exists $\beta \in \lambda_{\text{abr}}(p,q)$ with $p <_{\underline{P}} \beta(q)$. Suppose there exists $s \in P$ with $p <_{\underline{P}} s <_{\underline{P}} \beta(q)$. Then there exists $\gamma \in \Gamma$ with $\gamma(s) \in Y$ and then $\beta = \gamma^{-1}\gamma\beta \in \lambda(p, \gamma(s)) \circ \lambda(\gamma(s), q)$ and hence $\beta \notin \lambda_{\text{abr}}(p,q)$, a contradiction. Therefore p is lower neighbor of $\beta(q)$. Furthermore there exist $\beta_1 \in \lambda_Y(p,r)$ and $\beta_2 \in \lambda_Y(r,q)$ with $p <_{\underline{P}} \beta_1(r)$ and $r <_{\underline{P}} \beta_2(q)$. Now we define $x := \beta_1(r)$ and $\alpha := \beta\beta_2^{-1}\beta_1^{-1}$. Then $p <_{\underline{P}} x$ and $\alpha(x) <_{\underline{P}} \beta(q)$ as required.

($\Longleftarrow$) Let $\alpha, \beta \in \Gamma$ and $x \in P$ such that p is lower neighbor of $\beta(q)$ in $\underline{P}$, $p <_{\underline{P}} x$ and $\alpha(x) <_{\underline{P}} \beta(q)$. Let $y \in Y$ with $p <_Y y <_Y q$ and suppose $\beta \in \lambda(p,y)\lambda(y,q)$. Then there exist $\beta_1 \in \lambda(p,y)$ and $\beta_2 \in \lambda(y,q)$ with $\beta = \beta_1\beta_2$. Then $p <_{\underline{P}} \beta_1(y)$ and $\beta_1(y) <_{\underline{P}} \beta_1\beta_2(q)$ in contradiction to p being a lower neighbor of $\beta(q)$. Therefore there exists an abridged annotation function λ_{abr} of $\underline{Y}$ such that $\beta \in \lambda_{\mathrm{abr}}(p,q)$ and by Proposition 1.5.7 every abridged annotation function λ_{abr} satisfies $\lambda_{\mathrm{abr}}(p,q) \neq \emptyset$. Furthermore there exists $\gamma \in \Gamma$ such that $\gamma(x) \in Y$. Then $p <_Y \gamma(x)$ and $\gamma(x) <_Y q$ and thus (p,q) is a long edge of $\underline{Y}$. $\square$

1.6 A `GAP` package for orbifolds

To illustrate the rather theoretical view on group-annotated preordered sets an implementation in `GAP`, a computer algebra system specialized in group theory, has been made providing basic functionalities together with general binary relation structures and group-annotated binary relation structures. The package is named `ctxorb` and is freely available through `http://www.math.tu-dresden.de/~borch/math/ctxorb/`. The purpose of this paragraph is to give an impression of this package by means of an example.

Example 1.6.1 This example intents to show the basic usage of `ctxorb` when dealing with binary relation structures and binary relation structure orbifolds.
After starting `GAP` from the command line we firstly load `ctxorb`

```
gap> LoadPackage("ctxorb");
GAP package for context orbifolds in version 0.1
true
gap>
```

Now we are able to construct binary relation structures:

```
gap> brs := BinaryRelationStructure([1,2,3],[1,2,3],\<);
Binary Relation Structure
gap> Display(brs);
Binary Relation Structure
  Source:      [ 1, 2, 3 ]
  Sourcenames: [ 1, 2, 3 ]
  Range:       [ 1, 2, 3 ]
  Rangenames:  [ 1, 2, 3 ]
  Relation:
       .xx
       ..x
       ...

gap>
```

This constructs a binary relation structure `brs` as[2] $(\{1,2,3\},\{1,2,3\},<)$ and the command `Display` gives us a more detailed output of the binary relation structure. The output given may be a little bit confusing since the set $\{1,2,3\}$, here printed as `[ 1, 2, 3 ]`,

[2]The backslash $\backslash$ in front of the $<$ sign is necessary for `GAP` to interpret the $<$ sign as a variable.

is displayed four times, whereas we only would expect it twice. This comes from a basic principle of the package `ctxorb`, which it has borrowed from the package `grape` (see [GRAPE]), a `GAP` package for working with graphs and on which `ctxorb` is implemented: every element of the source and the range of the binary relation structure, that is every element of the first and second argument of the constructor, is assigned an index to, starting with 1. This index is the internal representation of this element as a member of the binary relation structure object and all computations are done internally only by means of these indices.

```
gap> brs2 := BinaryRelationStructure([7,8,9],[10,11,12],\<);
Binary Relation Structure
gap> Display(brs2);
Binary Relation Structure
  Source:      [ 1, 2, 3 ]
  Sourcenames: [ 7, 8, 9 ]
  Range:       [ 4, 5, 6 ]
  Rangenames:  [ 10, 11, 12 ]
  Relation:
       xxx
       xxx
       xxx

gap>
```

Here we see that the element 7 gets the index 1, the element 8 gets the index 2 and so on, so finally the source and the range of the binary relation structure `brs2` consist of all the indices $\{1,2,3\}$ and $\{4,5,6\}$ respectively. In contrast to this are the *names* of the indices the original arguments given to the constructor. They can be retrieved easily:

```
gap> BinaryRelationStructureSourceName(brs2, 2);
8
gap> BinaryRelationStructureRangeName(brs2,6);
12
gap>
```

Sometimes it is even desirable to have more complex objects as elements of the source and the range of a binary relation structure. For this one has to know that the constructor for binary relation structures expects *sets* as arguments, which are represented in `GAP` as sorted lists. If the constructor does not get proper sets it may issue a confusing error:

```
gap> brs3 := BinaryRelationStructure([7,8,9],[9,8,7],\<);
<obj> must be a function (not a boolean)
gap>
```

This error can be eliminated with the `GAP` function `AsSet`, which converts lists into `GAP` sets:

```
gap> brs3 := BinaryRelationStructure(AsSet([7,8,9]),AsSet([9,8,7]),\<);
Binary Relation Structure
gap>
```

Now that we have constructed binary relation structures we are able to compute their automorphism groups and fold them by subgroups of their automorphism groups.

```
gap> brs4 := BinaryRelationStructure([1..10],[1..10],
>            function(x,y)
>              return (x+y) mod 2 = 0;
>            end);
Binary Relation Structure
gap> Display(brs4);
Binary Relation Structure
  Source:      [ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ]
  Sourcenames: [ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ]
  Range:       [ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ]
  Rangenames:  [ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ]
  Relation:
       x.x.x.x.x.
       .x.x.x.x.x
       x.x.x.x.x.
       .x.x.x.x.x
       x.x.x.x.x.
       .x.x.x.x.x
       x.x.x.x.x.
       .x.x.x.x.x
       x.x.x.x.x.
       .x.x.x.x.x

gap> AutomorphismGroup(brs4);
Group([ (8,10), (6,8), (4,6), (2,4), (7,9), (5,7), (3,5),
(1,2)(3,4)(5,6)(7,8)(9,10) ])
gap> StructureDescription(last);
"(S5 x S5) : C2"
gap>
```

The function `StructureDescription` gives a mathematical interpretation of a given group. In our case the string `"(S5 x S5) : C2"` means that the automorphism group of `brs4` is isomorphic to $(S_5 \times S_5) \rtimes C_2$, where C_2 is the cyclic group with 2 elements.

Now if we want to compute the binary relation structure orbifold of `brs4` under its automorphism group we get a group annotated binary relation structure with a full annotation function (in form of a table). This table might be very large and therefore unsuitable for output. To get an abridged annotation of the orbifold we use the method `AbridgedAnnotation` on `brs4`. This abridged annotation will also be displayed instead of the full annotation on further calls to `Display`.

```
gap> fld_brs := FoldBinaryRelationStructure(brs4);
Group Annotated Binary Relation Structure with Size 1x1x1
gap> Display(fld_brs);
# lots of output
gap> AbridgedAnnotation(fld_brs);
[ [ [ 1, 1 ], [ (), (1,3) ] ] ]
gap> Display(fld_brs);
Group Annotated Binary Relation Structure:
  Underlying Binary Relation Structure:
Binary Relation Structure
  Source:      [ 1 ]
  Sourcenames: [ 1 ]
  Range:       [ 1 ]
  Rangenames:  [ 1 ]
  Relation:
       x

  Group: Group( [ ( 8,10), ( 6, 8), ( 4, 6), ( 2, 4), ( 7, 9), ( 5, 7),
  ( 3, 5), ( 1, 2)( 3, 4)( 5, 6)( 7, 8)( 9,10) ] )
  Abridged Annotation:
    eta(1, 1) = [ (), ( 1, 3) ]

gap>
```

We shall consider a more complicated example in section 2.3 to give more insight into the usage of `ctxorb`. For more information one might also want to consult the help system coming with `ctxorb` which is available online or directly from the `GAP` commandline by issuing

```
gap> ?ctxorb
```

and navigating forward and backward with `?>` and `?<` respectively, or by directly asking for information on particular functions with

```
gap> ?BinaryRelationStructure
```

◇

2 Context Orbifolds

What we are now going to do is to transform most of the results we have obtained in chapter 1 from preordered sets to (formal) contexts to get *context orbifolds* and *group-annotated contexts*. As we shall see this transformation is mostly straight forward and nearly all of the proofs can already be found in the previous chapter. But we shall also see that we can do more: it is possible to define a special kind of context derivation for context orbifolds which is strongly correlated to the derivation in original contexts. By the help of this notion we are even able to construct the lattice orbifold of the concept lattice of a context out of the context orbifold of the same context.

2.1 Context Orbifolds and Group-annotated Contexts

To fold contexts we simply do the same what we have done when folding preordered sets. To have a concise notation let us first introduce some conventions: Let $\mathbb{K} = (G, M, I)$ be a context and $\Gamma \leq \underline{\mathrm{Aut}}(\mathbb{K})$. Then every $\alpha \in \Gamma$ is traditionally interpreted as a pair of permutations acting on the objects and attributes respectively. For the purpose of clarity we may here consider a context automorphism differently, that is as a permutation on $G \mathbin{\dot\cup} M$ with

$$\alpha[G] = G, \alpha[M] = M.$$

Furthermore for all $g \in G, m \in M$ it holds

$$g\ I\ m \iff \alpha(g)\ I\ \alpha(m).$$

Now we see that context automorphisms are a special kind of automorphisms of the binary relation structure $(G \mathbin{\dot\cup} M, G \mathbin{\dot\cup} M, \tilde{I})$ with

$$(g, m) \in I \iff (g, m) \in \tilde{I},$$

thus we can easily transform the ideas of the preceding chapter to contexts and context orbifolds.

2.1.1 Context Orbifolds and Group-annotated Contexts

Definition 2.1.1 (Context Orbifold) Let $\mathbb{K} = (G, M, I)$ be a context and $\Gamma \leq \underline{\mathrm{Aut}}(\mathbb{K})$. Let Y_G a transversal of the orbits of Γ on G and Y_M a transversal of the orbits of Γ on M. We further define

$$\begin{array}{rccc} \eta: & Y_G \times Y_M & \longrightarrow & \mathfrak{P}(\Gamma) \\ & (g, m) & \longmapsto & \{\, \alpha \in \Gamma \mid g\ I\ \alpha(m) \,\} \end{array}$$

and

$$I_{\text{rep}} = \{\, (g, m) \in Y_G \times Y_M \mid \eta(g, m) \neq \emptyset \,\}.$$

Then the tuple

$$\text{rep}_\Gamma(\mathbb{K}) := (Y_G, Y_M, I_{\text{rep}}, (\Gamma_g)_{g \in Y_G}, (\Gamma_m)_{m \in Y_M}, \eta)$$

is called a *context orbifold (or representation) of* $\mathbb{K}$ *under* Γ. $\diamondsuit$

Example 2.1.2 1) We may have a look at the context shown in Figure 2.1. The context

$$\mathbb{K} = \begin{array}{|c|ccc|ccc|} \hline I & 1 & 2 & 3 & 5 & 6 & 7 \\ \hline 1 & \times & & & \times & \times & \times \\ 2 & & \times & & \times & \times & \times \\ 3 & & & \times & \times & \times & \times \\ \hline 5 & & & & \times & & \\ 6 & & & & & \times & \\ 7 & & & & & & \times \\ \hline \end{array} \qquad \begin{array}{l} \alpha = (\alpha_G, \alpha_M) = ((123)(567), (123)(567)), \\ \Gamma = \langle \alpha \rangle = (\{\, \text{id}, \alpha, \alpha^2 \,\}, \circ) \end{array}$$

Figure 2.1: Standard context of the lattice $\underline{L}$ depicted in Figure 1.1.

$\mathbb{K} = (G, M, I)$ is the standard context of the lattice $\underline{L}$ of Figure 1.1 together with a context automorphism α and the group Γ generated by α. We want to compute a context orbifold of $\mathbb{K}$ under Γ. For this we choose as transversals of the orbits of Γ on G and M, respectively, the sets

$$Y_G = \{\, 1, 5 \,\} \quad \text{and} \quad Y_M = \{\, 1, 5 \,\}$$

and we obtain the Γ-annotated context shown in Figure 2.2.

	1	5
1	$\{\,\text{id}\,\}$	Γ
5	$\emptyset$	$\{\,\text{id}\,\}$

Figure 2.2: A context orbifold of 2.1 under Γ.

2) Let M be a set and Δ_M be the diagonal relation on M, that is

$$\Delta_M = \{\, (m, m) \mid m \in M \,\}.$$

The context $\mathbb{K} = (M, M, \Delta_M)$ has as context automorphisms the mappings α for each $\alpha \in S_M$ acting on the objects and the attributes simultaneously. Defining Γ

as the group of automorphisms induced by S_M and choosing one arbitrary element $m \in M$ the Γ-annotated context

	m
m	Γ_m

represents a context orbifold of $\mathbb{K}$ by Γ.

