Finite and Algorithmic Model Theory

Lecture 1 (Dresden 12.10.22)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

Today's agenda

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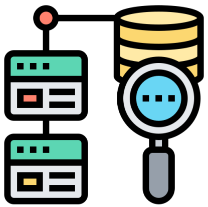

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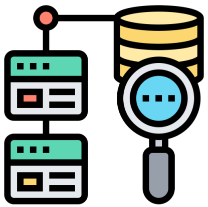

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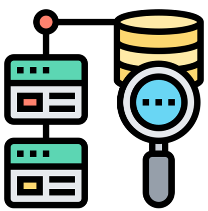

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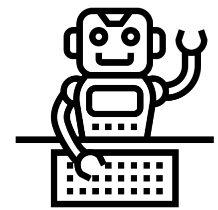

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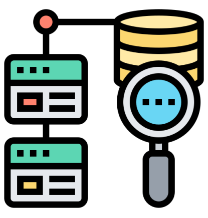

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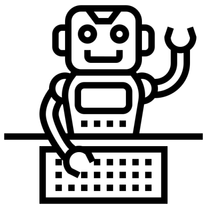

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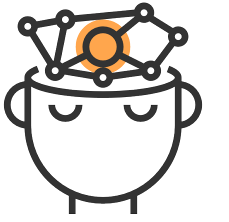

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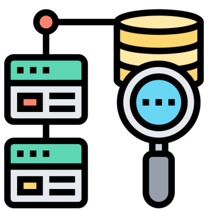

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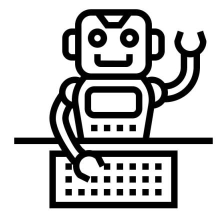

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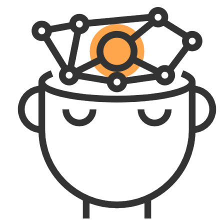

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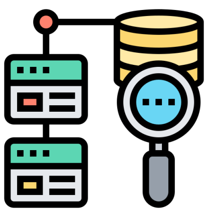

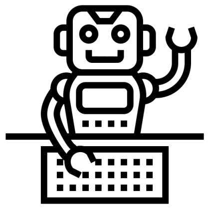
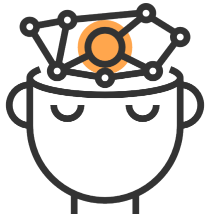

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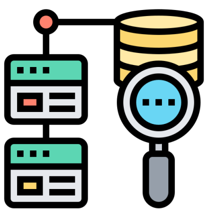
Query languages? 

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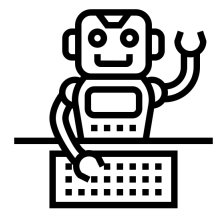
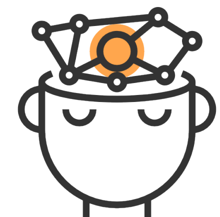
Formal languages? 

Complexity? 

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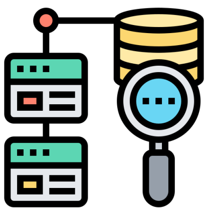

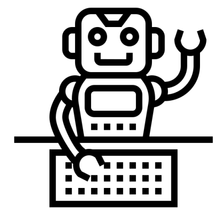
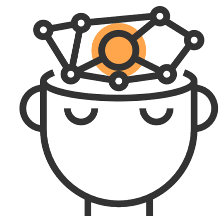

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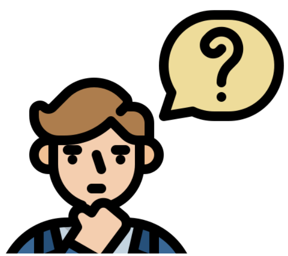

Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

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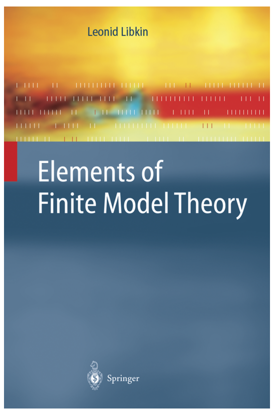

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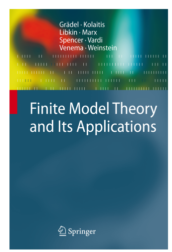

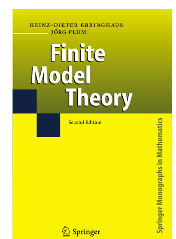

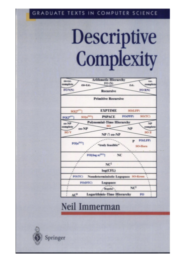

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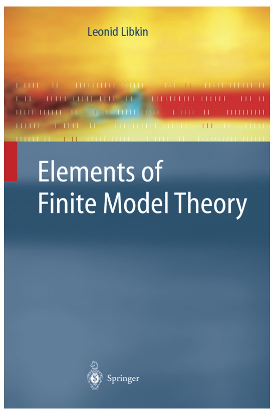

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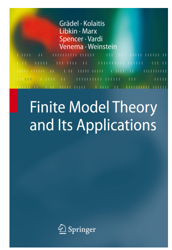

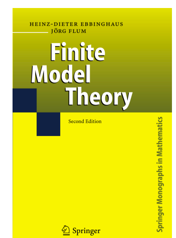

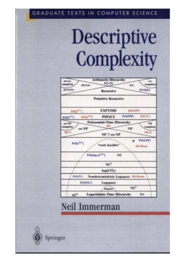

Last but Not Least: I offer MSc/PHD research projects for motivated students!

2–3. Examples and Motivations

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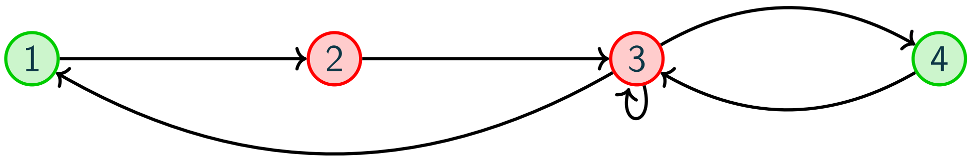
$\mathfrak{A} :=$ 

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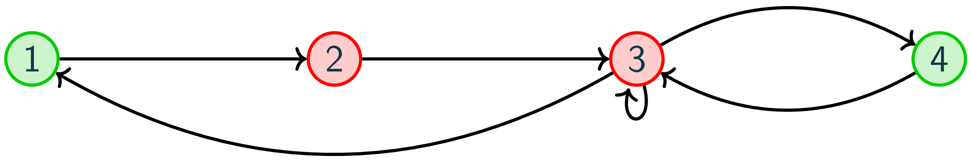
$\mathfrak{A} :=$ 

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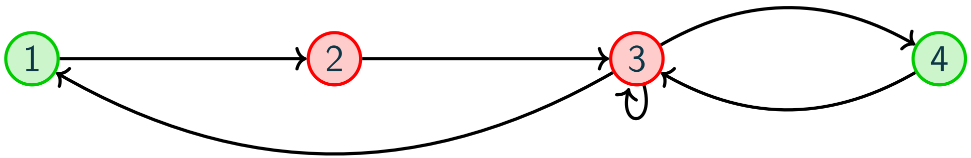

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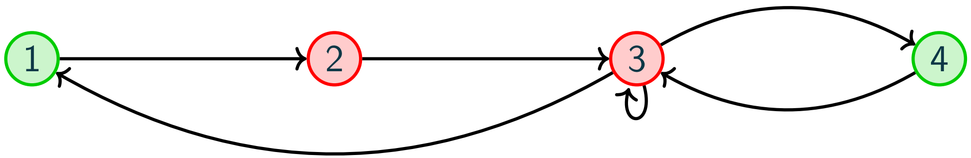
$\mathfrak{A} :=$ 

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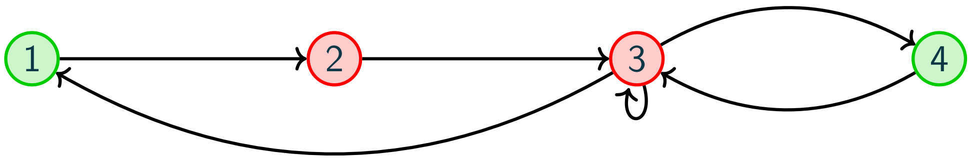
$\mathfrak{A} :=$ 

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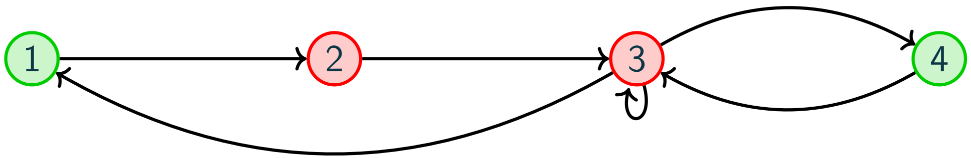
$\mathfrak{A} :=$ 

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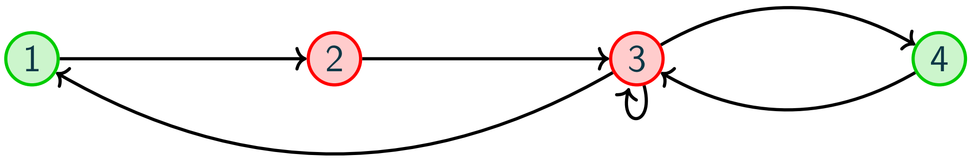
$\mathfrak{A} :=$ 

