Exercise 9: Semi-Positive Datalog

Database Theory

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- Then for every fact φ over the signature of P, we have that P' entails φ over an instance D iff P entails φ over D.

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- Then for every fact φ over the signature of P, we have that P' entails φ over an instance D iff P entails φ over D.
- The size of P' is polynomial in the size of P, and P' is safe.

Exercise 2

Exercise. Assume that the database uses a binary EDB predicate *edge* to store a directed graph. Try to express the following properties in semi-positive Datalog programs with a successor ordering, or explain why this is not possible.

1. The database contains an even number of elements.
2. The graph contains a node with two outgoing edges.
3. The graph is 3-colourable.
4. The graph is *not* connected (*).
5. The graph does not contain a node with two outgoing edges.
6. The graph is a chain.

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Solution.

1.

$$\begin{aligned} \mathrm{Odd}(x) &\leftarrow \mathrm{first}(x) \\ \mathrm{Odd}(y) &\leftarrow \mathrm{Even}(x), \mathrm{succ}(x, y) \\ \mathrm{Even}(y) &\leftarrow \mathrm{Odd}(x), \mathrm{succ}(x, y) \\ \mathrm{EvenParity}() &\leftarrow \mathrm{Even}(x), \mathrm{last}(x) \end{aligned}$$

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\begin{aligned}
<(x,y) &\leftarrow \mathrm{succ}(x,y)\\
<(x,z) &\leftarrow <(x,y), \mathrm{succ}(y,z)\\
<>(x,y), <>(y,x) &\leftarrow <(x,y)\\
\mathrm{TwoOutgoingEdges}() &\leftarrow \mathrm{edge}(x,y), \mathrm{edge}(x,z), <>(y,z)
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3. This is (most likely) not expressible (unless $\mathrm{P} = \mathrm{NP}$), since 3-colourability is NP-complete and Datalog has P data complexity.

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Solution.

4. $\mathrm{D}(x, y, k)$ x and y are not reachable via a path of length at most k
   $\mathrm{N}(x, y, z, k)$ there is no path of length $k + 1$ from x to z via y

$$\mathrm{D}(x, y, \ell), \mathrm{D}(y, x, \ell) \leftarrow \neg\mathrm{edge}(x, y), \mathrm{first}(\ell), <>(x, y)$$

$$\mathrm{N}(x, y, z, k) \leftarrow \mathrm{first}(y), \mathrm{D}(x, y, k) \qquad \mathrm{N}(x, y, z, k) \leftarrow \mathrm{first}(y), \mathrm{D}(y, z, k)$$

$$\mathrm{N}(x, y', z, k) \leftarrow \mathrm{succ}(y, y'), \mathrm{N}(x, y, z, k), \mathrm{D}(x, y', k) \qquad \mathrm{N}(x, y', z, k) \leftarrow \mathrm{succ}(y, y'), \mathrm{N}(x, y, z, k), \mathrm{D}(y', z, k)$$

$$\mathrm{D}(x, z, k') \leftarrow \mathrm{succ}(k, k'), \mathrm{D}(x, z, k), \mathrm{last}(y), \mathrm{N}(x, y, z, k) \qquad \mathrm{Ans}() \leftarrow \mathrm{D}(x, y, k), \mathrm{last}(k)$$

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Solution.

5.

$$\begin{aligned}
\text{oneEdge}(x,y) &\leftarrow \text{first}(y), \text{edge}(x,y) \\
\text{oneEdge}(x,z) &\leftarrow \text{noEdge}(x,y), \text{succ}(y,z), \text{edge}(x,z) \\
\text{oneEdge}(x,z) &\leftarrow \text{oneEdge}(x,y), \text{succ}(y,z), \neg\text{edge}(x,z) \\
\text{r}(x) &\leftarrow \text{last}(y), \text{noEdge}(x,y) \\
\text{r}(x) &\leftarrow \text{last}(y), \text{oneEdge}(x,y) \\
\text{NoTwoOutEdges}() &\leftarrow \text{s}(x), \text{last}(x)
\end{aligned}$$

$$\begin{aligned}
\text{noEdge}(x,y) &\leftarrow \text{first}(y), \neg\text{edge}(x,y) \\
\text{noEdge}(x,z) &\leftarrow \text{noEdge}(x,y), \text{succ}(y,z), \neg\text{edge}(x,z) \\
\text{s}(x) &\leftarrow \text{first}(x), \text{r}(x) \\
\text{s}(y) &\leftarrow \text{succ}(x,y), \text{s}(x), \text{r}(y)
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$$\mathrm{Chain}() \leftarrow \mathrm{Connected}(), \mathrm{NoTwoInEdges}(), \mathrm{NoTwoOutEdges}(), \mathrm{NoCycle}()$$