3) Let us consider the ordinal scale $(\mathbb{Z}, \mathbb{Z}, \leq)$ and the context automorphism $\alpha = (\mathbb{Z}, \mathbb{Z}, x \longmapsto x + 2)$. Let $\Gamma = \langle \alpha \rangle$. If we choose as a transversal of the orbits of Γ the set $\{ 0, 1 \}$ we get the context orbifold

	0	1
0	$\{ (\mathbb{Z}, \mathbb{Z}, x \longmapsto x + 2k) \mid k \geq 0 \}$	$\{ (\mathbb{Z}, \mathbb{Z}, x \longmapsto x + 2k) \mid k \geq 0 \}$
1	$\{ (\mathbb{Z}, \mathbb{Z}, x \longmapsto x + 2k) \mid k > 0 \}$	$\{ (\mathbb{Z}, \mathbb{Z}, x \longmapsto x + 2k) \mid k \geq 0 \}$

which is "the corresponding" context orbifold of the preorder orbifold of Example 1.2.2, 2). $\diamondsuit$

Now we can, in nearly the same way as we did for preorder orbifolds, define an abstract structure for context orbifolds.

Definition 2.1.3 (Group-annotated Context) Let $\mathbb{K} = (G, M, I)$ be a context and $\underline{H}$ be a group. Let $\eta : G \times M \longrightarrow \mathfrak{P}(H)$ satisfying $\eta(g, m) \neq \emptyset$ if and only if $g\ I\ m$ and let $(\underline{H}_g)_{g \in G}$, $(\underline{H}_m)_{m \in M}$ be families of subgroups of $\underline{H}$ such that

$$\bigcap_{h \in H, g \in G} h^{-1} H_g h \cap \bigcap_{h \in H, m \in M} h^{-1} H_m h = \{ e_{\underline{H}} \}$$

where $e_{\underline{H}}$ is the neutral element of $\underline{H}$. Then the tuple

$$(G, M, I, (\underline{H}_g)_{g \in G}, (\underline{H}_m)_{m \in M}, \eta)$$

is said to be an $\underline{H}$*-annotated context of* $\mathbb{K}$. The triple $((\underline{H}_g)_{g \in G}, (\underline{H}_m)_{m \in M}, \eta)$ is called the corresponding $\underline{H}$*-annotation* of the $\underline{H}$-annotated context. $\diamondsuit$

Proposition 2.1.4 *Let* $\mathbb{Y} = (Y_G, Y_M, I_{\text{rep}}, (\Gamma_g)_{g \in Y_G}, (\Gamma_m)_{m \in Y_M}, \eta)$ *be a context orbifold of* $\mathbb{K} = (G, M, I)$ *under the group* $\Gamma \leq \underline{\mathrm{Aut}}(\mathbb{K})$*. Then* $\mathbb{Y}$ *is an* Γ*-annotated context.*

Proof Clearly it is $y_g\ I_{\text{rep}}\ y_m \iff \eta(y_g, y_m) \neq \emptyset$ by definition for all $y_g \in Y_G, y_m \in Y_M$. Furthermore it is

$$\bigcap_{\beta \in \Gamma, y \in G \dot\cup M} \beta^{-1} \Gamma_y \beta = \bigcap_{\beta \in \Gamma, y \in G \dot\cup M} \Gamma_{\beta(y)} = \{ \mathrm{id} \}$$

since the only automorphism of $\mathbb{K}$ fixing all $g \in G$ and $m \in M$ must be the identity mapping. Hence $\mathbb{Y}$ is a Γ-annotated context. $\square$

Remark 2.1.5 Because of group-annotated contexts being (in some sense) relation transversals we are able to give a representation of $\eta(g,m)$ as disjoint union of double cosets, i.e.

$$\eta(g,m) = \dot{\bigcup_{\alpha \in R}} H_g \alpha H_m$$

where R is a set of representatives. We therefore may define the notion of an *abridged annotation function* η_{abr} as a set of double coset representatives of $\eta(g,m)$ for every pair $(g,m) \in G \times M$. Note that in contrast to λ_{abr} we have that

$$\eta(g,m) = \eta(g,m) \setminus \bigcup_{gIyIm} \eta(g,y) \circ \eta(y,m)$$

since no such y exists, as we always consider the sets G and M to be disjoint, therefore a (useful) reduction as has been done for λ_{abr} is not possible.

2.1.2 Isomorphy of Group-annotated Contexts

Now we can ask the same question we have already asked for preorder orbifolds: Are two context orbifolds of one context to be considered the same or are they inherently different? Because of the results we have gotten for preorder orbifolds we can expect them to be isomorphic in some sense. The definition of isomorphy and all corresponding proofs are very similar as in the case of preorder orbifolds. We start by considering isomorphy of group annotated contexts.

Definition 2.1.6 (Isomorphy of Group-annotated Contexts) Let $\mathbb{Y}_i = (G_i, M_i, I_i, (\underline{H}_{i,g})_{g \in G}, (\underline{H}_{i,m})_{m \in M}, \eta_i)$ be an $\underline{H}_i$-annotated context for $i \in \{1, 2\}$. The two contexts $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are said to be isomorphic iff

- there exists a context isomorphism $\alpha : (G_1, M_1, I_1) \longrightarrow (G_2, M_2, I_2)$,
- there exists a group isomorphism $\delta : \underline{H}_1 \longrightarrow \underline{H}_2$ and
- there exists a mapping $\varphi : G_1 \dot{\cup} M_1 \longrightarrow \underline{H}_2$

such that

$$\delta[\eta_1(g,m)] = \varphi(g)^{-1} \eta_2(\alpha(g), \alpha(m)) \varphi(m)$$

for $g \in G_1, m \in M_1$ and

$$\delta[\underline{H}_y] = \varphi(y)^{-1} \underline{H}_{\alpha(y)} \varphi(y)$$

for $y \in G_1 \dot{\cup} M_1$. ♢

This definition is indeed very similar to the one of isomorphy of binary relation transversals and therefore the following results are not a big surprise.

Theorem 2.1.7 *Let* $\mathbb{K}_1 = (G_1, M_1, I_1)$ *and* $\mathbb{K}_2 = (G_2, M_2, I_2)$ *be contexts and* $\alpha : \mathbb{K}_1 \longrightarrow \mathbb{K}_2$ *a context isomorphism. Furthermore let* $\Gamma_1 \leq \underline{\mathrm{Aut}}(\mathbb{K}_1), \Gamma_2 \leq \underline{\mathrm{Aut}}(\mathbb{K}_2)$ *such that*

$$\begin{array}{rccc} \delta : & \Gamma_1 & \longrightarrow & \Gamma_2 \\ & \beta & \longmapsto & \alpha \circ \beta \circ \alpha^{-1} \end{array}$$

is a group isomorphism. Then $\mathrm{rep}_{\Gamma_1}(\mathbb{K}_1) \cong \mathrm{rep}_{\Gamma_2}(\mathbb{K}_2)$.

Proof The proof is the same as for Theorem 1.3.2. The only thing we have to show is that $\bar{\alpha} = \varphi_x \circ \alpha$ defined in the proof is actually a context automorphism, i.e. we have to show that $\bar{\alpha}[Y_{G_1}] = Y_{G_2}, \bar{\alpha}[Y_{M_1}] = Y_{M_2}$, if $Y_{G_1}, Y_{G_2}, Y_{M_1}, Y_{M_2}$ are the chosen transversals. For this we observe that $\bar{\alpha}[Y_{G_1}] = \varphi_x[\alpha[Y_{G_1}]] \subseteq \varphi_x[G_2] \subseteq Y_{G_2}$ and likewise for Y_{M_1} and Y_{M_2}. Now if $y \in Y_{G_2}$ there is a unique element $\tilde{y} \in G_2$ such that $y \in \Gamma_2(\tilde{y})$ and $\alpha(x) = \tilde{y}$ for some $x \in Y_{G_1}$. But for this element it is $\bar{\alpha}(x) = \varphi_x(\tilde{y}) = y$. □

Lemma 2.1.8 *Let* $\mathbb{K}_1 = (G_1, M_1, I_1)$ *and* $\mathbb{K}_2 = (G_2, M_2, I_2)$ *be contexts and* $\Gamma_1 \leq \underline{\mathrm{Aut}}(\mathbb{K}_1)$, $\Gamma_2 \leq \underline{\mathrm{Aut}}(\mathbb{K}_2)$. *If* $\mathrm{rep}_{\Gamma_1}(\mathbb{K}_1) \cong \mathrm{rep}_{\Gamma_2}(\mathbb{K}_2)$ *then there exists a context automorphism* $\psi : \mathbb{K}_1 \longrightarrow \mathbb{K}_2$ *such that* $\Gamma_2 = \psi\Gamma_1\psi^{-1}$.

Proof Again the proof is very similar to the one of Lemma 1.3.5, whereat we have to show that the mapping ψ defined in the proof is a context isomorphism. We see with a similar argument as in the previous proof that $\psi[G_1] \subseteq G_2$ and $\psi[M_1] \subseteq M_2$. With the mapping ψ^{-1} we further see $\psi^{-1}[G_2] \subseteq G_1$ and $\psi^{-1}[M_2] \subseteq M_2$ and therefore $\psi[G_1] = G_2$ and $\psi[M_1] = M_2$. □

This gives us the same characterization as for isomorphic preorder orbifolds.

Corollary 2.1.9 *Let* $\mathbb{K}_1$ *and* $\mathbb{K}_2$ *be contexts and let* $\Gamma_1 \leq \underline{\mathrm{Aut}}(\mathbb{K}_1), \Gamma_2 \leq \underline{\mathrm{Aut}}(\mathbb{K}_2)$. *Then the following conditions are equivalent*

1. $\mathrm{rep}_{\Gamma_1}(\mathbb{K}_1) \cong \mathrm{rep}_{\Gamma_2}(\mathbb{K}_2)$ *and*
2. *there exists a context isomorphism* $\alpha : \mathbb{K}_1 \longrightarrow \mathbb{K}_2$ *such that* $\Gamma_2 = \alpha\Gamma_1\alpha^{-1}$. □

2.1.3 Unfolding Group-annotated Contexts

The next step is to transform the notion of unfolding to group-annotated contexts. Again this is very similar to unfolding group-annotated preordered sets and bears no surprises.

Definition 2.1.10 (Unfolding Group-annotated Contexts) Let $\mathbb{Y} = (G, M, I, (\underline{H}_g)_{g \in G}, (\underline{H}_m)_{m \in M}, \eta)$ an $\underline{H}$-annotated context. Then the *unfolding (or reconstruction) of* $\mathbb{Y}$ *under* $\underline{H}$ is the triple

$$\mathrm{rec}_{\underline{H}}(\mathbb{Y}) = (\dot{\bigcup_{g \in G}} \underline{H}/\underline{H}_g, \dot{\bigcup_{m \in M}} \underline{H}/\underline{H}_m, I_{\mathrm{rec}})$$

where

$$g_1 H_g \; I_{\mathrm{rec}} \; m_1 H_m :\Longleftrightarrow \; g_1^{-1} m_1 \in \eta(g, m). \qquad \diamondsuit$$

With the same argumentation as in the case of unfolding group-annotated preordered sets the definition of unfolding group-annotated contexts is well-defined. The following result even shows that there is actually no difference in unfolding group-annotated contexts and unfolding relation transversals.

Corollary 2.1.11 *Let* $\mathbb{K}$ *be a context,* $\Gamma \leq \underline{\mathrm{Aut}}(\mathbb{K})$*,* $\underline{H}$ *be a group and* $\mathbb{Y} = (Y_G, Y_M, I_{\mathrm{rep}}, (\underline{H}_{y_g})_{y_g \in Y_G}, (\underline{H}_{y_m})_{y_m \in Y_M}, \eta)$ *be an* $\underline{H}$*-annotated context. Then*

1) $\mathrm{rec}_\Gamma(\mathrm{rep}_\Gamma(\mathbb{K})) \cong \mathbb{K}$ *and*

2) $\mathrm{rep}_{\iota[\underline{H}]}(\mathrm{rec}_{\underline{H}}(\mathbb{Y})) \cong \mathbb{Y}$.

Proof To show 1) we observe that with a slight modification of Proposition 1.4.1 the mapping Ψ is a relation structure isomorphism between the sets

$$(\dot{\bigcup_{y \in G \dot\cup M}} \Gamma/\Gamma_y, I_{\mathrm{rec}}) \quad \text{and} \quad (G \mathbin{\dot\cup} M, I).$$

It therefore remains to show that Ψ is actually a context automorphism, that is $\Psi[\dot\bigcup_{g \in G} \Gamma/\Gamma_g] = G$ and $\Psi[\dot\bigcup_{m \in M} \Gamma/\Gamma_m] = M$. But this is easy to see since

$$\Psi[\dot{\bigcup_{g \in G}} \Gamma/\Gamma_g] = \{\, \gamma(g) \mid \gamma \in \Gamma \,\} = G$$

and likewise for M.

To show 2) we argue similarly to the proof of Theorem 1.4.13: Let $\mathrm{rep}_{\iota[\underline{H}]}(\mathrm{rec}_{\underline{H}}(\mathbb{Y})) = (Z_G, Z_M, I_Z, (\underline{H}_{z_g})_{z_g \in Z_G}, (\underline{H}_{z_m})_{z_m \in Z_M}, \eta_Z)$. Then for every $x \in Y_G \mathbin{\dot\cup} Y_M$ we choose $\gamma_x \in \Gamma$ such that $\gamma_x \underline{H}_x \in Z_G \mathbin{\dot\cup} Z_M$ and define

$$\begin{array}{rccc} \alpha: & Y_G \mathbin{\dot\cup} Y_M & \longrightarrow & Z_G \mathbin{\dot\cup} Z_M \\ & x & \longmapsto & \gamma_x \underline{H}_x. \end{array}$$

Then the proof of Theorem 1.4.13 tells us that α yields the isomorphy between the binary relation structures $(Y_G \mathbin{\dot\cup} Y_M, I)$ and $(Z_G \mathbin{\dot\cup} Z_M, I_Z)$. Thus it again remains to show that α is a context automorphism. But this follows immediately with

$$\alpha[Y_G] = \{\, \gamma_x \underline{H}_x \mid x \in Y_G \,\} \subseteq Z_G$$

and

$$\alpha^{-1}[Z_G] = \alpha^{-1}[\{\, \gamma \underline{H}_x \mid x \in Y_G, \gamma \in \Gamma \text{ such that } \gamma \underline{H}_x \in Z_G \,\}] \subseteq Y_G,$$

hence $\alpha[Y_G] = Z_G$, and again likewise for Y_M and Z_M. □

Finally we have a look at one extended example.