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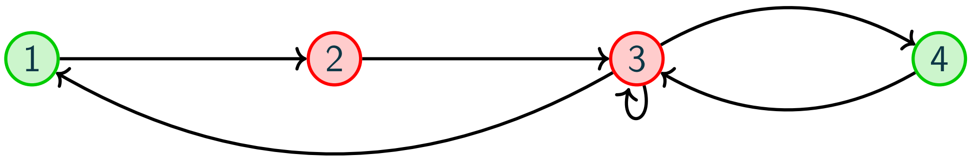
$\mathfrak{A} :=$ 

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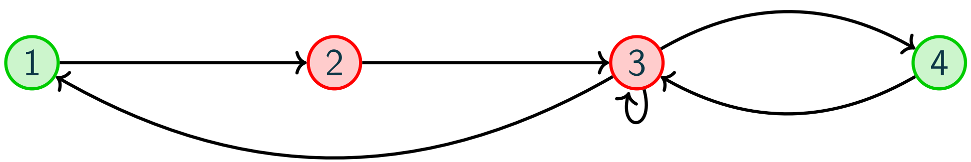
$\mathfrak{A} :=$ 

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Constants $\approx$ elements, unary relations $\approx$ colours, binary (resp. higher-arity) relations $\approx$ (hyper)edges

What is a "logic"? A running example.

Naively: a "formal language" for expressing properties of relational structures ($\approx$ hypergraphs).

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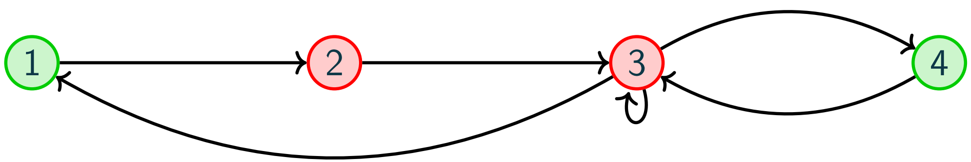
$\mathfrak{A} :=$ 

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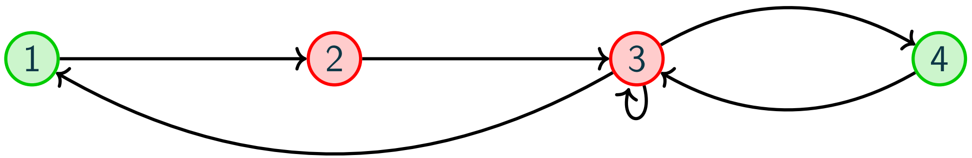
$\mathfrak{A} :=$ 

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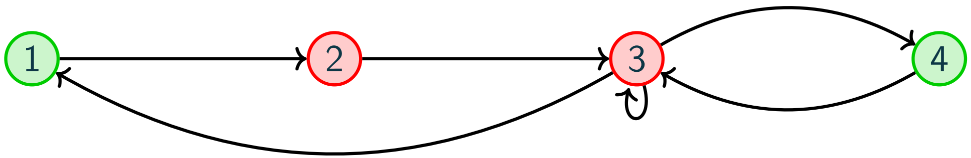
$\mathfrak{A} :=$ 

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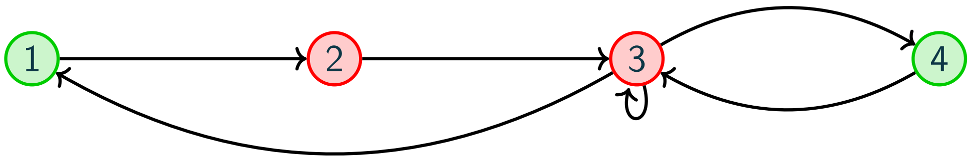
$\mathfrak{A} :=$ 

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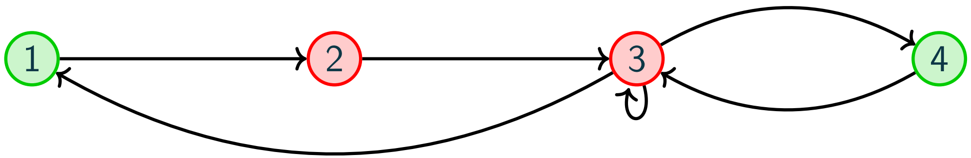
$\mathfrak{A} :=$ 

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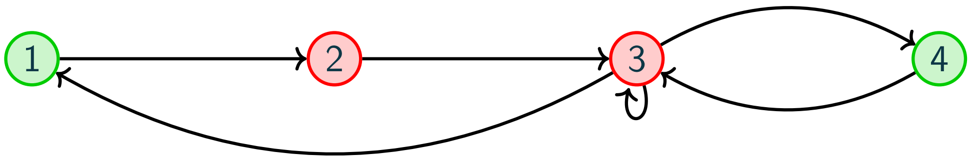
$\mathfrak{A} :=$ 

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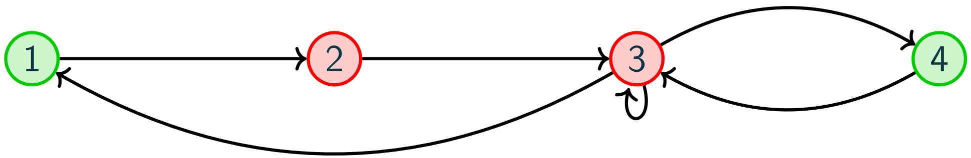
$\mathfrak{A} :=$ 

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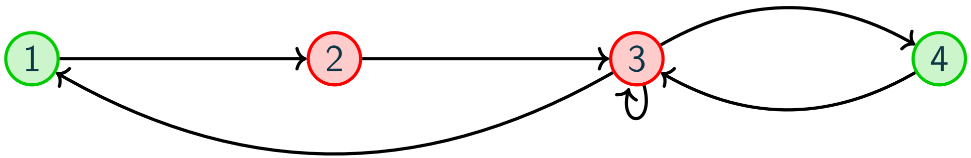
$\mathfrak{A} :=$ 

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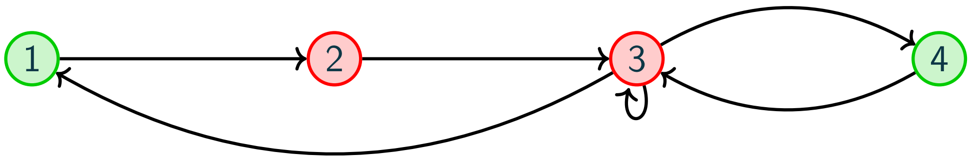
$\mathfrak{A} :=$ 

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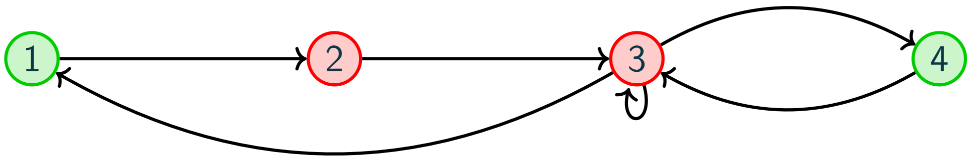

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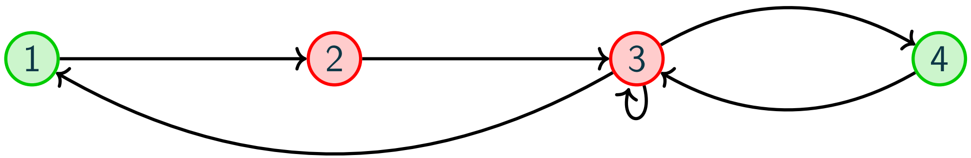

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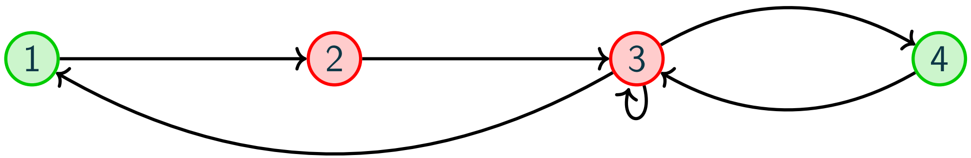
$\mathfrak{A} :=$ 

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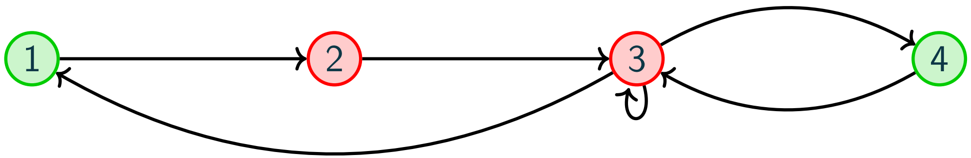
$\mathfrak{A} :=$ 

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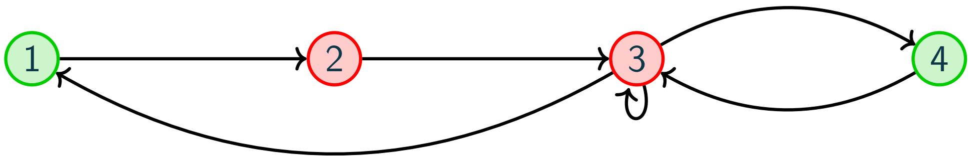
$\mathfrak{A} :=$ 