$$\mathrm{Conn}(x), \mathrm{Reachable}(x) \leftarrow \mathrm{first}(x) \qquad \mathrm{Reachable}(x) \leftarrow \mathrm{Reachable}(x), \mathrm{succ}(x, y), \mathrm{Conn}(y)$$

$$\mathrm{Conn}(y) \leftarrow \mathrm{Conn}(x), \mathrm{edge}(x, y) \qquad \mathrm{Conn}(y) \leftarrow \mathrm{Conn}(x), \mathrm{edge}(y, x)$$

$$\mathrm{Connected}() \leftarrow \mathrm{last}(x), \mathrm{Reachable}(x)$$

$$\mathrm{NoInEdge}(x, y) \leftarrow \mathrm{first}(x), \neg\mathrm{edge}(x, y)$$

$$\mathrm{NoOutEdge}(x, y) \leftarrow \mathrm{first}(x), \neg\mathrm{edge}(y, x)$$

$$\mathrm{NoInEdge}(x', y) \leftarrow \mathrm{succ}(x, x'), \mathrm{NoInEdge}(x, y), \neg\mathrm{edge}(x', y)$$

$$\mathrm{NoOutEdge}(x', y) \leftarrow \mathrm{succ}(x, x'), \mathrm{NoOutEdge}(x, y), \neg\mathrm{edge}(y, x')$$

$$\mathrm{NoCycle}() \leftarrow \mathrm{last}(x), \mathrm{NoInEdge}(\mathrm{x}, \mathrm{y}), \mathrm{NoOutEdge}(\mathrm{x}, \mathrm{z})$$

with NoTwoOutEdges() defined as in 5., and NoTwoInEdges() defined analogously.

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Exercise. A Horn logic program is in *propHorn2* if every rule it contains is of the form $H \leftarrow$ or $H \leftarrow B_1 \wedge B_2$.
It was claimed that entailment checking in *propHorn2* is P-hard. To support this claim, explain how entailment in propositional Horn logic can be reduced to entailment in *propHorn2*. Argue how this reduction can be accomplished in logarithmic space.

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- For all propositional variables V occurring in P, we have that $P \models V$ if and only if $P' \models V$.

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- The program P' can be computed with a $\textsc{LogSpace}$ transducer:
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  - count number of rules to have fresh identifiers for every newly translated rule, and

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  - count the length of any propositional variable name to have unique identifiers

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Solution.

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Exercise. Prove that entailment checking in propositional Horn logic is P-hard.

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  - $\mathrm{S}(\mathrm{cell}_{0,i}, w_i)$ for $1 \leq i \leq n$, and $\mathrm{S}(\mathrm{cell}_{0,i}, b)$ for $n+1 \leq i \leq n^k$, and
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- Rules:
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  - $T(z, q') \leftarrow T(x, q) \wedge S(x, \gamma) \wedge \mathrm{future}(x, y) \wedge \mathrm{right}(z, y)$, $S(y, \gamma') \leftarrow T(x, q) \wedge S(x, \gamma) \wedge \mathrm{future}(x, y)$ for all $\langle q, \gamma, q', \gamma', L \rangle \in \delta$,
  - and similarly for all $\langle q, \gamma, q', \gamma', R \rangle \in \delta$

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- The grounding of P is a Propositional Horn Logic program that is polynomial in the size of P (which is polynomial in the size of n).

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- The grounding of P is a Propositional Horn Logic program that is polynomial in the size of P (which is polynomial in the size of n).
- $\mathsf{ground}(P)$ can be computed by a $\mathrm{LogSpace}$ transducer.

Exercise 5

Exercise. Show that the following property cannot be expressed in Datalog: The edge predicate has a *proper* cycle, i.e., a cycle that is not of the form edge(a, a).
Can you express this property using ...

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3. Since $\approx$ can be axiomatised using $x \approx x \leftarrow$, Datalog with an equality predicate is not more expressive than Datalog.
4. $$\text{properEdge}(x, y) \leftarrow \text{edge}(x, y) \wedge x \not\approx y \qquad \text{properPath}(x, y) \leftarrow \text{properEdge}(x, y)$$
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