Example 2.1.12 Let M be a set and let $(A, B) \in \mathfrak{P}(M)^2$, called an *implication on M* and often written as $A \longrightarrow B$. We say that $A \longrightarrow B$ *respects* a subset $X \subseteq M$ if and only if

$$A \nsubseteq X \text{ or } B \subseteq X.$$

Furthermore let

$$\models := \{\, (X, A \longrightarrow B) \in M \times \mathfrak{P}(M)^2 \mid A \longrightarrow B \text{ respects } X \,\}.$$

Now we are interested in folding the context

$$(\mathfrak{P}(M), \mathfrak{P}(M)^2, \models).$$

Before we do so we observe that an implication $A \longrightarrow B$ respects a set X if and only if *if* $A \subseteq X$ *then* $B \subseteq X$. With this we can equivalently consider the implication $A \longrightarrow B \setminus A$. Thus if we define

$$\mathrm{Imp}(M) := \{\, A \longrightarrow B \mid A \cap B = \emptyset \,\}$$

and consider folding the context

$$\underline{\mathrm{Imp}(M)} := (\mathfrak{P}(M), \mathrm{Imp}(M), \models)$$

called the *implication context of M*, by a group of context automorphisms (where the relation $\models$ is now restricted to $\mathfrak{P}(M) \times \mathrm{Imp}(M)$ without actually changing the name; this is an abuse of notation).

As the group Γ of automorphisms we want to consider all mappings induced by permutations on M, that is for every $\alpha \in S_M$ we take the mapping

$$\begin{array}{rccc} \hat{\alpha}: & \mathfrak{P}(M) \times \mathrm{Imp}(M) & \longrightarrow & \mathfrak{P}(M) \times \mathrm{Imp}(M) \\ & (X, A \longrightarrow B) & \longmapsto & (\alpha[X], \alpha[A] \longrightarrow \alpha[B]), \end{array}$$

which is a context automorphism of $\underline{\mathrm{Imp}(M)}$. This is because $\hat{\alpha}$ is a bijective mapping and

$$\begin{aligned} X \models A \longrightarrow B &\iff A \nsubseteq X \vee B \subseteq X \\ &\iff \alpha[A] \nsubseteq \alpha[X] \vee \alpha[B] \subseteq \alpha[X] \\ &\iff \alpha[X] \models \alpha[A] \longrightarrow \alpha[B]. \end{aligned}$$

Now let Γ be the group of context automorphisms induced by S_M. Then we may identify every $\alpha \in \Gamma$ with the inducing permutation, that means we shall write

$$\alpha[M] \text{ instead of } \alpha(M).$$

We then have

$$\begin{aligned} \Gamma(X) &= \{\, Y \in \mathfrak{P}(M) \mid |X| = |Y| \,\}, \\ \Gamma(A \longrightarrow B) &= \{\, C \longrightarrow D \mid |A| = |C| \wedge |B| = |D| \,\}, \end{aligned}$$

the latter because S_M acts transitively on $\mathrm{Imp}(M)$.

From now on let M be a finite set, $n = |M|$ and $(m_i)_{0\le i<n}$ an enumeration of M. We choose as transversal of the orbits of Γ on $\mathfrak{P}(M)$ the sets

$$M_i := \{\, m_j \mid j < i \,\}$$

and as transversal of the orbits of Γ on $\mathrm{Imp}(M)$ the implications

$$M_i \longrightarrow M_j \setminus M_i \text{ with } i < j.$$

Note that we have in particular $M_n = M$. We then get as stabilizers

$$\begin{aligned}
\Gamma_{M_i} &= \{\, \alpha \in \Gamma \mid \alpha[M_i] = M_i \,\},\\
\Gamma_{M_i \longrightarrow M_j \setminus M_i} &= \{\, \alpha \in \Gamma \mid \alpha[M_i] = M_i \wedge \alpha[M_j \setminus M_i] = M_j \setminus M_i \,\}\\
&= \{\, \alpha \in \Gamma \mid \alpha[M_i] = M_i \wedge \alpha[M_j] = M_j \,\}\\
&= \Gamma_{M_i} \cap \Gamma_{M_j}.
\end{aligned}$$

The annotation function η is given by

$$\begin{aligned}
\eta(M_i, M_j \longrightarrow M_k \setminus M_j) &= \{\, \alpha \in \Gamma \mid M_i \models \alpha[M_j] \longrightarrow \alpha[M_k \setminus M_j] \,\}\\
&= \{\, \alpha \in \Gamma \mid \alpha[M_j] \nsubseteq M_i \vee \alpha[M_k \setminus M_j] \subseteq M_i \,\}\\
&= \bigcup_{\alpha \in \tilde{\eta}(M_i, M_j \longrightarrow M_k \setminus M_j)} \Gamma_{M_i} \alpha \Gamma_{M_j \longrightarrow M_k \setminus M_j}
\end{aligned}$$

for some representatives $\tilde{\eta}(M_i, M_j \longrightarrow M_k \setminus M_j)$. Now let $i, j, k \in \{\, 1, \dots, n \,\}$ and let $\alpha \in \eta(M_i, M_j \longrightarrow M_k \setminus M_j)$. We define the function

$$\begin{aligned}
\zeta^{n_1,n_2}_{i,j,k} :\quad M &\longrightarrow M\\
m_s &\longmapsto \begin{cases}
m_s & \text{if } 0 \le s < n_1\\
m_{s+i} & \text{if } n_1 \le s < j\\
m_{s+(n_1-j)} & \text{if } j \le s < j + n_2\\
m_{s+i-(n_1+n_2)} & \text{if } j + n_2 \le s < k\\
m_{s+(n_1+n_2-k)} & \text{if } k \le s < k + i - (n_1 + n_2)\\
m_s & \text{if } k + i - (n_1 + n_2) \le s < n
\end{cases}
\end{aligned}$$

for every $n_1 \le j, n_2 \le k - j$ and $n_1 + n_2 \le i$. Figure 2.3 shows how these functions act on M.

We now have two cases:

Case 1: $\alpha[M_j] \nsubseteq M_i$. Let

$$M_j^1 \mathbin{\dot\cup} M_j^2 \mathbin{\dot\cup} M_k^1 \mathbin{\dot\cup} M_k^2 \mathbin{\dot\cup} M_n^1 \mathbin{\dot\cup} M_n^2 = M$$

a decomposition of M into disjoint sets such that

- $M_s = M_s^1 \mathbin{\dot\cup} M_s^2$ for all $s \in \{\, j, k, n \,\}$,

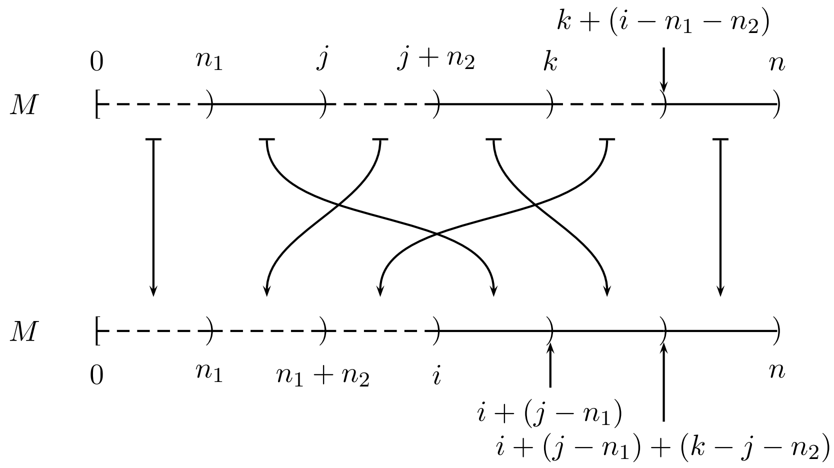


Figure 2.3: Action of $\zeta_{i,j,k}^{n_1,n_2}$ on M.

- $\alpha[M_s^1] \subseteq M_i$ and $\alpha[M_s^2] \subseteq M_i^c$ for all $s \in \{ j, k, n \}$.

That is, we write the sets $M_j, M_k \setminus M_j$ and $M_n \setminus M_k$ as disjoint union of sets which are mapped into M_i under α and which are mapped into $M_i^c = M \setminus M_i$ under α. In particular we have

$$|M_j^1| + |M_k^1| + |M_n^1| = i$$

and since $\alpha[M_j] \not\subseteq M_i$ we also have $M_j^2 \neq \emptyset$. This implies that

$$\begin{aligned} 0 \leq |M_j^1| &\leq j - 1 \\ 0 \leq |M_k^1| &\leq k - j \\ i - (n - k) \leq |M_j^1| + |M_k^1| &\leq i. \end{aligned}$$

Now there exists a mapping $\sigma_1 \in \Gamma_{M_j} \cap \Gamma_{M_k}$ such that

$$\sigma_1[M_j^1] < \sigma_1[M_j^2] < \sigma_1[M_k^1] < \sigma_1[M_k^2] < \sigma_1[M_n^1] < \sigma_1[M_n^2],$$

ordered elementwise[1]. If we now apply $\zeta := \zeta_{i,j,k}^{|M_j^1|,|M_k^1|}$ to these sets we get

$$\underbrace{\zeta\sigma_1[M_j^1] < \zeta\sigma_1[M_k^1] < \zeta\sigma_1[M_n^1]}_{=M_i} < \underbrace{\zeta\sigma_1[M_j^2] < \zeta\sigma_1[M_k^2] < \zeta\sigma_1[M_n^2]}_{=M_i^c},$$

that is we move the blocks of M which are mapped into M_i with the help of ζ to the front of M. Now there exists a (unique) mapping $\sigma_2 \in \Gamma_{M_2}$ such that

$$\alpha[M_s^t] = \sigma_2\zeta\sigma_1[M_s^t] \text{ for } s \in \{ i, j, k \}, t \in \{ 1, 2 \}.$$

[1] We set $m_i < m_j$ if $i < j$.

The function σ_2 simply reorders the elements in M_i and M_i^c to obtain the same result as applying α to M.

Therefore

$$\alpha \in \bigcup_{\substack{i-(n-k)\leq s+t\leq i,\\ 0\leq s<j, 0\leq t\leq k-j}} \Gamma_{M_i}\zeta_{i,j,k}^{s,t}\Gamma_{M_j\longrightarrow M_k\setminus M_j}.$$

It remains to show that $\beta := \zeta_{i,j,k}^{s,t} \in \eta(M_j, M_j \longrightarrow M_k \setminus M_j)$ for all pairs (s,t) allowed in the above union. For this we have to show that β is well defined, i.e. the parameters (s,t) are valid and $\beta[M_j] \not\subseteq M_i$ or, equivalently, $s < j$. The second is obviously true such that we only have to show that (s,t) are valid parameters for $\zeta_{i,j,k}^{\cdot,\cdot}$. For this we have to show that the sums

$$s + t + (i - s - t) = i \quad \text{and} \quad (j - s) + (k - j - t) + (n - k - i + s + t) = n - i$$

hold (which is obvious) and are sums of non-negative integers since these sums are the lengths of the segments which are moved around by $\zeta_{i,j,k}^{s,t}$. Clearly we have $0 \leq s, t$ by definition and $0 \leq i-s-t$ since $s+t \leq i$. Furthermore we have $0 \leq j-s$ and $t \leq k - j$ also by definition and thus only $0 \leq n - k - i + s + t$ remains to be shown. But this follows directly from $i - (n - k) \leq s + t$ and hence (s,t) are valid parameters.

The whole situation is shown graphically in Figure 2.4.

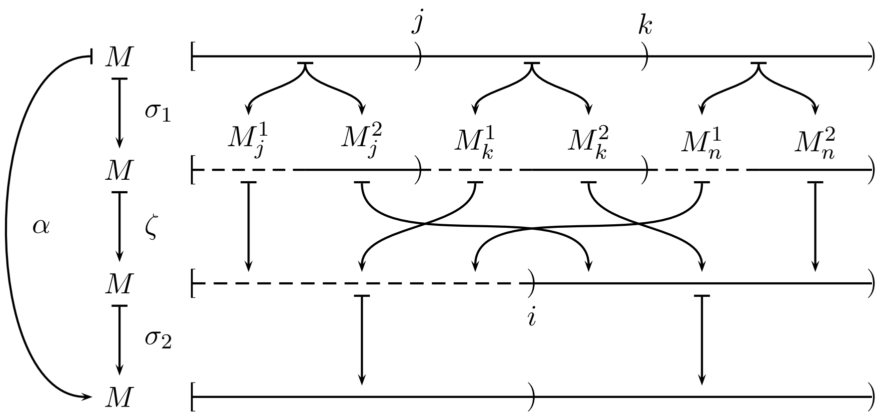


Figure 2.4: Situation of Case 1.

Case 2: $\alpha[M_j] \subseteq M_i$ **and** $\alpha[M_k \setminus M_j] \subseteq M_i$**.** Then $|\alpha[M_j] + \alpha[M_k \setminus M_j]| \leq i$, that is $k \leq i$. Thus the decomposition of M into disjoint sets as above simplifies to

$$M_j^1 \mathbin{\dot\cup} M_k^1 \mathbin{\dot\cup} M_n^1 \mathbin{\dot\cup} M_n^2 = M.$$

Then there exists a mapping $\sigma_1 \in \Gamma_{M_j} \cap \Gamma_{M_k}$ such that

- $\sigma_1|_{M_k} = \mathrm{id}$ and
- $\sigma_1[M_n^1] < \sigma_1[M_n^2]$,

i.e. $\sigma_1[M_n^1] \subseteq M_i$. Now we can again reorder the elements of the set $M_i = \sigma_1[M_j^1] \mathbin{\dot\cup} \sigma_1[M_k^1] \mathbin{\dot\cup} \sigma_1[M_n^1]$ and $M_i^c = \sigma_1[M_n^2]$ with a mapping $\sigma_2 \in \Gamma_{M_i}$ to obtain the same result as applying α to M. Therefore we have shown that

$$\alpha \in \Gamma_{M_i}\Gamma_{M_j \longrightarrow M_k \setminus M_j}$$

and that id is a representative of a double coset. Note that $\mathrm{id} = \zeta_{i,j,k}^{j,k}$.