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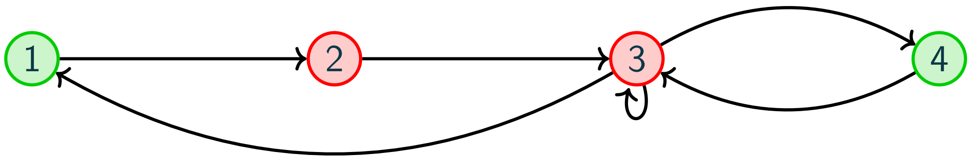
$\mathfrak{A} :=$ 

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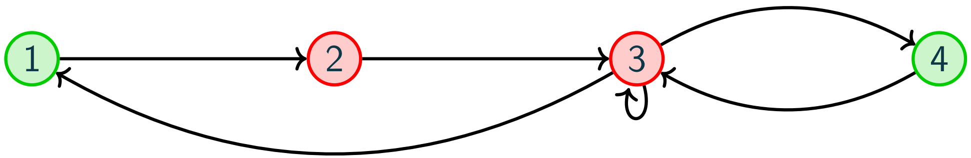
$\mathfrak{A} :=$ 

over a signature $\tau := \{\mathrm{G}^{(1)}, \mathrm{R}^{(1)}, \mathrm{E}^{(2)}\}$

$\mathrm{G}^{\mathfrak{A}} := \{1, 4\}, \qquad \mathrm{R}^{\mathfrak{A}} := \{2, 3\}$

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A signature contains (at most countably* many) constant and relation symbols (each with a fixed arity).

Structure = Domain + interpretation of symbols, e.g. $\mathfrak{A} := (A, \cdot^{\mathfrak{A}})$ depicted above,

where $A = \{1, 2, 3, 4\}$ and $\cdot^{\mathfrak{A}}(\mathrm{G}), \cdot^{\mathfrak{A}}(\mathrm{R}), \cdot^{\mathfrak{A}}(\mathrm{E})$ are as above.

Example (of a First-Order Logic (FO) Formula)

(in a coloured graph:) Any node is either green or red.

$\varphi := \forall x\ (\mathrm{G}(x) \vee \mathrm{R}(x)) \wedge (\mathrm{G}(x) \leftrightarrow \neg \mathrm{R}(x))$

We write $\mathfrak{A} \models \varphi$ to indicate that $\mathfrak{A}$ satisfies φ or $\mathfrak{A}$ is a model of φ.

Formulae often employ: Variables: $x, y, z, X, Y, \ldots$ Boolean connectives: $\wedge, \vee, \neg, \leftrightarrow, \bigvee_{i=0}^{\infty}, \ldots$

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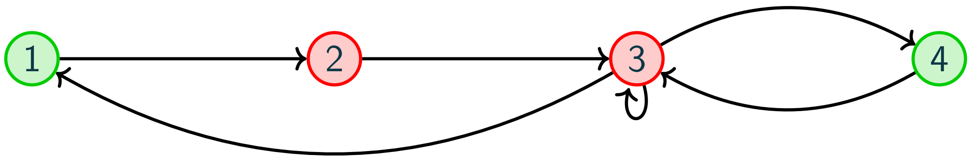
$\mathfrak{A} :=$ 

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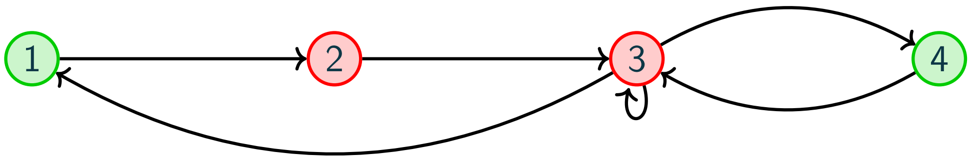
$\mathfrak{A} :=$ 

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More examples I.

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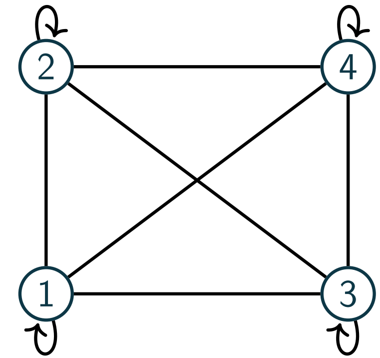

$\models \varphi_{\textsf{4clique}}$

More examples I.

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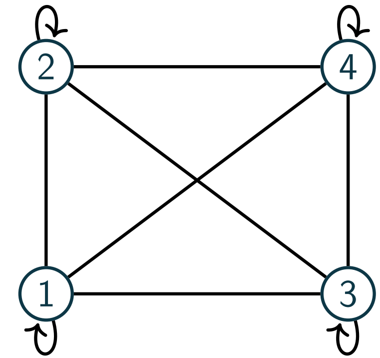

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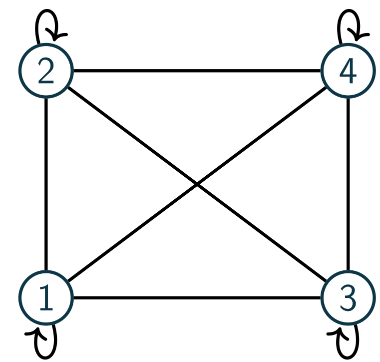

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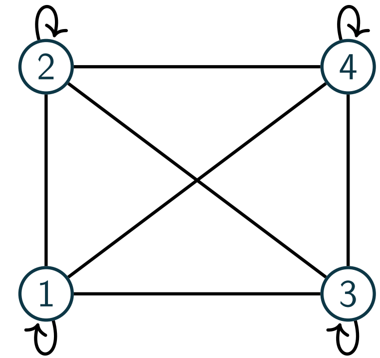
2. and any two of them are linked by E.



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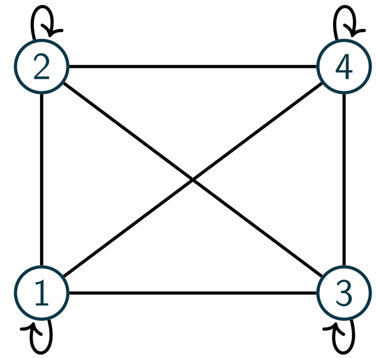

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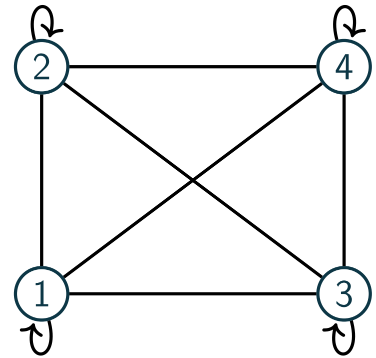

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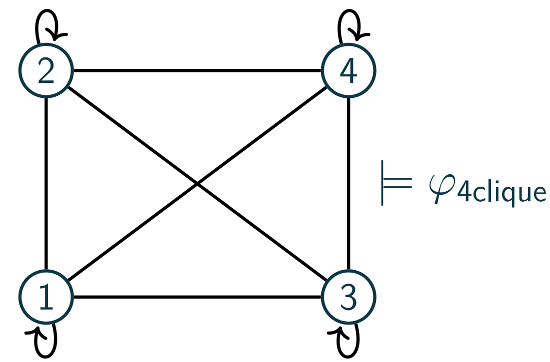

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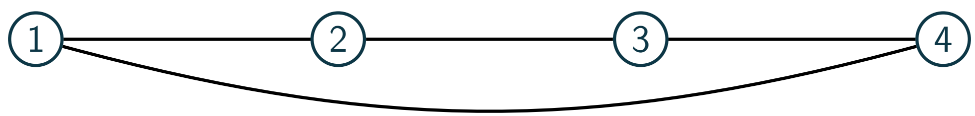
$\mathfrak{G} :=$ 

More examples I.

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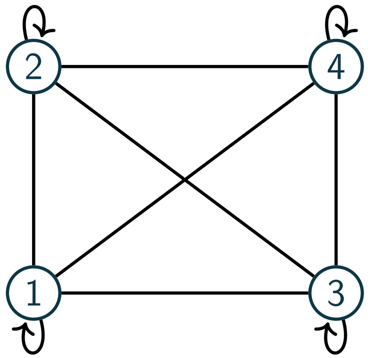

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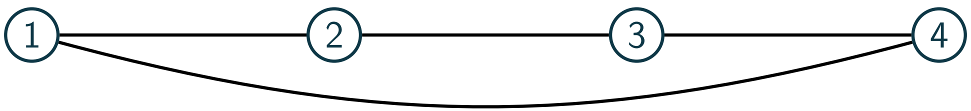
$\mathfrak{G} :=$ 

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More examples I.

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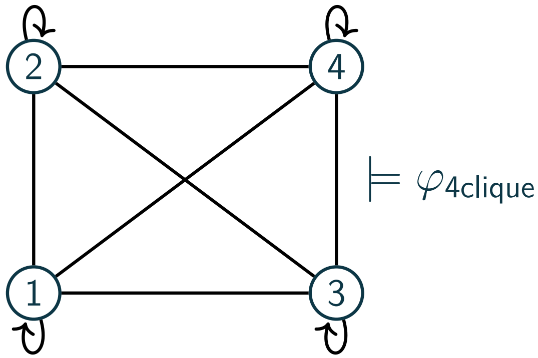

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More examples I.

