Complexity Theory

Exercise 9: Circuit Complexity

Exercise 9.1. Denote with \( \text{add} : \{0, 1\}^{2n} \to \{0, 1\}^{n+1} \) the function that takes two binary \( n \)-bit numbers \( x \) and \( y \) and returns their \( n + 1 \)-bit sum. Show that \( \text{add} \) can be computed with size \( \mathcal{O}(n) \) circuits.

Exercise 9.2. Define the function \( \text{maj}_n : \{0, 1\}^n \to \{0, 1\}^n \) by

\[
\text{maj}_n(x_1, \ldots, x_n) := \begin{cases} 0 & \text{if } \sum x_i < n/2 \\ 1 & \text{if } \sum x_i \geq n/2. \end{cases}
\]

Devise a circuit to compute \( \text{maj}_3 \) and test it on the example input 101 and 010.

Exercise 9.3. Show \( \text{NC}^1 \subseteq \text{L} \).

Exercise 9.4. Show that every Boolean function with \( n \) variables can be computed with a circuit of size \( \mathcal{O}(n \cdot 2^n) \).

Exercise 9.5. Show that every language \( L \subseteq \{1^n \mid n \in \mathbb{N}\} \) is contained in \( \text{P/poly} \). Conclude that \( \text{P/poly} \) contains undecidable languages.

Exercise 9.6. Find a decidable language in \( \text{P/poly} \) that is not contained in \( \text{P} \).

Hint: Take a language over \( \{0, 1\} \) that is \( 2\text{ExpTime} \)-hard and consider its unary encoding.

Exercise 9.7. Show how to compute \( \text{maj}_n \) with circuits of size \( \mathcal{O}(n \log n) \).

Exercise 9.8. Show that \( \text{NC} \neq \text{PSPACE} \).