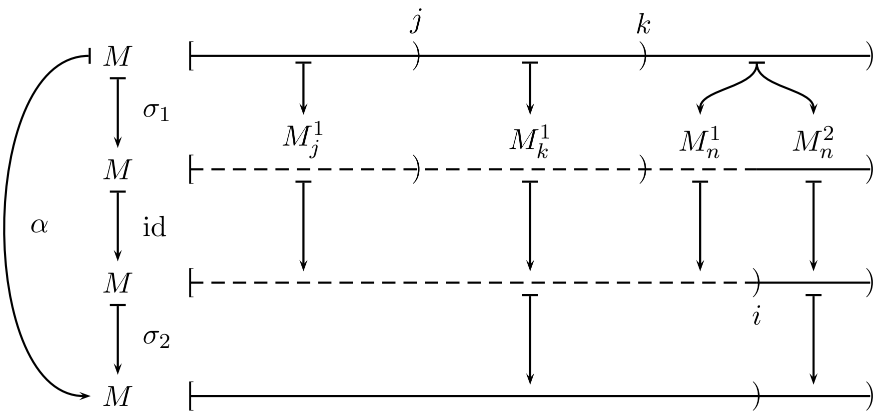


Figure 2.5: Situation of Case 2.

It remains to show that the double cosets generated by the mappings $\zeta_{i,j,k}^{n_1,n_2}$ are different. For this to see let $\alpha \in \Gamma_{M_i}\zeta_{i,j,k}^{n_1,n_2}\Gamma_{M_j \longrightarrow M_k \setminus M_j}$. Then it can easily be seen that with the decomposition into disjoint cosets of M as above it is

$$n_1 = |M_j^1| \text{ and } n_2 = |M_k^1|$$

hence α is in exactly one double coset. Therefore the given mappings are representatives. (Note that id can only be a representative in the first case if $i < j$, i.e. $M_j \not\subseteq M_i$, which means that $k \not\leq i$ and the double coset of the second case does not exist.)

Now that we have found representatives we can concisely write down the context orbifold of $\underline{\mathrm{Imp}(M)}$ under Γ by simply writing down the table

$$(\{0,\ldots,n\}, \{(j,k) \mid 0 \leq j < k \leq n\}, (\bar\eta(i,(j,k))))$$

with

$$\bar\eta(i,(j,k)) := \{(s,t) \mid 0 \leq s < j, 0 \leq t \leq k-j, i-(n-k) \leq s+t \leq i\} \\ \cup \begin{cases} \emptyset & \text{if } k > i \\ \{(j,k)\} & \text{if } k \leq i \end{cases},$$

that is instead of writing down $\zeta_{i,j,k}^{s,t}$ we only store the pair (s,t).

But we can go one step further if we make the following observation: The context $\underline{\mathrm{Imp}(M)}$ can be attribute-reduced to the context

$$\underline{\mathrm{Imp}(M)^*} = (\mathfrak{P}(M), \mathrm{Imp}(M)^*, \models)$$

where

$$\mathrm{Imp}(M)^* = \{\, A \longrightarrow \{\, b \,\} \mid b \notin A \,\}.$$

This means in our case, that $k = j + 1$ and thus the context orbifold of the attribute-reduced implication context $\underline{\mathrm{Imp}(M)^*}$ can be represented by

$$(\{\, 0, \ldots, n \,\}, \{\, 0, \ldots, n-1 \,\}, \bar{\eta}(i,j))$$

and

$$\begin{aligned}\bar{\eta}(i,j) := {} & \{\, (s,t) \mid 0 \le s < j, 0 \le t \le 1, i + j + 1 - n \le s + t \le i \,\} \\ & \cup \begin{cases} \emptyset & \text{if } j \ge i \\ \{\, (j, j+1) \,\} & \text{if } j < i \end{cases} .\end{aligned}$$

An example of such a table is shown in Figure 2.6. ♢

	0	1	2
0	$\{\,\}$	$\{\,(0,0)\,\}$	$\{\,(0,0)\,\}$
1	$\{\,(0,1)\,\}$	$\{\,(0,0),(0,1)\,\}$	$\{\,(0,0),(0,1),(1,0)\,\}$
2	$\{\,(0,1)\,\}$	$\{\,(0,0),(0,1),(1,2)\,\}$	$\{\,(0,0),(0,1),(1,0),(1,1)\,\}$
3	$\{\,(0,1)\,\}$	$\{\,(0,0),(0,1),(1,2)\,\}$	$\{\,(0,1),(1,0),(1,1),(2,3)\,\}$
4	$\{\,(0,1)\,\}$	$\{\,(0,1),(1,2)\,\}$	$\{\,(1,1),(2,3)\,\}$
5	$\{\,(0,1)\,\}$	$\{\,(1,2)\,\}$	$\{\,(2,3)\,\}$

	3	4
0	$\{\,(0,0)\,\}$	$\{\,(0,0)\,\}$
1	$\{\,(0,0),(0,1),(1,0)\,\}$	$\{\,(0,1),(1,0)\,\}$
2	$\{\,(0,1),(1,0),(1,1),(2,0)\,\}$	$\{\,(1,1),(2,0)\,\}$
3	$\{\,(1,1),(2,0),(2,1)\,\}$	$\{\,(2,1),(3,0)\,\}$
4	$\{\,(2,1),(3,4)\,\}$	$\{\,(3,1)\,\}$
5	$\{\,(3,4)\,\}$	$\{\,(4,5)\,\}$

Figure 2.6: Example of a context orbifold of the implication context of $M = \{\, 1,2,3,4,5 \,\}$ under the group Γ.

2.1.4 Computing the Standard Context Orbifold from the Concept Lattice Orbifold

Now that we understand context orbifolds well enough we may ask if, given a concept lattice orbifold, we are able to compute the context orbifold of the standard context. The answer is yes and the algorithm to achieve this goal is easy to formulate: if we were able to find from the concept lattice orbifold all supremum and infimum irreducible elements we can write down the context

$$(J(\underline{L}), M(\underline{L}), \leq_{\underline{L}})$$

and fold this context to get the standard context orbifold. But we can even do better because we can observe that lattice automorphisms take supremum irreducible elements to supremum irreducible elements and likewise for infimum irreducible ones. Therefore, if we were able to identify orbit representatives of irreducible elements in the concept lattice orbifold, we automatically get representatives for the standard context orbifold. Furthermore, since the incidence relation in the standard context is $\leq_{\underline{L}}$, the computation of the annotation is as easy as it could be:

$$\eta(g,m) = \{\, \alpha \in \Gamma : g \leq_{\underline{L}} \alpha(m) \,\} = \lambda(g,m).$$

So what is left is to characterize supremum and infimum irreducible elements in lattice orbifolds. To do so we firstly identify infimum irreducible elements and secondly have a look on what happens to a lattice orbifold when folding the dual lattice. We shall also consider only finite lattices here since they allow an easy characterization of their irreducible elements. This will be needed to find these elements from a corresponding lattice orbifold.

We start with the following observation: Let $\underline{L} = (L, \leq_{\underline{L}})$ be a finite lattice. An element $x \in L$ is infimum irreducible if and only if it has exactly one upper neighbor $y \in L$. Let $\Gamma \leq \underline{\mathrm{Aut}}(\underline{L})$ and $\underline{Y} = (Y, \leq_Y, (\Gamma_z)_{z\in Y}, \lambda)$ be a lattice orbifold of $\underline{L}$ under Γ such that $x, y \in Y$. Then

$$\lambda(x,y) = \{\, \alpha \in \Gamma \mid x \leq_{\underline{L}} \alpha(y) \,\} = \Gamma_y.$$

This also gives us

$$\Gamma_y = \lambda(x,y) = \Gamma_x \lambda(x,y) \Gamma_y$$

and therefore $\Gamma_x \subseteq \Gamma_y$. So in sum we have

$$x \text{ infimum irreducible in } \underline{L} \implies \lambda(x,y) = \Gamma_x\Gamma_y \text{ and } \Gamma_x \subseteq \Gamma_y \text{ for } x \leq_Y y \text{ upper neighbor.}$$

We shall see in a moment that this statement can be transformed to be independent from the actual lattice orbifold $\underline{Y}$ (we do not even need a lattice) and, more importantly, that the implication actually gets an equivalence.

Lemma 2.1.13 *Let $\underline{P} = (P, \leq_{\underline{P}})$ be a preordered set, $\Gamma \leq \underline{\mathrm{Aut}}(\underline{P})$ and $\underline{Y} = (Y, \leq_Y, (\Gamma_y)_{y \in Y}, \lambda)$ a preorder orbifold of $\underline{P}$ under Γ. Let $x, y \in Y$ such that $x \leq_Y y$ and $\alpha \in \lambda(x, y)$.*

Then $\alpha(y)$ is the only element in $\Gamma(y)$ with $x \leq_{\underline{P}} \alpha(y)$ iff $\Gamma_x \subseteq \Gamma_{\alpha(y)}$ and $\lambda(x, y) = \Gamma_x \alpha \Gamma_y$.

Proof ($\Rightarrow$). Let $\alpha(y)$ be the only element in $\Gamma(y)$ such that $x \leq_{\underline{P}} \alpha(y)$. Then $\lambda(x, y) = \Gamma_{\alpha(y)}$. Now we can consider a preorder orbifold $\underline{\tilde{Y}} = (\tilde{Y}, \leq_{\tilde{Y}}, (\Gamma_y)_{y \in \tilde{Y}}, \tilde{\lambda})$ where we choose $\tilde{Y} = Y \setminus \{ y \} \cup \{ \alpha(y) \}$. Then $\underline{Y} \cong \underline{\tilde{Y}}$, $\tilde{\lambda}(x, \alpha(y)) = \Gamma_{\alpha(y)}$ and as above it follows $\Gamma_x \subseteq \Gamma_{\alpha(y)}$. By Proposition 1.5.7 we also see that the number of double cosets are the same for $\lambda(x, y)$ and $\tilde{\lambda}(x, \alpha(y))$ and since $\tilde{\lambda}(x, \alpha(y)) = \Gamma_x \Gamma_{\alpha(y)}$ the set $\lambda(x, y)$ contains exactly one double coset, where α is a representative of. Hence $\lambda(x, y) = \Gamma_x \alpha \Gamma_y$ as required.

($\Leftarrow$). Let $\Gamma_x \subseteq \Gamma_{\alpha(y)}$ and $\lambda(x, y) = \Gamma_x \alpha \Gamma_y$. Furthermore let $\beta(y) \in \Gamma(y)$ such that $x \leq_{\underline{P}} \beta(y)$. Then $\beta \in \lambda(x, y)$ and therefore there exist $\gamma \in \Gamma_x$, $\delta \in \Gamma_y$ with $\beta = \gamma \alpha \delta$. Then it is

$$\begin{aligned} \beta(y) &= \gamma \alpha \delta(y) \\ &= \gamma \alpha(y) \end{aligned}$$

because $\delta \in \Gamma_y$,

$$= \alpha(y)$$

because $\gamma \in \Gamma_x \subseteq \Gamma_{\alpha(y)}$. Therefore $\alpha(y)$ is the only element in $\Gamma(y)$ with $x \leq_{\underline{P}} \alpha(y)$. □

Together with the observation that neighbors in $\underline{P}$ are neighbors in every preorder orbifold $\underline{Y}$ and, if the neighbor is not given by a long edge, vice versa we get the following corollary.

Corollary 2.1.14 *Let $\underline{L}$ be a finite lattice, $\Gamma \leq \underline{\mathrm{Aut}}(\underline{L})$ and $\underline{Y} = (Y, \leq_Y, (\Gamma_y)_{y \in Y}, \lambda)$ be a lattice orbifold of $\underline{L}$ under Γ. Then an element $x \in Y$ is a representative of an orbit of infimum irreducible elements of $\underline{L}$ if and only if x has exactly one upper neighbor y in $(Y, \leq_Y)$ where $\lambda(x, y) = \Gamma_x \alpha \Gamma_y$ for some $\alpha \in \Gamma$ and $\Gamma_x \subseteq \Gamma_{\alpha(y)}$.* □

Now we could do all of the above again to find supremum irreducible elements y and prove that the condition has to be changed to

$$\lambda(x, y) = \Gamma_x \alpha \Gamma_y \text{ and } \Gamma_x \supseteq \Gamma_{\alpha(y)}.$$

But we may go another way and study what happens to preorder orbifolds $\underline{Y} = (Y, \leq_Y, (\Gamma_y)_{y \in Y}, \lambda)$ when we dualize the original preorder $\underline{P} = (P, \leq_{\underline{P}})$. Firstly we see that the stabilizers do not change since they do not depend on $\leq_{\underline{P}}$. Secondly the relation $\leq_Y$ will also get dualized since $x \leq_{\underline{P}} \alpha(y)$ implies $y \leq_{\underline{P}} \alpha^{-1}(x)$. Finally the annotation function is given by

$$\{ \alpha \in \Gamma \mid y \leq_{\underline{P}}^{-1} \alpha(x) \} = \{ \alpha \in \Gamma \mid x \leq_{\underline{P}} \alpha^{-1}(y) \} = \lambda(x, y)^{-1}.$$

So in sum we get the following result.

Lemma 2.1.15 *Let $\underline{P} = (P, \leq_{\underline{P}})$, $\Gamma \leq \underline{\mathrm{Aut}}(\underline{P})$ and $\underline{Y} = (Y, \leq_Y, (\Gamma_y)_{y \in Y}, \lambda)$ a preorder orbifold of $\underline{P}$ under Γ. Then $\tilde{\underline{Y}} = (Y, \leq_Y^{-1}, (\Gamma_y)_{y \in Y}, \tilde{\lambda})$ with*

$$\tilde{\lambda}(y, x) = \lambda(x, y)^{-1}$$

is a preorder orbifold of $\underline{P}^d = (P, \leq_{\underline{P}}^{-1})$ under Γ. □

With this we see that the representatives x of elements with exactly one lower neighbor of $\underline{L}$, which are the elements with exactly one upper neighbor of $\underline{L}^d$, are given by

$$\tilde{\lambda}(x, y) = \Gamma_x \alpha \Gamma_y \text{ and } \Gamma_x \subseteq \Gamma_{\alpha(y)}.$$

This gives

$$\lambda(y, x) = \Gamma_y \alpha^{-1} \Gamma_x \text{ and } \Gamma_x \subseteq \Gamma_{\alpha(y)}$$

and interchanging x and y, observing that $\Gamma_{\alpha(y)} = \alpha \Gamma_y \alpha^{-1}$ and renaming α^{-1} to α yields what we were looking for.