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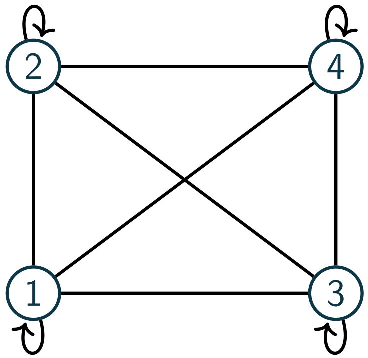

$\models \varphi_{\text{4clique}}$

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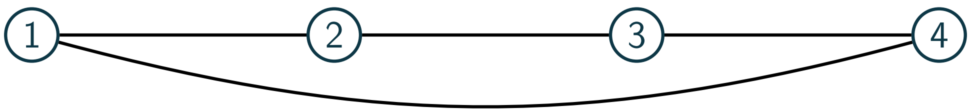
$\mathfrak{G} :=$ 

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Quantification over sets:

More examples I.

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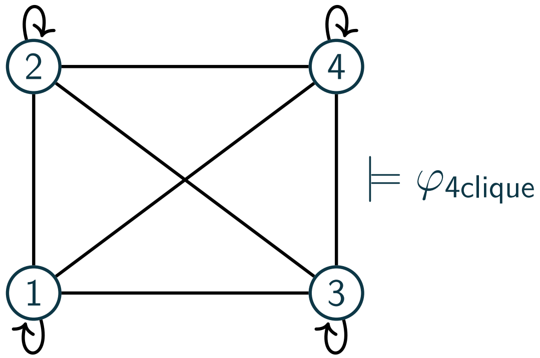

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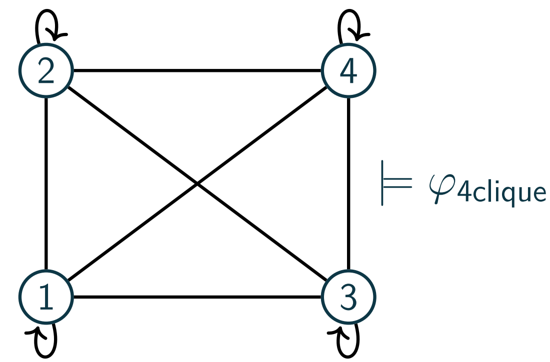

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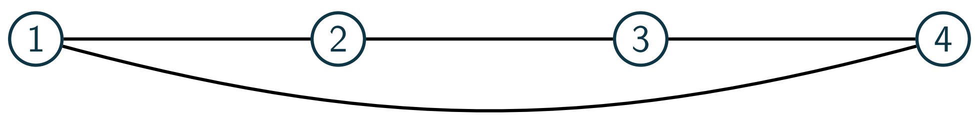

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There exists a colouring with G and R and it is correct

More examples I.

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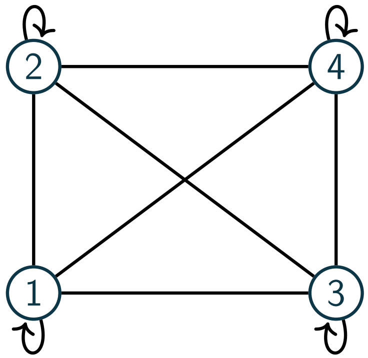

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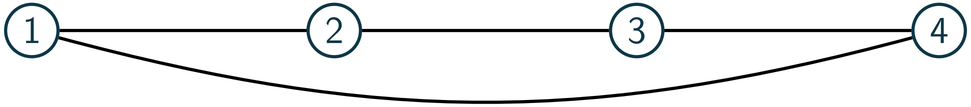
$\mathfrak{G} :=$ 

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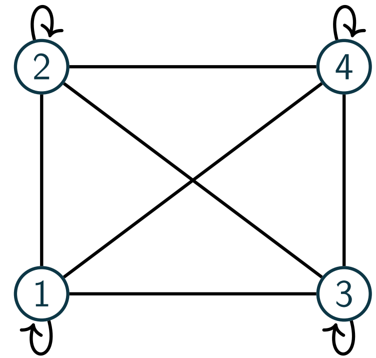

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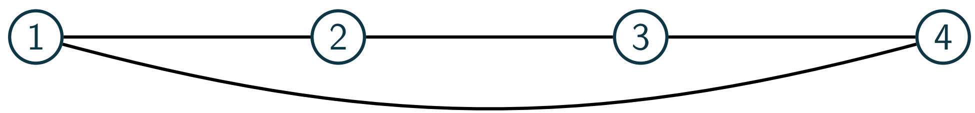

$\models \varphi_{2COL}$

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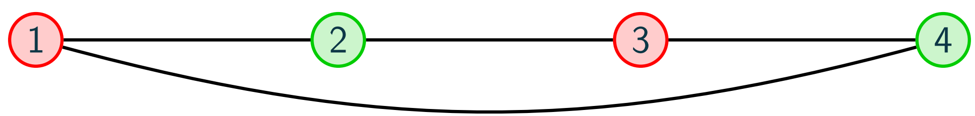

$\models \varphi_{ok}$

More examples II.

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No. And we will show it today!

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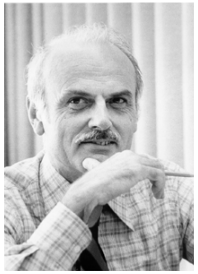
Basic SQL $\approx$ First-Order Logic



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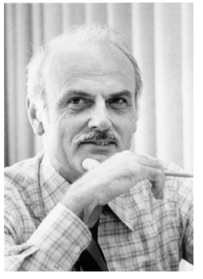

Other useful logic: Datalog $\approx$ SQL + recursion

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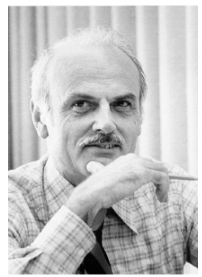

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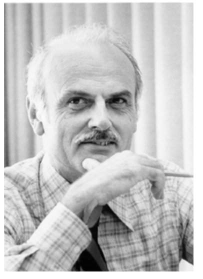

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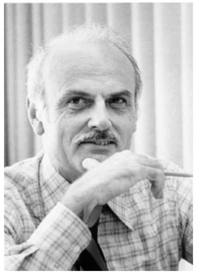

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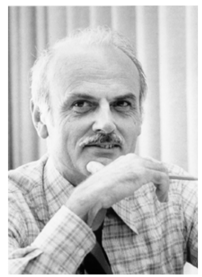

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Description logics: a family of logics for knowledge representation.

W3C® SNOMED CT The global language of healthcare protégé Dublin Core Metadata Initiative Making it easier to find information