Corollary 2.1.16 *Let $\underline{L}$ be a finite lattice, $\Gamma \leq \underline{\mathrm{Aut}}(\underline{L})$ and $\underline{Y} = (Y, \leq_Y, (\Gamma_y)_{y \in Y}, \lambda)$ be a lattice orbifold of $\underline{L}$ under Γ. Then an element $y \in Y$ is a representative of an orbit of supremum irreducible elements of $\underline{L}$ if and only if y has exactly one lower neighbor x in $(Y, \leq_Y)$ where $\lambda(x, y) = \Gamma_x \alpha \Gamma_y$ for some $\alpha \in \Gamma$ and $\Gamma_x \supseteq \Gamma_{\alpha(y)}$.* □

This immediately gives us an algorithm to compute the standard context orbifold directly from a concept lattice orbifold.

Theorem 2.1.17 *Let $\underline{L} = (L, \leq_{\underline{L}})$ be a finite lattice, $\Gamma \leq \underline{\mathrm{Aut}}(\underline{L})$ and $\underline{Y} = (Y, \leq_Y, (\Gamma_y)_{y \in Y}, \lambda)$ a lattice orbifold. Let*

$$J(\underline{Y}) := \{\, y \in Y \mid x \leq_Y y \text{ unique lower neighbor, } \exists \alpha \in \Gamma : \lambda(x, y) = \Gamma_x \alpha \Gamma_y, \Gamma_x \supseteq \Gamma_{\alpha(y)} \,\},$$
$$M(\underline{Y}) := \{\, x \in Y \mid x \leq_Y y \text{ unique upper neighbor, } \exists \alpha \in \Gamma : \lambda(x, y) = \Gamma_x \alpha \Gamma_y, \Gamma_x \subseteq \Gamma_{\alpha(y)} \,\}.$$

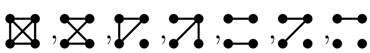
Then the Γ-annotated context

$$(J(\underline{Y}), M(\underline{Y}), \lambda|_{J(\underline{Y}) \times M(\underline{Y})})$$

is a standard context orbifold of $(J(\underline{L}), M(\underline{L}), \leq_{\underline{L}})$. □

Example 2.1.18 We want to consider Example 1.2.8 again, but this time we want to take all graphs on four vertices (not necessarily connected) into account. This resulting lattice orbifold $\underline{Y}$ has been considered in [GB] and is shown in Figure 2.7. In this example we would like to find the standard context orbifold of this lattice orbifold.

Let $\Gamma := S_4$. We are now going to identify the supremum and infimum irreducible elements of this lattice orbifold. For the supremum irreducible elements possible candidates are



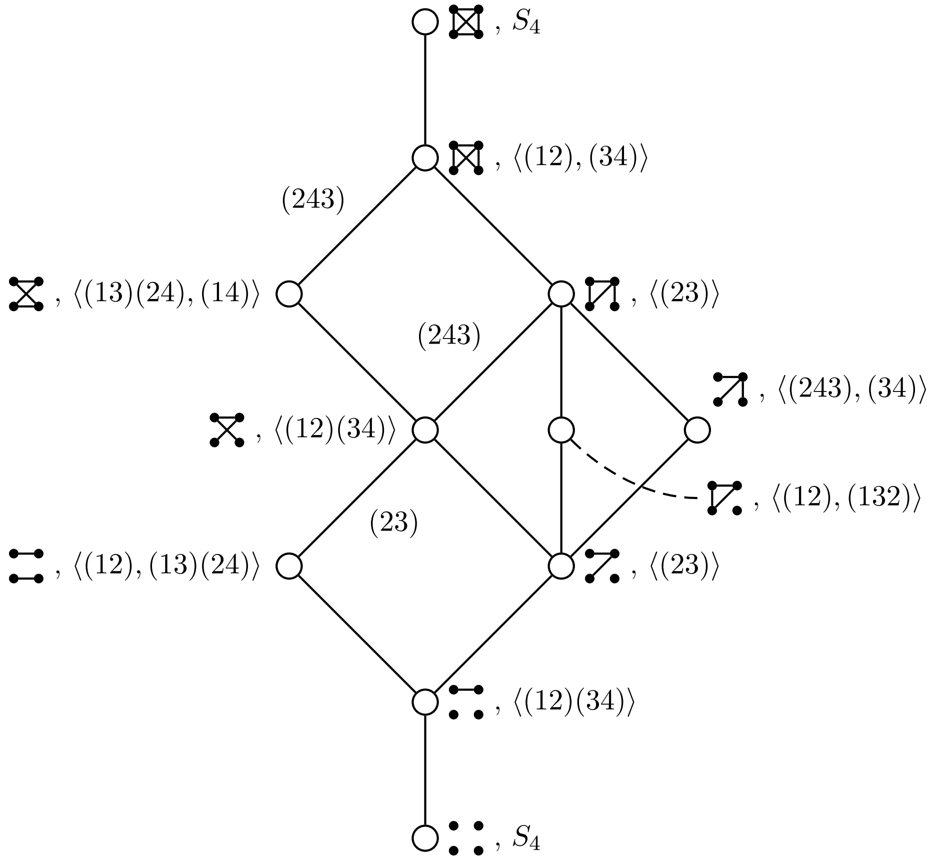
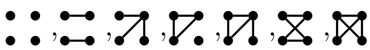


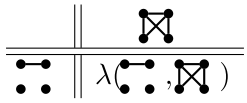
Figure 2.7: Lattice orbifold of the lattice of all graphs on four vertices by the automorphism group induced by the S_4, see [GB]. The diagram also shows the stabilizers of every representative.

since they only have one lower neighbor and the corresponding abridged annotation consists only of one element. Let us consider the graph . Its lower neighbor is , its stabilizer is $\langle(12),(13)(24)\rangle$ and the stabilizer of is $\langle(12)(34)\rangle$. Since the edge between the two is annotated with id we need $\langle(12)(34)\rangle \supseteq \langle(12),(13)(24)\rangle$ for to be a supremum irreducible element, which is not true. Therefore is not a supremum irreducible element of $\underline{Y}$. On the other hand if we consider we have as its lower neighbor, which has S_4 as stabilizer, hence is an irreducible element of $\underline{Y}$ and it turns out that it is the only one.

For the infimum irreducible elements we get as candidates the graphs



and as for the supremum irreducible elements actually only the graph turns out to be an infimum irreducible element of $\underline{Y}$, such that we get as standard context orbifold



where

λ( , ) $= \Gamma$ $\circ \{ \mathrm{id}, (23) \} \circ \Gamma$ . ♢

2.2 Context Derivation with Context Automorphisms

So now that we are able to fold contexts and unfold context orbifolds the next question we want to look at is whether it is possible to compute derivations A' in a context orbifold $\mathbb{Y}$ without actually unfolding it. This is indeed possible and gives first insight into the possibility of computing the concept lattice orbifold from a context orbifold directly.

2.2.1 η-Derivation

Example 2.2.1 We consider the context orbifold $\mathbb{Y} = (Y_G, Y_M, I_{\text{rep}}, (\Gamma_g)_{g \in Y_G}, (\Gamma_m)_{m \in Y_M}, \eta)$ depicted in Figure 2.2 on page 32 and the set $\{ 1 \}$, i.e. we want to compute the object concept of 1 in $\mathbb{K} = (G, M, I)$ directly from $\mathbb{Y}$. This is fairly simple: Denote with $\{ 1 \}^\eta$ the set

$$\begin{aligned}\{ 1 \}^\eta &= \{ \alpha(m) \mid m \in Y_M, \alpha \in \eta(1, m) \}\\ &= \{ 1, 5, \alpha(5), \alpha^2(5) \}\\ &= \{ 1, 5, 6, 7 \}\end{aligned}$$

that is we collect all $\alpha(m)$ such that $\alpha \in \eta(1, m)$ or equivalently $1 \; I \; \alpha(m)$. But this immediately gives $\{ 1 \}^\eta = \{ 1 \}'$. With this we also compute the derivation of the set $\{ 1, 5 \}$:

$$\begin{aligned}\{ 1, 5 \}^\eta &= \{ 1 \}^\eta \cap \{ 5 \}^\eta\\ &= \{ 1, 5, 6, 7 \} \cap \{ 5 \}\\ &= \{ 5 \}.\end{aligned}$$

But we can do even better: Since every $g \in G$ can be represented as $\alpha_1(g_1)$ with $g_1 \in Y_G$ and $\alpha_1 \in \Gamma$ we can set

$$\{ g \}^\eta = \alpha_1[\{ g_1 \}^\eta].$$

This is indeed well defined since $\alpha_1(g_1) = \alpha_2(g_2)$ implies

$$\begin{aligned}\alpha_1[\{ g_1 \}^\eta] &= \alpha_1[\{ \alpha(m) \mid m \in Y_M, \alpha \in \eta(g_1, m) \}]\\ &= \{ \alpha_1(\alpha(m)) \mid m \in Y_M, \alpha \in \eta(g_1, m) \}]\\ &= \{ \alpha(m) \mid m \in Y_M, \alpha \in \alpha_1\eta(g_1, m) \}\\ &= \{ \alpha(m) \mid m \in Y_M, \alpha \in \alpha_2\eta(g_1, m) \}\\ &= \alpha_2[\{ g_2 \}^\eta]\end{aligned}$$

and

$$\begin{aligned}
\alpha_1\eta(g_1, m) &= \alpha_1[\{\, \alpha \in \Gamma \mid g_1 \; I \; \alpha(m) \,\}] \\
&= \{\, \alpha_1\alpha \in \Gamma \mid g_1 \; I \; \alpha(m) \,\} \\
&= \{\, \alpha_1\alpha \in \Gamma \mid \alpha_1(g_1) \; I \; \alpha_1\alpha(m) \,\} \\
&= \{\, \alpha \in \Gamma \mid \alpha_1(g_1) \; I \; \alpha(m) \,\} \\
&= \{\, \alpha \in \Gamma \mid \alpha_2(g_2) \; I \; \alpha(m) \,\} \\
&= \alpha_2\eta(g_2, m).
\end{aligned}$$

With a very similar argumentation we also see that with $\alpha_1(g_1) = \alpha_2(g_2)$ and $\beta_1(m_1) = \beta_2(m_2)$ we have

$$\alpha_1\eta(g_1, m_1)\beta_1^{-1} = \alpha_2\eta(g_2, m_2)\beta_2^{-1}$$

that is the set $\alpha_1\eta(g_1, m_1)\beta_1^{-1}$ is determined uniquely by $\alpha_1(g_1)$ and $\beta_1(m_1)$ and neither by α_1, g_1, β_1 nor m_1.

Finally we have for $H \subseteq G$

$$\begin{aligned}
H^\eta &:= \bigcap_{g \in H} g^\eta \\
&= \bigcap_{\alpha_1(g_1) \in H} \alpha_1[\{\, g_1 \,\}^\eta] \\
&= \bigcap_{\alpha_1(g_1) \in H} \{\, \alpha_1\alpha(m) \mid m \in Y_M, \alpha \in \eta(g_1, m) \,\} \\
&= \bigcap_{\alpha_1(g_1) \in H} \{\, \alpha(m) \mid m \in Y_M, \alpha \in \alpha_1\eta(g_1, m) \,\} \\
&= \{\, \alpha(m) \mid m \in Y_M, \forall \alpha_1(g_1) \in H : \mathrm{id} \in \alpha_1\eta(g_1, m)\alpha^{-1} \,\}.
\end{aligned}$$

Of course we can also define the derivation on attributes $m \in Y_M$ with

$$\{\, m \,\}^\eta := \{\, \alpha(g) \mid g \in Y_G, \alpha^{-1} \in \eta(g, m) \,\}$$

motivated by $\alpha^{-1} \in \eta(g, m) \iff g \; I \; \alpha^{-1}(m) \iff \alpha(g) \; I \; m.$ $\diamondsuit$

With this motivation we are well prepared to define derivation in context orbifolds.

Definition 2.2.2 (η-Derivation in Group-annotated Contexts) Let $\mathbb{Y} = (G, M, I, (\underline{H}_g)_{g \in G}, (\underline{H}_m)_{m \in M}, \eta)$ be an $\underline{H}$-annotated context, $S \subseteq \dot{\bigcup}_{g \in G} \underline{H}/\underline{H}_g$ and $T \subseteq \dot{\bigcup}_{m \in M} \underline{H}/\underline{H}_m$. Then the *$\eta$-derivation in* $\mathbb{Y}$ of S and T, respectively, is defined as

$$\begin{aligned}
S^{\uparrow\eta} &:= \{\, bH_m \in \dot{\bigcup_{m \in M}} \underline{H}/\underline{H}_m \mid \forall aH_g \in S : e_{\underline{H}} \in a\eta(g, m)b^{-1} \,\} \\
T^{\downarrow\eta} &:= \{\, aH_g \in \dot{\bigcup_{g \in G}} \underline{H}/\underline{H}_g \mid \forall bH_m \in T : e_{\underline{H}} \in a\eta(g, m)b^{-1} \,\}
\end{aligned}$$

where $e_{\underline{H}}$ is the neutral element of $\underline{H}$. We may again write S^η and T^η instead of $S^{\uparrow\eta}$ and $T^{\downarrow\eta}$ if it is clear where S and T come from. We may also write $\gamma_\eta(g)$ and $\mu_\eta(m)$ to denote the object and attribute concepts of g and m, respectively. $\diamondsuit$

Remark 2.2.3 In the definition of $\uparrow\eta$ and $\downarrow\eta$ we think of the set comprehension as a *disjoint* set comprehension, that is we set

$$a_1 H_{g_1} \neq a_2 H_{g_2}$$

whenever $g_1 \neq g_2$ and likewise $b_1 H_{m_1} \neq b_2 H_{m_2}$ whenever $m_1 \neq m_2$.

Proposition 2.2.4 *Let $\mathbb{Y}$ be an $\underline{H}$-annotated context as above. Then the η-derivation in $\mathbb{Y}$ is well defined and the pair $(\uparrow\eta, \downarrow\eta)$ forms a Galois-connection.*

Proof Let $\mathbb{Y} = (G, M, I, (\underline{H}_g)_{g\in G}, (\underline{H}_m)_{m\in M}, \eta)$ be an $\underline{H}$-annotated context. Then $\uparrow\eta$ is well defined as already has been shown in Example 2.2.1; the proof for $\downarrow\eta$ being well defined is similar. To see that $(\uparrow\eta, \downarrow\eta)$ form a Galois-connection we show

- $\forall S_1 \subseteq S_2 \subseteq G : S_1^\eta \supseteq S_2^\eta$,
- $\forall T_1 \subseteq T_2 \subseteq M : T_1^\eta \supseteq T_2^\eta$,
- $\forall S \subseteq G : S \subseteq S^{\eta\eta}$ and
- $\forall T \subseteq M : T \subseteq T^{\eta\eta}$

where $S^{\eta\eta} = (S^\eta)^\eta$ and likewise for $T \subseteq M$. We shall only show the second and the fourth claim since the other two are similar to prove.

$\forall T_1 \subseteq T_2 \subseteq M : T_1^\eta \supseteq T_2^\eta$: Let $aH_g \in T_2^\eta$. Then $e_{\underline{H}} \in a\eta(g,m)b^{-1}$ for all $bH_m \in T_2$, hence $e_{\underline{H}} \in a\eta(g,m)b^{-1}$ for all $bH_m \in T_1$ and therefore $aH_g \in T_1^\eta$ as required.

$\forall T \subseteq M : T \subseteq T^{\eta\eta}$: We first note that

$$\begin{aligned}&bH_m \in T^{\eta\eta}\\ &\quad\iff \forall aH_g \in M : (\forall \tilde{b}H_{\tilde{m}} \in T : e_{\underline{H}} \in a\eta(g,m)\tilde{b}^{-1}) \implies e_{\underline{H}} \in a\eta(g,m)b^{-1}.\end{aligned}$$

But if $bH_m \in T$ then the implication on the right side is fulfilled and hence $bH_m \in T^{\eta\eta}$ as required.

One may note that more abstractly $(\uparrow\eta, \downarrow\eta)$ is induced by the well defined relation

$$\{\, (aH_g, bH_m) \mid e_{\underline{H}} \in a\eta(g,m)b^{-1} \,\}$$

and thus must be a Galois-connection. $\square$

Now a crucial fact is that derivation in context orbifolds is closely related to derivation in the original contexts, which can be expressed best with the help of the mapping Ψ from Proposition 1.4.1.

Lemma 2.2.5 *Let $\mathbb{K} = (G, M, I)$ be a context, $\Gamma \leq \underline{\mathrm{Aut}}(\mathbb{K})$ and $\mathbb{Y} = (Y_G, Y_M, I_{\mathrm{rep}}, (\underline{H}_g)_{g\in G}, (\underline{H}_m)_{m\in M}, \eta) = \mathrm{rep}_\Gamma(\mathbb{K})$. Then for every set $S \subseteq G$ it is*

$$S^I = \Psi[(\Psi^{-1}[S])^\eta]$$

and for every $T \subseteq M$

$$T^I = \Psi[(\Psi^{-1}[T])^\eta].$$

Proof We only show $\Psi^{-1}[S^I] = (\Psi^{-1}[S])^\eta$. For this we observe that

$$\alpha(g)\ I\ \beta(m) \iff \alpha^{-1}\beta \in \eta(g, m)$$

for all $g \in Y_G, m \in Y_M, \alpha, \beta \in \Gamma$. Therefore we see

$$\begin{aligned}
\beta\Gamma_m \in \Psi^{-1}[S^I] &\iff \forall \alpha\Gamma_g \in \Psi^{-1}[S] : \beta^{-1}\alpha \in \eta(g, m)\\
&\iff \forall \alpha\Gamma_g \in \Psi^{-1}[S] : \mathrm{id} \in \beta\eta(g, m)\alpha^{-1}\\
&\iff \beta\Gamma_m \in (\Psi^{-1}[S])^\eta
\end{aligned}$$

again for $g \in Y_G, m \in Y_M, \alpha, \beta \in \Gamma$. Note that this argumentation does not depend on the choice of α and β since the sets $\alpha\eta(g, m)\beta^{-1}$ are uniquely determined by $\alpha\Gamma_g$ and $\beta\Gamma_m$ as has been shown above. □

With this lemma we see that we are able to compute all concepts of a given context solely from a context orbifold. This will enable us, under some circumstances, to compute the lattice orbifold of the concept lattice of the unfolding of a context orbifold without actually computing the unfolding.

2.2.2 Computing the Concept Lattice Orbifold from the Context Orbifold

Now that we are able to compute derivations in context orbifolds we are of course able to define *concepts* in contexts orbifolds. By Lemma 2.2.5 we have that these concepts are nothing else but the concepts of the original contexts. So we see that – at least in principle – we are able to reconstruct the whole concept lattice solely from the context orbifold. But there is another question arising naturally, corresponding to an easy observation: Every context automorphism α acts on concepts on the concept lattice $\underline{\mathcal{B}}(\mathbb{K})$ of the context $\mathbb{K}$ by

$$\alpha((A, B)) := (\alpha[A], \alpha[B]) \quad \text{for all } (A, B) \in \mathcal{B}(\mathbb{K})$$

as a lattice automorphism. So we may ask whether we can find a "suitable" algorithm to compute the concept lattice orbifold of $\underline{\mathcal{B}}(\mathbb{K})$ by a group Γ of induced lattice automorphisms, given a context orbifold $\mathrm{rep}_\Gamma(\mathbb{K})$? Hereby suitable means that we want to avoid the obvious algorithm: unfold the context, compute the concept lattice and fold it by Γ. And indeed we are able to modify the Next Closure Algorithm from [GW] in such a way that we can compute a transversal of the orbits of Γ on $\underline{\mathcal{B}}(\mathbb{K})$ computing neither $\mathrm{rec}_\Gamma(\mathrm{rep}_\Gamma(\mathbb{K}))$ nor $\mathcal{B}(\mathbb{K})$.

For this we restate Theorem 51 from [GW, page 251]. Let $\mathbb{K} = (G, M, I)$ be a finite context, $<$ be an arbitrary order relation on $G = \{ g_1, \dots, g_{|G|} \}$ and $\Gamma \leq \underline{\mathrm{Aut}}(\mathbb{K})$. We want to compute from every orbit of concepts under Γ exactly one concept, and we choose the one with the lectically largest extent. We shall call such extent *orbit-maximal*. Then the following theorem gives us an algorithm to compute all orbit-maximal extents directly, i.e. without computing all extents and afterwards choosing the orbit-maximal ones.

Theorem 2.2.6 (Theorem 51 from [GW]) *Let $A \subseteq G$. The orbit-maximal extent being lectically greater than A is the set*

$$A^+ := A \oplus i := (A \cap \{ g_1, \dots, g_{(i-1)} \})'',$$

i being the largest element of G with $A <_i A \oplus i$ and $\alpha(A \oplus i) \leq A \oplus i$ for all $\alpha \in \Gamma$. □

So we see by the previous observation that orbits of extents of a context $\mathbb{K}$ under Γ correspond to orbits of concepts in the concept lattice. Hence if we compute extents one per orbit with Theorem 2.2.6 we are able to compute a transversal of the orbits of Γ on $\mathcal{B}(\mathbb{K})$. Therefore we have the following result.

Proposition 2.2.7 *Let $\mathbb{K}$ be a reduced context and $\Gamma \leq \underline{\mathrm{Aut}}(\mathbb{K})$. Then the set*

$$\{ (A, A') \mid A \text{ is orbit-maximal extent of } \mathbb{K} \}$$

is a transversal of the orbits of Γ on $\mathcal{B}(\mathbb{K})$. □

Now that we have seen how to compute a transversal of the orbits of Γ on the concepts of $\mathbb{K}$ we would like to be able to compute the corresponding annotation function λ from $\mathbb{K}$, too. To do this we need the following observation: Every concept lattice automorphism α is uniquely determined by its action on the supremum and infimum irreducible elements of the concept lattice. If this concept lattice is described by a reduced context there is a one-to-one correspondence between supremum irreducible elements of the concept lattice and the objects of the context, likewise for the infimum irreducible elements and the attributes of the context. By this we see that α induces a unique context automorphism, so we get the following result:

Proposition 2.2.8 *Let $\mathbb{K}$ be a reduced context. Then every context automorphism $\alpha \in \mathrm{Aut}(\mathbb{K})$ corresponds to exactly one concept lattice automorphism $\tilde{\alpha}$ where*

$$\tilde{\alpha}(A, B) = (\alpha[A], \alpha[B])$$

for all concepts $(A, B) \in \mathcal{B}(\mathbb{K})$. □

Therefore we shall from now on, for every reduced context, identify the automorphism group of the context and the automorphism group of the corresponding concept lattice.

With the preceding proposition we now can compute the annotation function λ given that $\mathbb{K} = (G, M, I)$ is reduced:

$$\begin{aligned}
\lambda((A_1, B_1), (A_2, B_2)) &= \{\, \alpha \in \Gamma \mid (A_1, B_1) \le \alpha((A_2, B_2)) \,\} \\
&= \{\, \alpha \in \Gamma \mid \bigvee_{g \in A_1} \gamma(g) \le \alpha(\bigwedge_{m \in B_2} \mu(m)) \,\} \\
&= \{\, \alpha \in \Gamma \mid \bigvee_{g \in A_1} \gamma(g) \le \bigwedge_{m \in B_2} \mu(\alpha(m)) \,\} \\
&= \{\, \alpha \in \Gamma \mid \forall g \in A_1, m \in B_2 : g \; I \; \alpha(m) \,\} \\
&= \bigcap_{\substack{\alpha(g) \in A_1, \beta(m) \in B_2, \\ g \in Y_G, m \in Y_M}} \alpha\eta(g, m)\beta^{-1}
\end{aligned}$$

where Y_G and Y_M are transversals of the orbits of Γ on G and M, respectively.

So summarizing what we have seen so far we get the following algorithm to compute the concept lattice orbifold out of the context orbifold of a reduced context.

Theorem 2.2.9 *Let $\mathbb{K} = (G, M, I)$ be a reduced context, $\Gamma \le \underline{\mathrm{Aut}}(\mathbb{K})$ and $\mathbb{Y} = (Y_G, Y_M, I_{\mathrm{rep}}, (\Gamma_{y_g})_{y_g \in Y_G}, (\Gamma_{y_m})_{y_m \in Y_M}, \eta)$ a context orbifold of $\mathbb{K}$ under Γ. Let*

$$Y := \{\, (A, A^\eta) \mid A \text{ is orbit maximal extent of } \mathbb{Y} \,\}$$

as can be computed with Theorem 2.2.6 and let

$$\lambda((A_1, B_1), (A_2, B_2)) := \bigcap_{\substack{\alpha(g) \in A_1, \beta(m) \in B_2, \\ g \in Y_G, m \in Y_M}} \alpha\eta(g, m)\beta^{-1}$$

with $(A_1, B_1), (A_2, B_2) \in Y$. Furthermore let $x \le_Y y :\iff \lambda(x, y) \neq \emptyset$ for $x, y \in Y$.

Then the Γ-annotated preordered set $(Y, \le_Y, (\Gamma_y)_{y \in Y}, \lambda)$ is a concept lattice orbifold of $\underline{\mathfrak{B}}(\mathbb{K})$ under Γ. □

One has not to overestimate this theorem: Although the computation of the set Y might be algorithmic the formula for the annotation function λ given in this theorem is not very useful. In practice the function λ will be computed differently, mostly with the help of the original definition. Especially the fact that we are not able to directly compute double coset representatives from an abridged annotation function η_{abr} for the corresponding concept lattice orbifold makes the above approach practically useless.

Finally we want to give an example of a context orbifold and its corresponding concept lattice orbifold to illustrate similarities and differences of context orbifolds and concept lattice orbifolds.

Example 2.2.10 We want to consider a finite, nonempty set M and the set $\mathrm{Part}(M)$ of all its partitions. This set can be ordered by fineness, that is given two partitions $X = \{\, y_1, \ldots, y_{n_X} \,\}$ and $Y = \{\, y_1, \ldots, y_{n_Y} \,\}$ we can set

$$X \le Y :\iff \forall x \in X \exists y \in Y : x \subseteq y.$$

This will enable us to comprehend the lattice $(\mathrm{Part}(M), \leq)$ as isomorphic to the concept lattice of the following formal context, as has been given in [GW, page 52, (8)]: Let $\binom{M}{2}$ denote the set of all two-elemental subsets of M and let $m \in M$ be an arbitrary element. Then define the formal context

$$\mathbb{K} := (\binom{M}{2}, \mathfrak{P}(M \setminus \{ m \}), \diamond)$$

where

$$\{ a, b \} \diamond T :\iff \{ a, b \} \subseteq T \text{ or } \{ a, b \} \subseteq M \setminus T.$$

The extents of $\mathbb{K}$ are exactly all equivalence relations on M and the intents are the corresponding partitions of M ordered by $\leq$. In the following we want to determine the context orbifold of $\mathbb{K}$ and the concept lattice orbifold of $\underline{\mathfrak{B}}(\mathbb{K})$ both by the automorphism groups induced by the full symmetric group S_M. Note that the context $\mathbb{K}$ is reduced up to the full column labeled with $\emptyset$ and thus both the automorphism group of $\mathbb{K}$ and the automorphism group of $\underline{\mathfrak{B}}(\mathbb{K})$ can be identified.

We start with the lattice $(\mathrm{Part}(M), \leq)$ and first consider the action of a permutation $\sigma \in S_M$ on $(\mathrm{Part}(M), \leq)$. For this let $X, Y \in \mathrm{Part}(M)$ with $X \leq Y$. Then obviously[2] $X^\sigma, Y^\sigma \in \mathrm{Part}(M)$ and we have for all $x \in X, y \in Y$ that

$$x \subseteq y \iff \sigma[x] \subseteq \sigma[y].$$

Therefore σ acts on $\mathrm{Part}(M)$ as an automorphism and we can fold $(\mathrm{Part}(M), \leq)$ by the group Γ of automorphisms induced by S_M.