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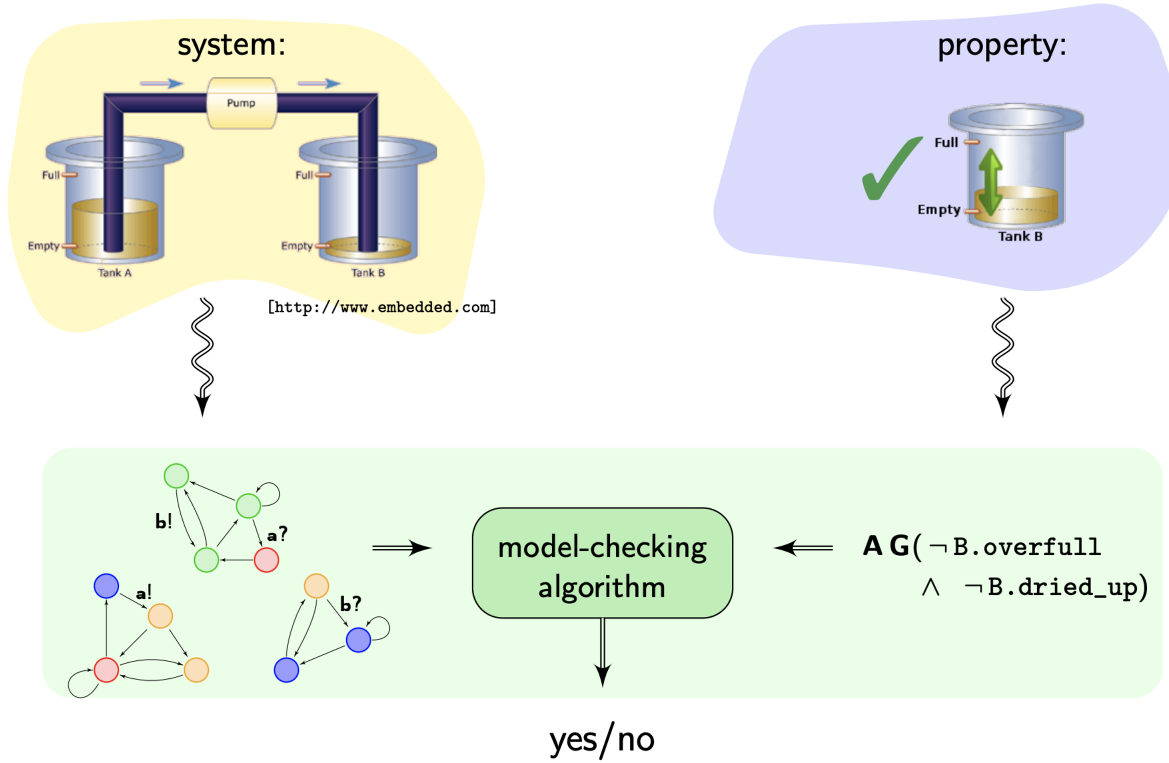
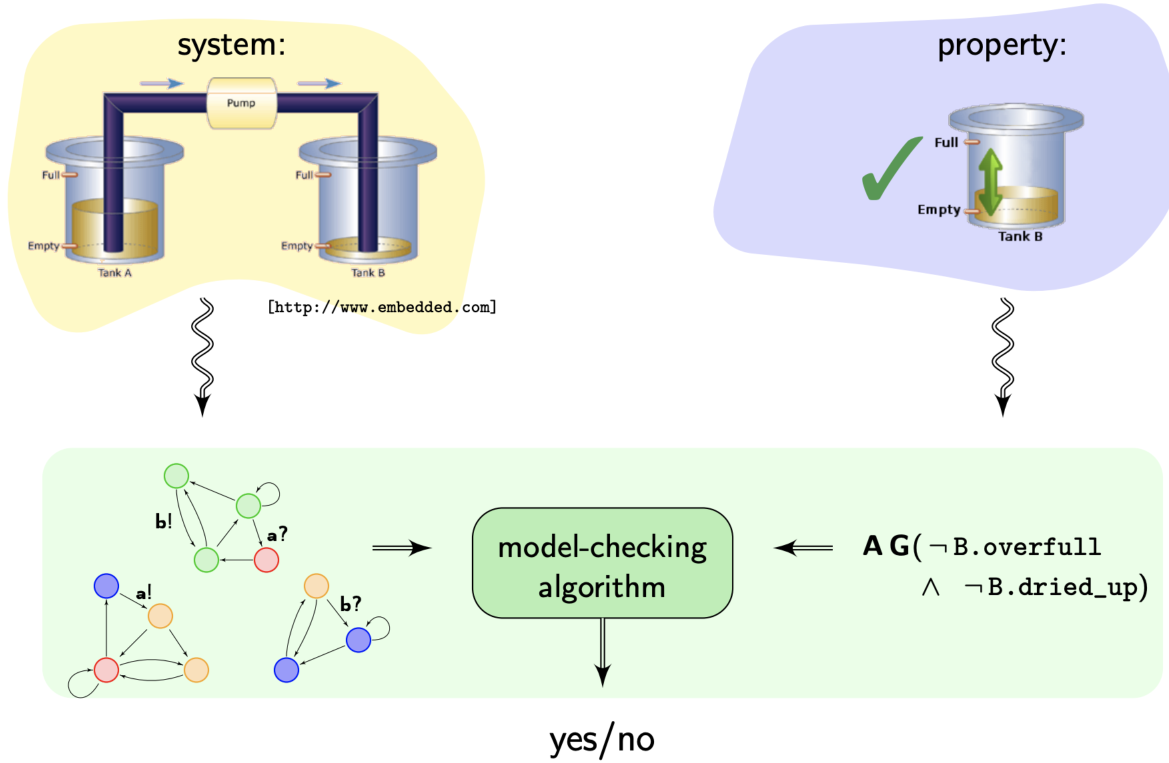


Motivations II: why do we care about logic?

1. Temporal logics as specification languages



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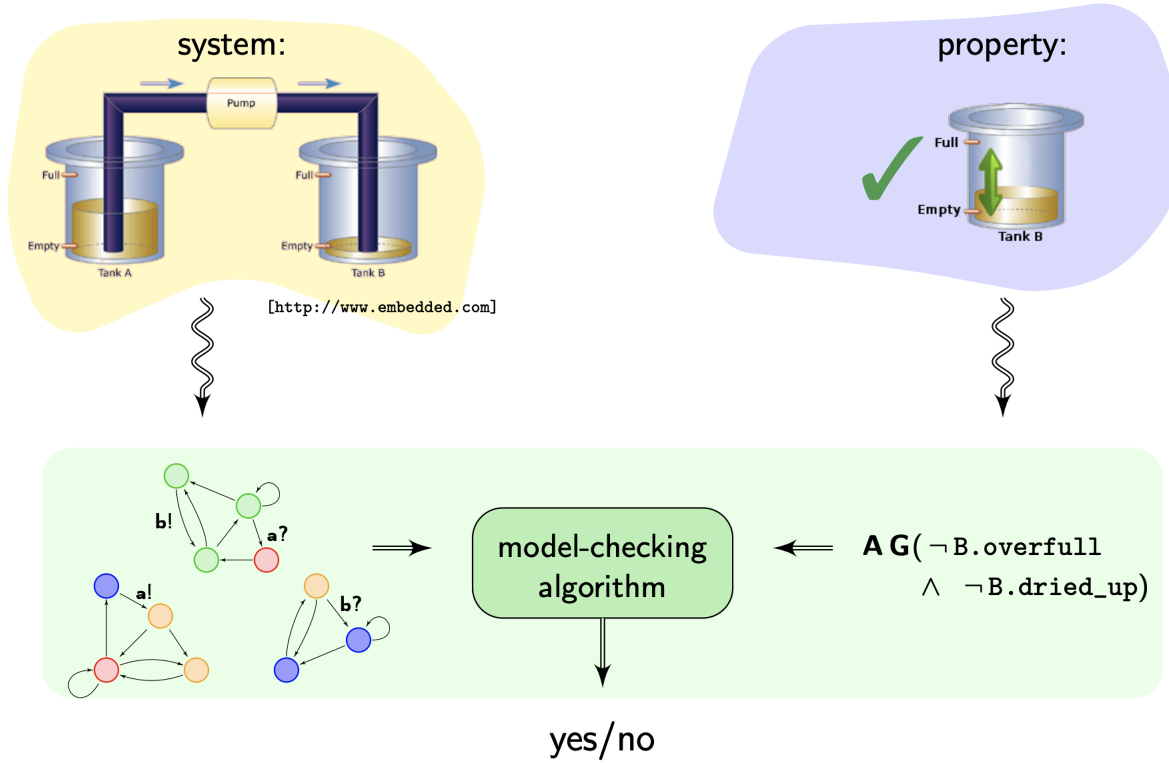

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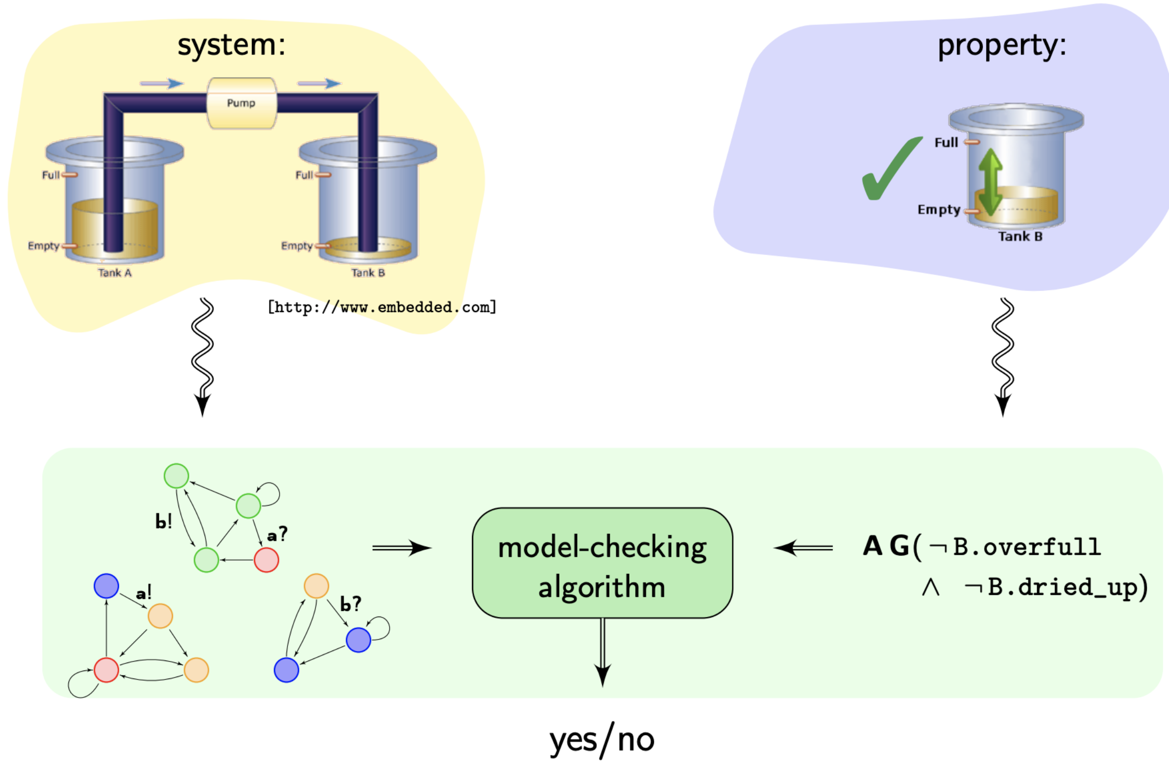