Our first step will now be to find a suitable transversal of the orbits of Γ on $\mathrm{Part}(M)$. For this we fix a linear ordering $<_M$ on M, that is without loss of generality we may assume $M = \{ 1, \ldots, |M| \}$. Then any partition $X = \{ x_1, \ldots, x_n \}$ can be reordered by a mapping $\sigma \in S_M$ such that $\sigma[x_1] <_M \sigma[x_2] <_M \ldots <_M \sigma[x_n]$, that is

$$\forall i, j \in \{ 1, \ldots, n \} \forall s \in x_i, t \in x_j : i < j \iff \sigma(s) <_M \sigma(t).$$

Thus we can restrict ourselves to partitions which are sequences of closed intervals of M. But since we are allowed to reorder the sequence of closed intervals in any possible way we can restrict ourselves even further to consider only those partitions of M where either the number of blocks is different or there are blocks of different sizes in each partition. For this let $X, Y \in \mathrm{Part}(M)$. Then we define

$$X \not\cong Y :\iff |X| \neq |Y| \text{ or } \exists x \in X \forall y \in Y : |x| \neq |y|.$$

Then X and Y are not in the same orbit if and only if $X \not\cong Y$.

We now may construct such a system of partitions as follows: Let $M_0 := \{ \{ m \} \mid m \in M \}$ the finest partition of M, that is the least element in $(\mathrm{Part}(M), \leq)$. Now we observe that the direct upper neighbors of M_0 are all those partitions where exactly two blocks are unified to one block. Since we are only interested in those partitions which are not in one

[2] We may use for this example the notation $X^\sigma = \{ \sigma[x] \mid x \in X \}$ in contrast to $\sigma[x]$ and $\sigma(x)$.

orbit under Γ we may take only those partitions where we have unified adjacent intervals and which are in the relation $\ngeqq$. This algorithm can be done easily. Thus we come from the level of M_0 with $|M|$ elements per partition to the next level with $|M|-1$ elements per partition and so on. We do this until we reach the one-partition $\{\{M\}\}$ with 1 element per partition. The resulting system of partitions created by this little algorithm now fulfills the requirement given above and additionally we have

$$\begin{aligned} X \lessdot Y \iff& \\ &\exists x, \tilde{x} \in X, y \in Y : x \cup \tilde{x} = y \text{ and } \forall \hat{x} \in X \setminus \{x, \tilde{x}\} \exists \hat{y} \in Y \setminus \{y\} : \hat{x} = \hat{y}, \quad (2.1) \end{aligned}$$

where $\lessdot$ denotes direct neighborhood. Hence we see that two partitions being directly neighbored are the same up to two sets which have been unified to one.

Example 2.2.11 We may consider the set $M = \{1,2,3,4\}$. The algorithm may produce the following results:

level	partitions
0	$\{\{1\},\{2\},\{3\},\{4\}\}$
1	$\{\{1,2\},\{3\},\{4\}\}$
2	$\{\{1,2,3\},\{4\}\},\{\{1,2\},\{3,4\}\}$
3	$\{\{1,2,3,4\}\}$

$\diamondsuit$

Now let $\mathcal{M}$ denote such a set of partitions of M. Then the order on $\mathcal{M}$ being the order of the lattice orbifold of $(\mathrm{Part}(M), \leq)$ is the restriction of $\leq$ on $\mathcal{M}$. Furthermore we can identify with every element $X \in \mathcal{M}$ a partition of $|M|$ in a unique way. Hence we see that the base set of the lattice orbifold of $(\mathrm{Part}(M), \leq)$ is isomorphic to the ordered set $(|M|, \leq_{|M|})$, where two partitions of $|M|$ are directly neighbored if and only if the upper arises from the lower by adding two summands. Figure 2.8 shows the base structure of the lattice orbifold of the set $\{1,\ldots,5\}$.

It now remains to compute the annotation function λ (or equivalently an abridged annotation function λ_{abr}) of this lattice orbifold.

For this we first see that by Proposition 1.5.9 there will be no long edges in the lattice orbifold of $(\mathrm{Part}(M), \leq)$. Hence we only have to compute an abridged annotation function λ_{abr} for partitions $X, Y \in \mathcal{M}$ with $X \lessdot Y$. For this let $\sigma \in S_M$ with $X \lessdot Y^\sigma$. Since $X \lessdot Y$ we can write Y as

$$Y = \{X_1 \cup X_2, X_3, \ldots, X_n\}.$$

The same argument also applies to Y^σ, thus we have

$$\begin{aligned} Y^\sigma &= \{X_{i_1} \cup X_{i_2}, X_{i_3}, \ldots, X_{i_n}\} \\ &= \{\sigma[X_1] \cup \sigma[X_2], \sigma[X_3], \ldots, \sigma[X_n]\}. \end{aligned}$$

Now we have to consider two different cases:

i) $\sigma[X_1] \cup \sigma[X_2] = X_{i_1} \cup X_{i_2}$ and

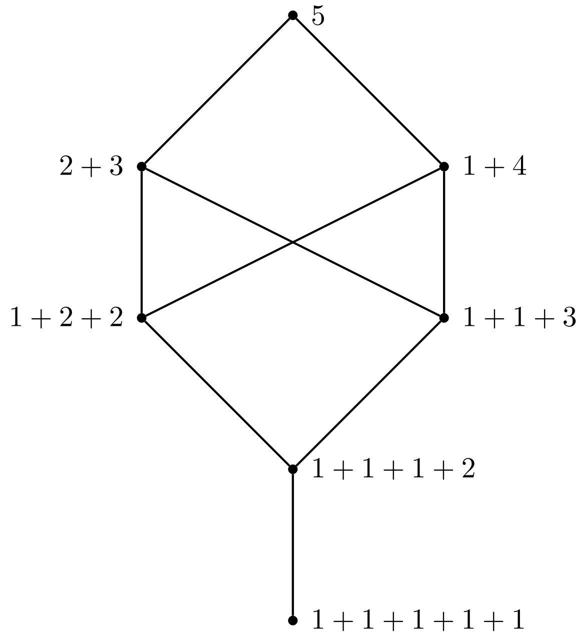


Figure 2.8: The base structure of lattice orbifold of the partition lattice of the set $\{1, \ldots, 5\}$.

ii) $\sigma[X_1] \cup \sigma[X_2] = X_j, \sigma[X_k] = X_{i_1} \cup X_{i_2}$ for some $X_j \in Y^\sigma$ and $X_k \in Y$.

Case i: We may assume that $\sigma[X_j] = X_{i_j}$ for all $j = 3, \ldots, n$, that is σ behaves like an element of the stabilizer of X except for the sets X_1, X_2, where σ is allowed to map elements arbitrarily to X_{i_1} and X_{i_2}. But we can decompose σ into a product of two certain mappings stabilizing X and Y. For this we can, without loss of generality, assume that $|X_1| = |X_{i_1}|$ and $|X_2| = |X_{i_2}|$. Then we move all elements from X_2, which are mapped under σ to X_{i_1}, to X_1 and vice versa. The mapping realizing this, say σ_1, stabilizes Y (note that σ_1 is not uniquely determined). Now we can simply map the elements of $\sigma_1[X_1]$ to X_{i_1} and of $\sigma_2[X_2]$ to X_{i_2} as σ does, and on all other sets X_j we map the elements to $\sigma[X_j]$ again as σ does. The mapping realizing this, say σ_2, can be defined by

$$\sigma_2(\sigma_1(x)) = \sigma(x)$$

and is then an element of the stabilizer of X and we have

$$\sigma = \sigma_2 \circ \sigma_1 \in \Gamma_X \circ \Gamma_Y.$$

Hence $\lambda_{\text{abr}}(X, Y)$ could simply be $\{\text{id}\}$.

Case ii: This case is similar to case 1, as σ acts on nearly all sets as a stabilizer of X and indeed a decomposition $\sigma = \sigma_2 \circ \sigma_1 \in \Gamma_X \circ \Gamma_Y$ is again possible. Firstly we observe that $|X_k| = |X_j| = |X_{i_1} \cup X_{i_2}| = |\sigma[X_1] \cup \sigma[X_2]|$ and without loss of generality we may assume that $|\sigma[X_1]| = |X_{i_1}|$ and $|\sigma[X_2]| = |X_{i_2}|$. Now the mapping σ_1 shall act on Y by fixing all elements but those in X_1, X_2 and X_k. The elements of $X_1 \cup X_2$ are mapped to X_k under σ_1 and the elements of X_k which are mapped to X_{i_1} are mapped to X_1 and the rest is mapped to X_2. Now σ_2 can be, as above, defined by

$$\sigma_2(\sigma_1(x)) = \sigma(x).$$

Then σ_2 maps the elements of X_k to X_j, the elements of X_1 to X_{i_1} and the elements of X_2 to X_{i_2}, hence $\sigma_2 \in \Gamma_X$. Therefore we can again choose $\lambda_{\mathrm{abr}}(X, Y)$ to be $\{ \mathrm{id} \}$.

So in sum we have proven that $\lambda_{\mathrm{abr}}(X, Y) = \{ \mathrm{id} \}$ for $X \lessdot Y$ and $\emptyset$ elsewhere is an abridged annotation function for the lattice orbifold of $(\mathrm{Part}(M), \leq)$ by Γ. Therefore Figure 2.8 can already be considered as the lattice orbifold of the partition lattice of $\{ 1, \dots, 5 \}$.

Contrary to this the context orbifold of $\mathbb{K}$ by the automorphism group induced by S_M can be computed easier. For this we firstly examine the action of S_M on $\mathbb{K}$ as a group of automorphisms and afterwards compute an abridged annotation function η_{abr}.

For this let σ be a permutation on M. Then we define $\tilde{\sigma}$ by

$$\tilde{\sigma}(\{ a, b \}) := \{ \sigma(a), \sigma(b) \} \quad (\{ a, b \} \in \binom{M}{2})$$
$$\tilde{\sigma}(T) := \begin{cases} \sigma[T] & \text{if } m \notin \sigma[T] \\ M \setminus \sigma[T] & \text{otherwise} \end{cases} \qquad (T \in \mathfrak{P}(M \setminus \{ m \}))$$

where m has been the element used in the construction of $\mathbb{K}$. Then $\tilde{\sigma}$ is a context automorphism of $\mathbb{K}$ since for every $\{ a, b \} \in \binom{M}{2}, T \in \mathfrak{P}(M \setminus \{ m \})$ it is

$$\begin{aligned} \{ a, b \} \diamond T &\iff \{ a, b \} \subseteq T \text{ or } \{ a, b \} \subseteq M \setminus T \\ &\iff \{ \sigma(a), \sigma(b) \} \subseteq \sigma[T] \text{ or } \{ \sigma(a), \sigma(b) \} \subseteq M \setminus \sigma[T] \\ &\iff \tilde{\sigma}(\{ a, b \}) \subseteq \tilde{\sigma}(T) \text{ or } \tilde{\sigma}(\{ a, b \}) \subseteq M \setminus \tilde{\sigma}(T) \\ &\iff \tilde{\sigma}(\{ a, b \}) \diamond \tilde{\sigma}(T). \end{aligned}$$

Let again denote Γ the set of all context automorphisms induced by S_M, that is

$$\Gamma = \{ \tilde{\sigma} \mid \sigma \in S_M \}.$$

Now a system of representatives of the orbits of Γ on the sets $\binom{M}{2}$ and $\mathfrak{P}(M \setminus \{ m \})$ can be computed easily. Since Γ acts on $\binom{M}{2}$ transitively there is only one orbit and we may choose any two distinct elements $a, b \in M$ for a representative $\{ a, b \}$ of the orbits of Γ on $\binom{M}{2}$. For the orbits of Γ on $\mathfrak{P}(M \setminus \{ m \})$ we first observe that for a given set $T \in \mathfrak{P}(M \setminus \{ m \})$ the orbit $\Gamma(T)$ contains all sets of cardinality $|T|$ and $|M| - |T|$. Hence we have

$$n := \left\lfloor \frac{|M|}{2} \right\rfloor$$

orbits of Γ on $\mathfrak{P}(M \setminus \{ m \})$ and we may choose for each of the cardinalities $\{ 0, \dots, n \}$ exactly one set $M_i \in \mathfrak{P}(M \setminus \{ m \})$ with this cardinality, i.e. $|M_i| = i$. So in sum we get a context orbifold of the form

$$\begin{array}{c||cccc} & M_0 & M_1 & \dots & M_n \\ \hline\hline \{ a, b \} & \eta(\{ a, b \}, M_0) & \eta(\{ a, b \}, M_1) & \dots & \eta(\{ a, b \}, M_n) \end{array} .$$

It now only remains to compute double coset representatives for the sets $\eta(\{ a, b \}, M_i)$, that is to compute an abridged annotation function η_{abr} for this context orbifold.

For this let $T \in \{ M_0, \dots, M_n \}$, $\{ a, b \}$ be the object representative and let $\sigma \in S_M$ such that $\{ a, b \} \diamond \tilde{\sigma}(T)$, that is

$$\{ a, b \} \subseteq \sigma[T] \text{ or } \{ a, b \} \subseteq M \setminus \sigma[T].$$

Let us first consider the case $\{ a, b \} \subseteq \sigma[T]$. Let $s, t \in T$ be two distinct, fixed elements of T. Now there exist $x, y \in T$ such that $\sigma(x) = a, \sigma(y) = b$. We define

$$\begin{aligned} \sigma_1 &:= (xs)(yt), \\ \sigma_2 &:= (sa)(tb), \\ \sigma_3 &:= \sigma \circ \sigma_1^{-1} \circ \sigma_2^{-1} = \sigma \circ (xs)(yt)(sa)(tb). \end{aligned}$$

Then $\sigma_1 \in \Gamma_T$ and it is $\sigma_3(a) = \sigma(x) = a, \sigma_3(b) = \sigma(y) = b$ and thus $\sigma_3 \in \Gamma_{\{ a,b \}}$. Obviously it is $\sigma = \sigma_3 \circ \sigma_2 \circ \sigma_1$ and hence

$$\sigma \in \Gamma_{\{ a,b \}} \circ (sa)(tb) \circ \Gamma_T.$$

Now the case $\{ a, b \} \subseteq M \setminus \sigma[T]$ works exactly the same with $s, t \in M \setminus T$ and gives different double cosets except for one case: if $|M|$ is even and $|T| = \frac{|M|}{2}$, then there exists a permutation $\tau \in S_M$ such that

$$\tau[T] = M \setminus T,$$

and hence $\tilde{\tau}(T) = T$ (since $m \in M \setminus T$). This means that for $\sigma \in S_M$ we get

$$\begin{aligned} \{ a, b \} \subseteq M \setminus \sigma[T] &\iff \{ a, b \} \subseteq M \setminus \sigma[\tau[M \setminus T]] \\ &\iff \{ a, b \} \subseteq \sigma[\tau[T]] = (\sigma \circ \tau)[T], \end{aligned}$$

and thus

$$\sigma \circ \tau \in \Gamma_{\{ a,b \}} \circ (sa)(tb) \circ \Gamma_T$$

with $s, t \in T$ as above. Since $\tau \in \Gamma_T$ we get $\tau^{-1} \in \Gamma_T$ and

$$\sigma \in \Gamma_{\{ a,b \}} \circ (sa)(tb) \circ \Gamma_T,$$

hence only one double coset exists in this case.

So in sum we have shown that

$$\eta(\{ a, b \}, T) = \Gamma_{\{ a,b \}} \circ \{ (s_1 a)(t_1 b), (s_2 a)(t_2 b) \} \circ \Gamma_T$$

with $s_1, t_1 \in T$, $s_2, t_2 \in M \setminus T$ distinct and provided that T and $M \setminus T$ have at least two elements and different cardinality. If the cardinalities are equal one of the two representatives can be chosen since they generate the same double coset. If $|T| \leq 1$ or $|M \setminus T| \leq 1$ then the corresponding double coset simply does not exist. Figure 2.9 shows the partition context orbifold for the set $M = \{1, \dots, 8\}$ and $M = \{1, \dots, 9\}$, respectively.

	$\emptyset$	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3,4\}$
$\{1,2\}$	$(17)(28)$	$(17)(28)$	$(1), (17)(28)$	$(1), (17)(28)$	$(17)(28)$

	$\emptyset$	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3,4\}$
$\{1,2\}$	$(18)(29)$	$(18)(29)$	$(1), (18)(29)$	$(1), (18)(29)$	$(1), (18)(29)$

Figure 2.9: Partition context orbifolds of the sets $M = \{1, \dots, 8\}$ and $M = \{1, \dots, 9\}$, respectively, under the automorphism group induced by S_M.

We see that folding the context of the partition lattice was easier than folding the partition lattice itself. So it would be, also in general, desirable to have an efficient way to compute the concept lattice orbifold directly from the context orbifold. In particular such an algorithm should be able to compute an abridged annotation function for the lattice orbifold directly from an abridged annotation function of the context orbifold itself. $\diamondsuit$

2.3 A Sample Computation in GAP

The GAP-package `ctxorb` mentioned in chapter 2.1 is also capable – albeit to a limited extent – to deal with group-annotated contexts and context orbifolds. In particular the package is able to fold given contexts by subgroups of their automorphism group. We give an extended example here and refer to the package documentation for more information on this topic.

Example 2.3.1 We want to compute in GAP the context orbifold of the context

$$(\mathfrak{P}(M), \mathrm{Imp}(M)^*, \models)$$

with $M = \{1, \dots, 5\}$ with the definitions from Example 2.1.12.

To do this we first have to compute the sets $\mathfrak{P}(M)$ and $\mathrm{Imp}(M)^*$. This can be done in GAP as follows

```
gap> M := [1 .. 5];
gap> Powerset_M := AsSet(Combinations(M));
[ [  ], [ 1 ], [ 1, 2 ], [ 1, 2, 3 ], [ 1, 2, 3, 4 ], [ 1, 2, 3, 4, 5 ],
```

```
  ...
  [ 4, 5 ], [ 5 ] ]
gap> Implications_M := AsSet(Filtered(Cartesian(Powerset_M,M),
                             pair_set_elt ->
                               not pair_set_elt[2] in pair_set_elt[1]));
[ [ [  ], 1 ], [ [  ], 2 ], [ [  ], 3 ], [ [  ], 4 ], [ [  ], 5 ],
  ...
  [ [ 5 ], 1 ], [ [ 5 ], 2 ], [ [ 5 ], 3 ], [ [ 5 ], 4 ] ]
gap>
```

Now we have to implement the relation $\models$. As the package `ctxorb` requires the relation to be implemented as a function we could write:

```
gap> satisfies := function(set, impl)
                    return (not IsSubset(set, impl[1]))
                        or impl[2] in set;
                  end;
gap>
```

Note that the function `IsSubset` in `GAP` needs its arguments in reverse order, that is

$$\texttt{IsSubset}(X, Y) \iff Y \subseteq X.$$

Now we can compute the context $(\mathfrak{P}(M), \mathrm{Imp}(M)^*, \models)$ with

```
gap> ctx := Context(Powerset_M, Implications_M, satisfies);
Context
gap>
```

A `Display(ctx)` yields a more detailed output of `ctx`. To fold this context we issue the command

```
gap> ctx_orb := ContextOrbifold(ctx);
Context Orbifold with Size 6x5x29
gap>
```

The annotation computed by `ContextOrbifold` is the full annotation which might be too much to be displayed (as can be seen by `Display(ctx_orb)`). To get an abridged annotation we can use the method `AbridgedAnnotation`

```
gap> AbridgedAnnotation(ctx_orb);
[ [ [ 1, 7 ], [  ] ], [ [ 1, 8 ], [ () ] ], [ [ 1, 9 ], [ () ] ]
  ...
  [ [ 6, 11 ], [ () ] ] ]
gap>
```

Hereby the annotation itself is of the form $\{\, ((a, b), \eta_{\mathrm{abr}}(a, b)), \ldots \}$. A `Display(ctx_orb)` now prints the abridged annotation computed before:

```
gap> Display(ctx_orb);
Context Orbifold
      Underlying Binary Relation Structure:
    Binary Relation Structure
      Source:      [ 1, 2, 3, 4, 5, 6 ]
      Sourcenames: [ 1, 2, 3, 4, 5, 6 ]
      Range:       [ 7, 8, 9, 10, 11 ]
      Rangenames:  [ 35, 40, 44, 47, 49 ]
      Relation:
           .xxxx
           xxxxx
           xxxxx
           xxxxx
           xxxxx
           xxxxx

      Group: Group( ... )
      Abridged Annotation:
        eta(1, 8) = [ () ]
        eta(1, 9) = [ () ]
        eta(1, 10) = [ () ]
        eta(1, 11) = [ () ]
        eta(2, 7) = [ () ]
        ...
        eta(6, 10) = [ () ]
        eta(6, 11) = [ () ]
gap>
```

We now could compare this to the results obtained in Example 2.1.12. To do this we first store the abridged annotation

```
gap> eta := AbridgedAnnotation(ctx_orb);;
gap>
```

Now by Proposition 1.5.7 we have that the number of representatives is invariant and thus we can compare the number of representatives from Figure 2.6 with the number of representatives in `eta`

```
gap> for entry in eta do
       Print("  entry ", entry[1], " has size ", Size(entry[2]), "\n");
     od;
  entry [ 1, 7 ] has size 0
  entry [ 1, 8 ] has size 1
  entry [ 1, 9 ] has size 1
  entry [ 1, 10 ] has size 1
```

```
  entry [ 1, 11 ] has size 1
  entry [ 2, 7 ] has size 1
  entry [ 2, 8 ] has size 2
  entry [ 2, 9 ] has size 3
  entry [ 2, 10 ] has size 3
  entry [ 2, 11 ] has size 2
  entry [ 3, 7 ] has size 1
  entry [ 3, 8 ] has size 3
  entry [ 3, 9 ] has size 4
  entry [ 3, 10 ] has size 4
  entry [ 3, 11 ] has size 2
  entry [ 4, 7 ] has size 1
  entry [ 4, 8 ] has size 3
  entry [ 4, 9 ] has size 4
  entry [ 4, 10 ] has size 3
  entry [ 4, 11 ] has size 2
  entry [ 5, 7 ] has size 1
  entry [ 5, 8 ] has size 2
  entry [ 5, 9 ] has size 2
  entry [ 5, 10 ] has size 2
  entry [ 5, 11 ] has size 1
  entry [ 6, 7 ] has size 1
  entry [ 6, 8 ] has size 1
  entry [ 6, 9 ] has size 1
  entry [ 6, 10 ] has size 1
  entry [ 6, 11 ] has size 1
gap>
```

This shows that the number of representatives in Figure 2.6 and in `eta` are the same.

One question which may arise is where the numbers $1, \ldots, 11$ come from and how they are connected to the original context `ctx`, that is now that we have computed the context orbifold `ctx_orb` how can we see which elements have been chosen for the transversals of the objects and the attributes? For this one has to know that the package `ctxorb` internally maps all structures to positive integers (as has been explained in Example 1.6.1), so called *indices*; a `Display(ctx)` shows some of these indices.

```
gap> Display(ctx);
Context
  Objects:         [ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13,
  14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28,
  29, 30, 31, 32 ]
  Objectsnames:    [ [  ], [ 1 ], [ 1, 2 ], [ 1, 2, 3 ],
  [ 1, 2, 3, 4 ], [ 1, 2, 3, 4, 5 ], [ 1, 2, 3, 5 ], [ 1, 2, 4 ],
  [ 1, 2, 4, 5 ], [ 1, 2, 5 ], [ 1, 3 ], [ 1, 3, 4 ], [ 1, 3, 4, 5 ],
  ...
```

```
gap>
```

Now when ContextOrbifold folds the context ctx it treats the indices as objects of ctx and maps them to new indices, in our case $1, \ldots, 11$.

```
gap> Display(ctx_orb);
Context Orbifold
      Underlying Binary Relation Structure:
    Binary Relation Structure
      Source:       [ 1, 2, 3, 4, 5, 6 ]
      Sourcenames: [ 1, 2, 3, 4, 5, 6 ]
      Range:        [ 7, 8, 9, 10, 11 ]
      Rangenames:  [ 35, 40, 44, 47, 49 ]
...
gap>
```

That is if we want to know which representatives have been used in ctx_orb we have to follow the indices *twice*. The functions Context and ContextObjectName can be used to do this: first we get the underlying context of ctx_orb with Context and then translate indices, say 4 and 5, twice:

```
gap> underlying_ctx := Context(ctx_orb);
Context
gap> ContextObjectName(ctx, ContextObjectName(underlying_ctx, 4));
[ 1, 2, 3 ]
gap> ContextObjectName(ctx, ContextObjectName(underlying_ctx, 5));
[ 1, 2, 3, 4 ]
gap>
```

Again if you want to see more information on a particular function you may write

```
gap> ?ContextOrbifold
```
$\diamondsuit$

3 Conclusion and Outlook

In the previous chapters we have developed a precise understanding of orbifolds of binary relation structures in general and preorder and context orbifolds in particular, based on the work by Monika Zickwolff ([Zw]). We have seen that preorder orbifolds of a given preordered set are all isomorphic under a suitable definition of isomorphy and that they allow easy visualizations in terms of a slight generalization of order diagrams together with a short but complete annotation. We were also able to transform the results obtained for preorder orbifolds to context orbifolds and moreover discovered that context orbifolds allow an own kind of derivation, which is closely related to the derivation in formal contexts. This gave us a first idea how to compute the concept lattice orbifold directly from a context orbifold of a given formal context.

But although we have seen some first, old and new, results concerning orbifolds, there are a lot of open questions and unexamined tasks related to preorder orbifolds, context orbifolds and binary relation structure orbifolds:

- Given a concept lattice orbifold, we know that the base structure is at least a preordered set. In the case of a lattice how can we find a context describing this base structure or, in general, how can we find the context of the Dedekind-MacNeille-Completion of the base structure, ideally from a context orbifold of the given lattice orbifold?
- One big drawback of the idea of folding is that we always need a group of automorphisms. Mostly it is not feasible to compute all of the automorphisms, but in some cases it might be easier to compute some of the automorphisms and then get a “partial orbifold” of the given structure. The question arising naturally now is how automorphisms of the original structure not considered in the orbifold can be seen in the orbifold? Can they possibly be understood as “automorphisms of the orbifold” and can we therefore develop a new, precise notion of “repetitive folding”, which at the end yields a result comparable to a complete orbifold?
- The original motivation of considering context orbifolds was to develop an algorithm for rule exploration, that is for finding certain implications with variables that describe a whole set of implications holding in a given context. Context orbifolds might be able to provide such an algorithm and a modification of the next-closure algorithm that directly works on context orbifolds has already been given. The problem which remains is to generalize the closure operator used in classical attribute exploration to context orbifolds, which seems difficult at least. One of the problems here might be that the automorphisms and the lectic order used for attribute exploration seem to be incompatible in such a way that a straight forward adaption of the closure operator is not possible.

- As we have seen in the previous chapter it is theoretically possible to compute the concept lattice orbifold directly from a context orbifold. But to be practically feasible one needs the ability to compute from the abridged annotation of the context orbifold an abridged annotation of the concept lattice orbifold. More generally one has to investigate whether all operations involving the full annotation function λ can be modified in such a way that they only need an abridged annotation function λ_{abr}. This might in particular be interesting when computing derivations in context orbifolds. One can then have the hope that some algorithms can be speeded up enormously.

- In this work we have never looked at certain properties the group Γ can have, although this might be very helpful. For instance, can we formulate folding and unfolding by means of generators of the group Γ? What if Γ has some interesting properties, like being abelian, soluble, cyclic or polycyclic? Can we then expose certain properties the orbifold has which can be used in algorithms?

- Automorphisms on a structure induce a partition of the base set by means of their orbits. This partition corresponds to an equivalence relation and one might be tempted to ask whether more general relations are suitable for computing orbifolds of certain kinds of relational structures?

As can be seen the list of interesting questions is long and by far incomplete. This work considers itself as a first investigation on the idea of folding structures by their symmetries, tries to give a first insight on what can be expected from this idea and what might be difficult to achieve using the technique of folding.

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ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema „Context Orbifolds“ unter Betreuung von Prof. Dr. Bernhard Ganter selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

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