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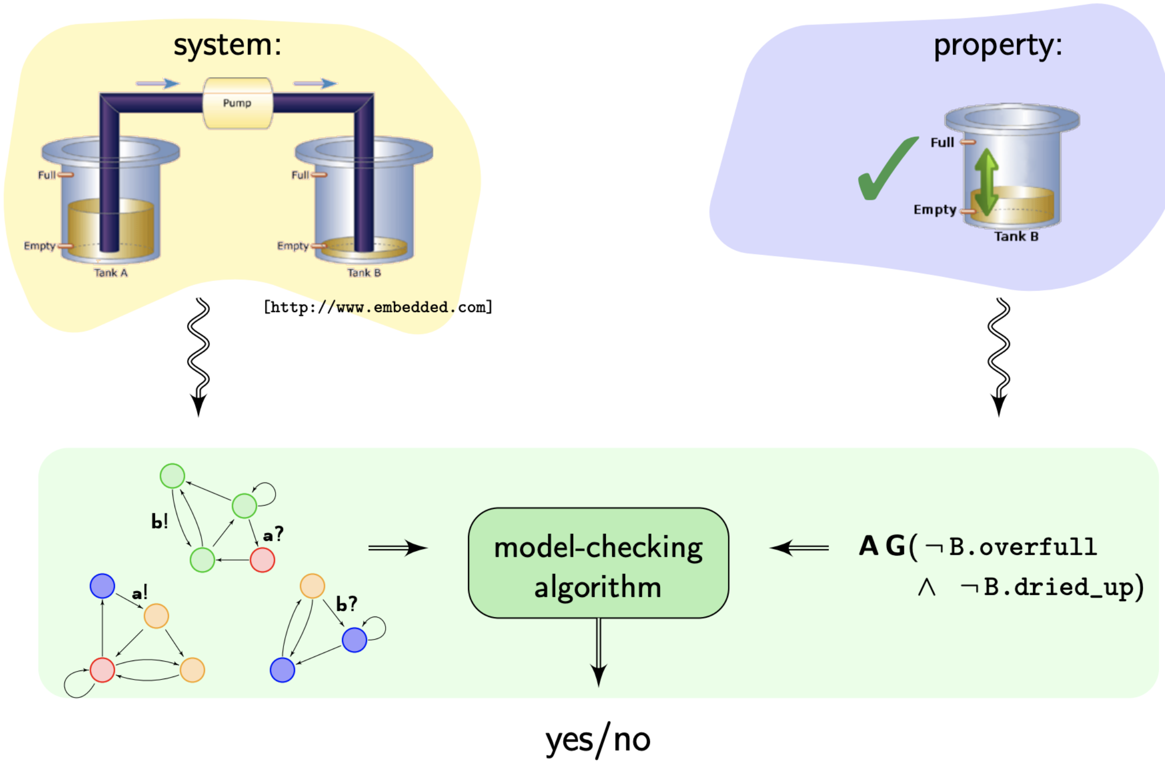

Motivations II: why do we care about logic?

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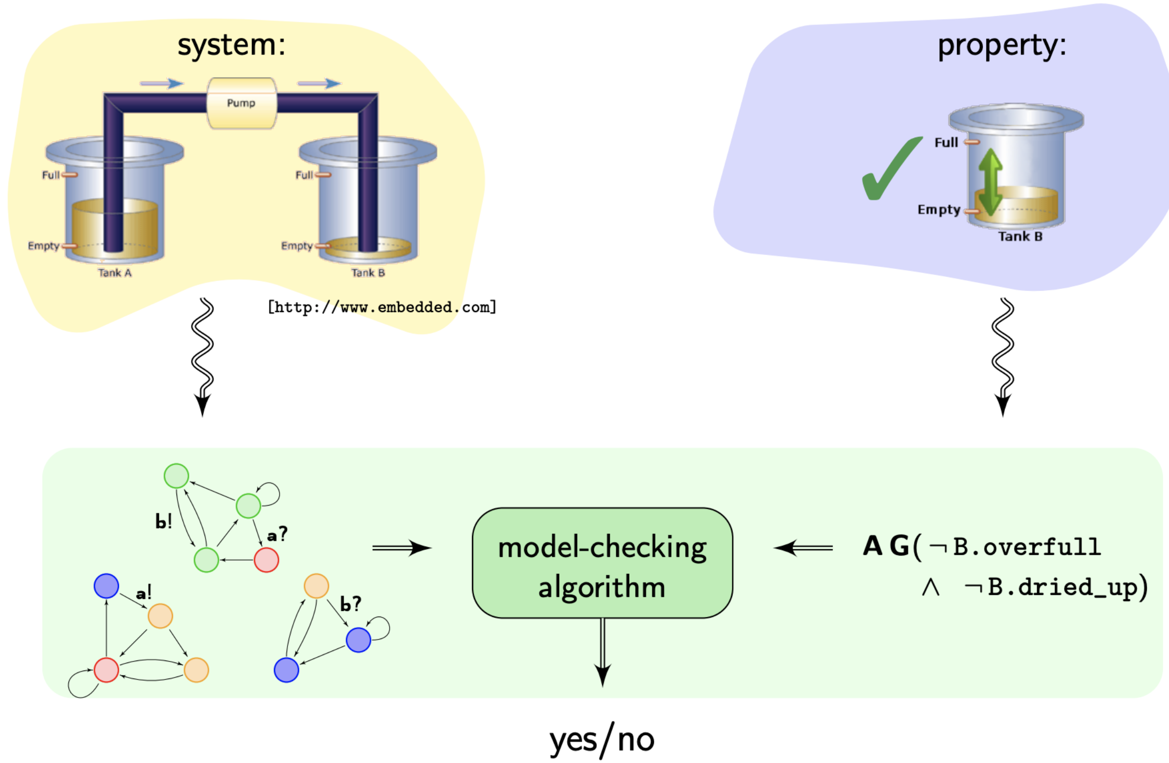

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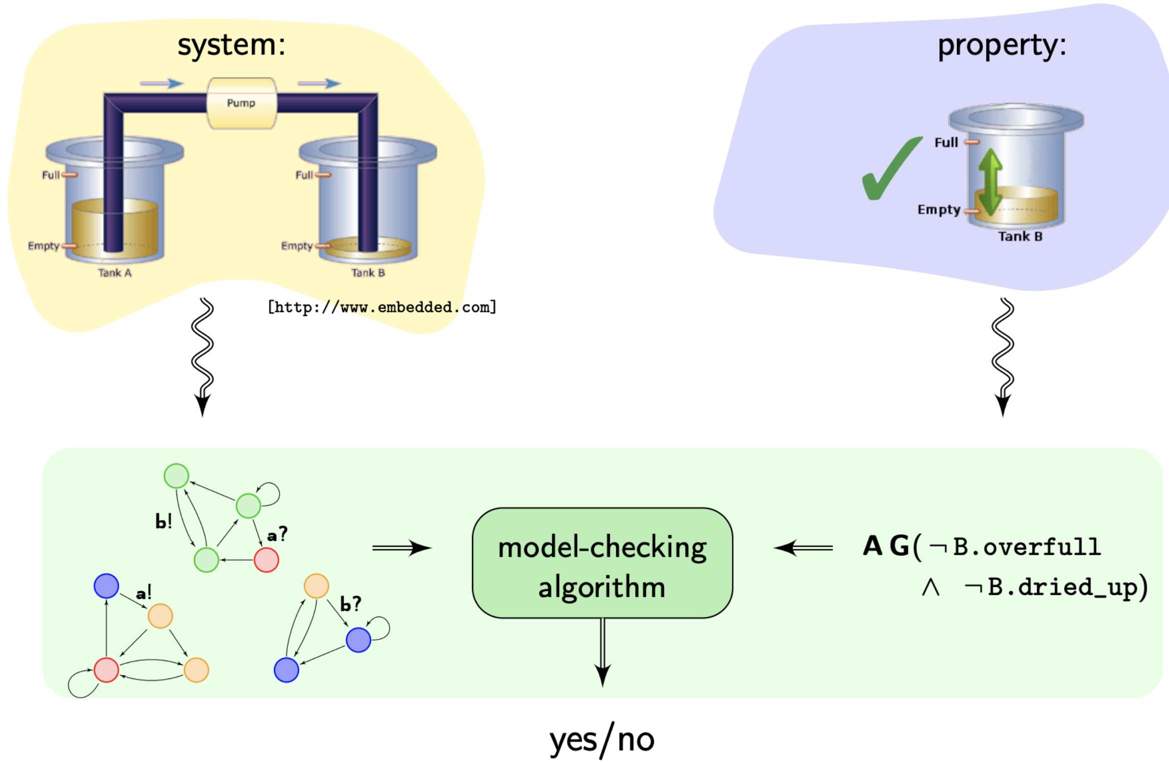

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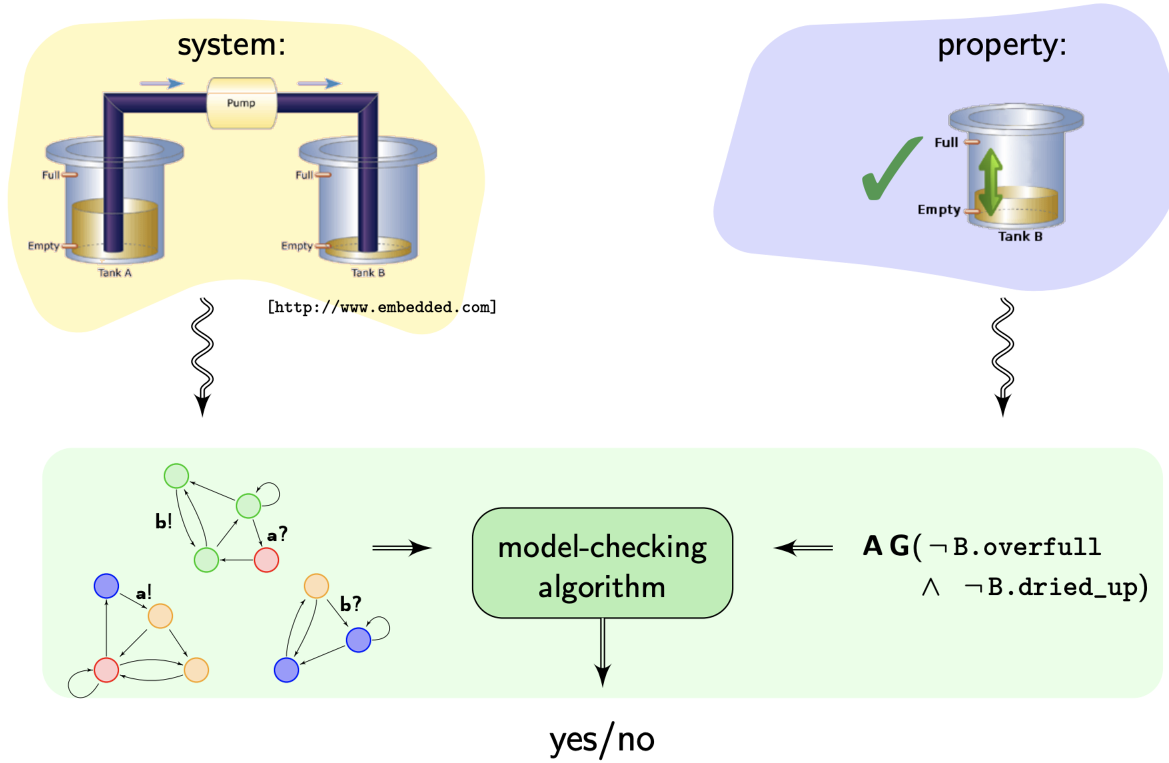
vim hello.c

```c
// hello.c
#include <stdlib.h>

void test() {
  int *s = NULL;
  *s = 42;
}
```



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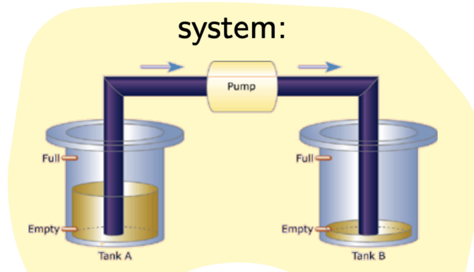

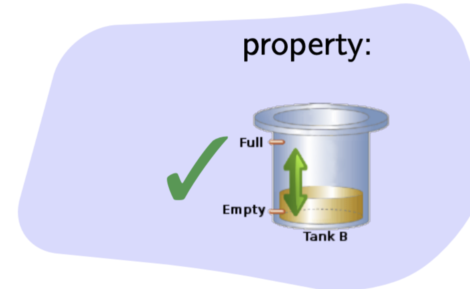


```
vim hello.c
// hello.c
#include <stdlib.h>

void test() {
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```
bartoszbednarczyk@Minsky-Machine: ~/Downloads/Infer
$ infer run -- gcc -c hello.c

Capturing in make/cc mode...
Found 1 source file to analyze in /Users/bartoszbednarczyk/Downloads/Infer/infer-out


Analysis finished in 775mss

Found 1 issue

hello.c:6: error: NULL_DEREFERENCE
  pointer `s` last assigned on line 5 could be null and is dereferenced at line 6, column 3.
  4.   void test() {
  5.     int *s = NULL;
  6. >   *s = 42;
  7.   }


Summary of the reports

  NULL_DEREFERENCE: 1
```

Motivations III: why do we care about logic?

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In "standard" computational complexity we measure resources, e.g. space and time.

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$O(n)$ time

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$\Theta(n \log(n))$ memory?

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solvable in PSpace?

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decidable?

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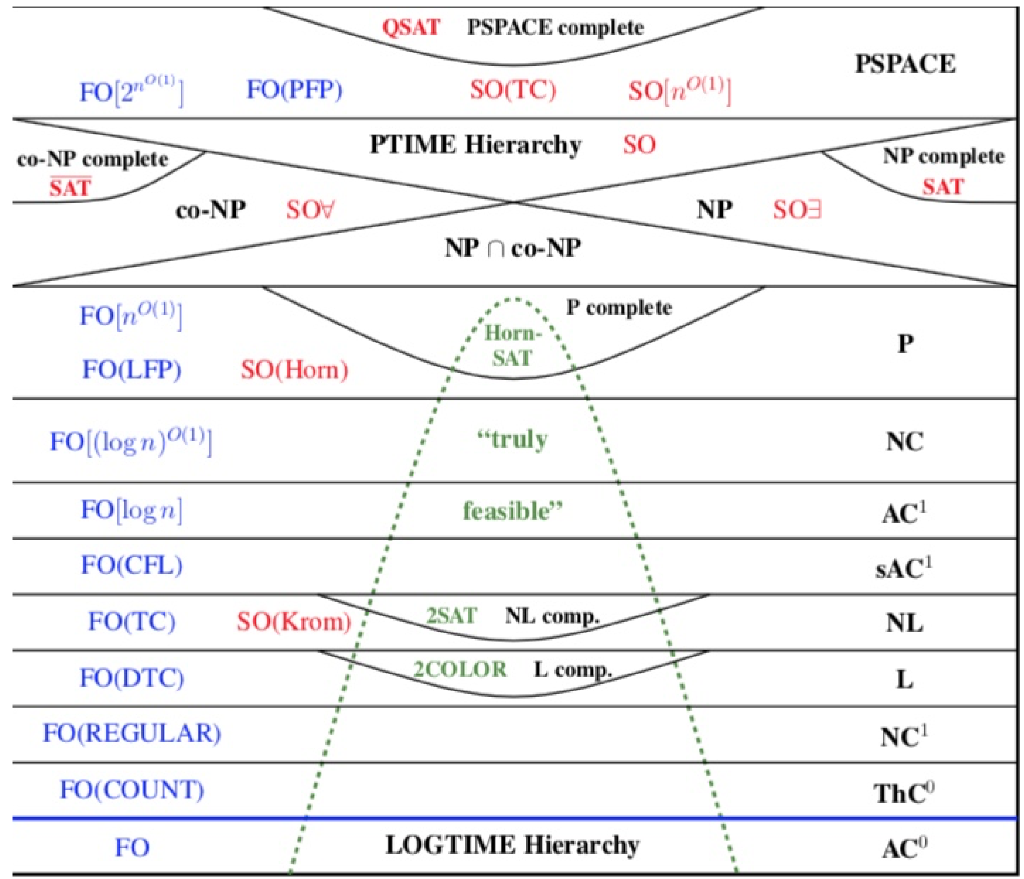

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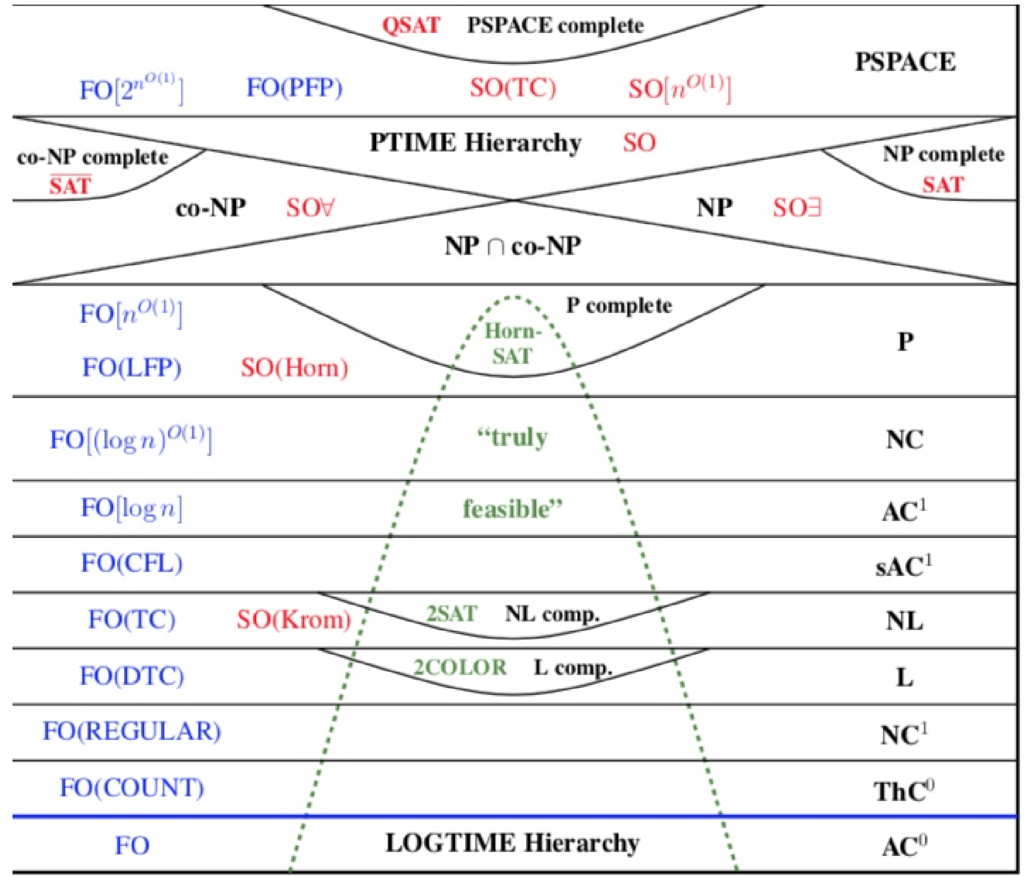
A logic $\mathcal{L}$ characterises the complexity class $\mathcal{C}$ iff for every property of finite structures $\mathcal{P}$:



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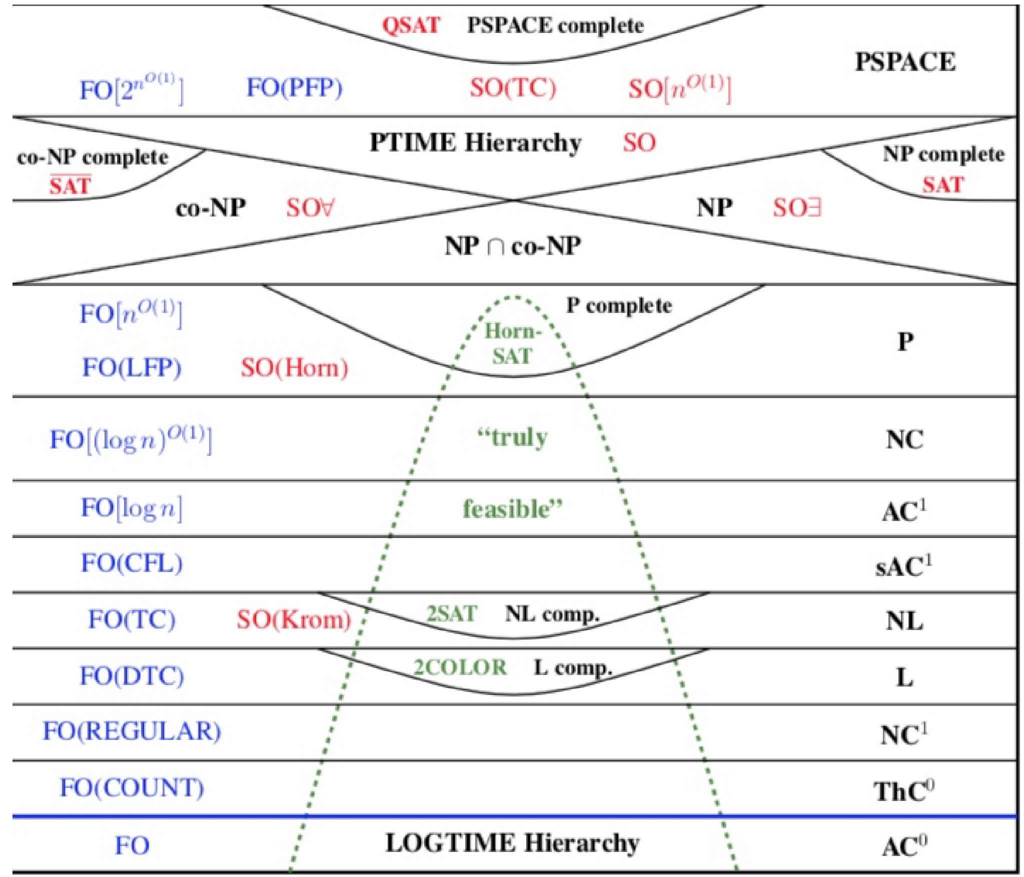

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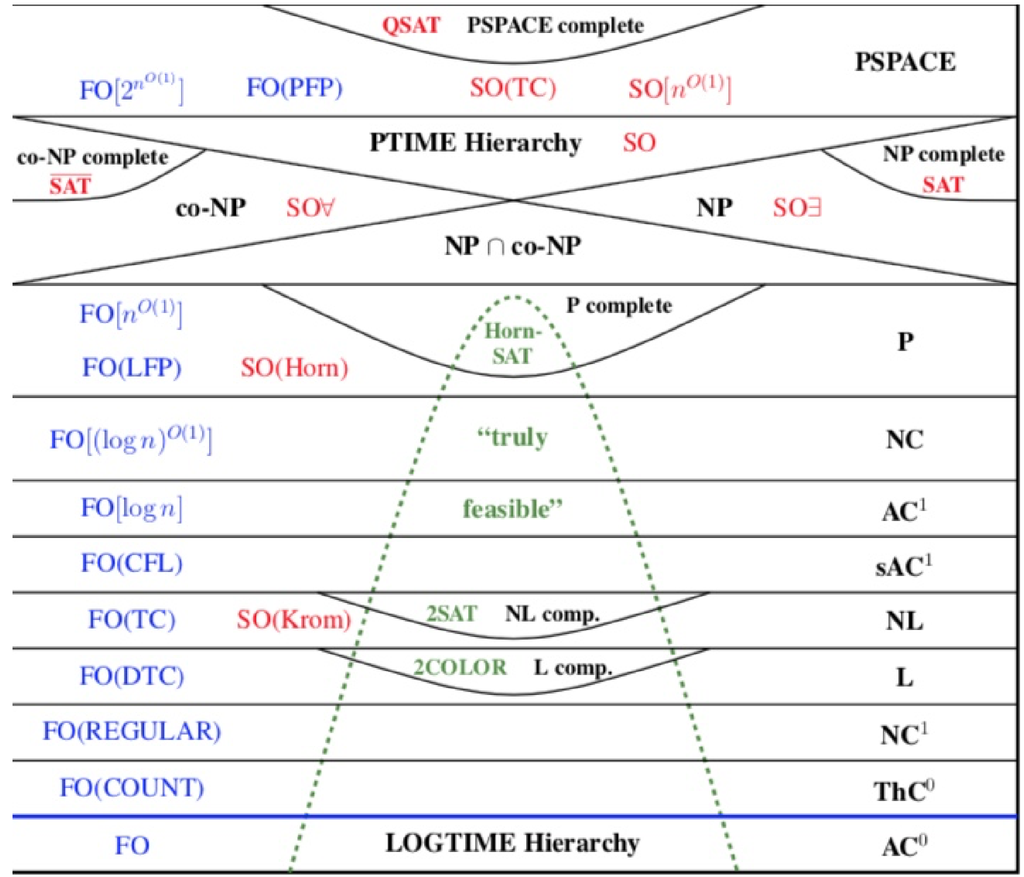

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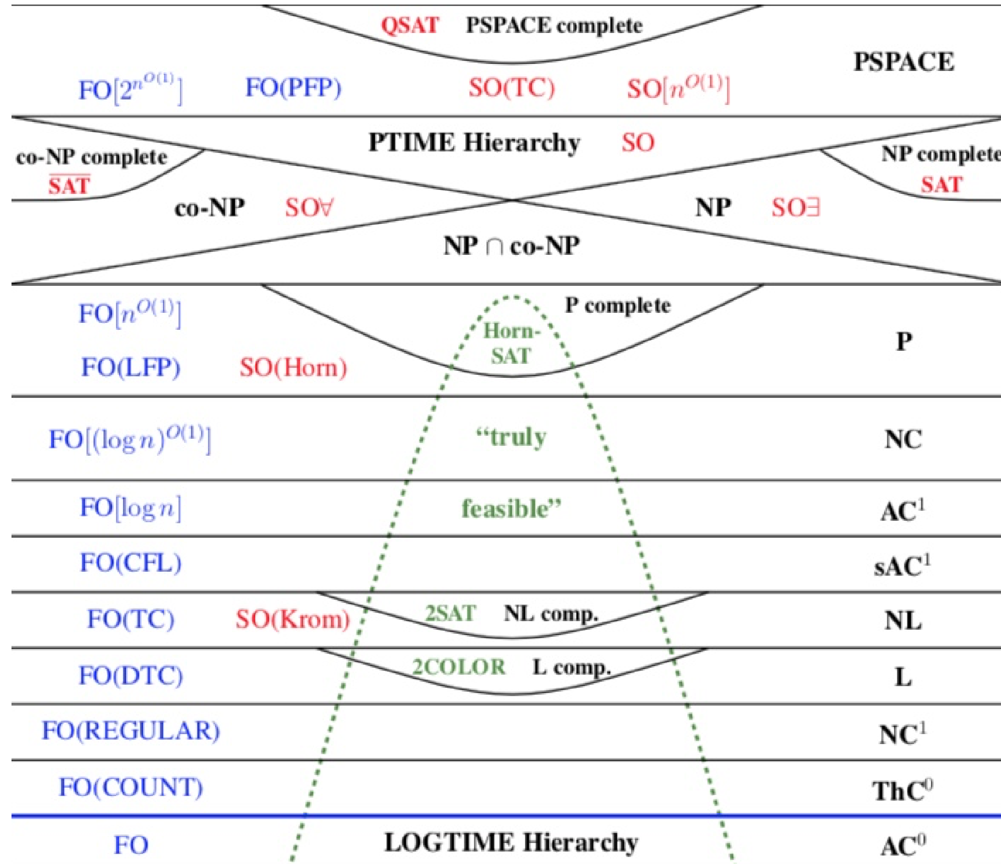

Theorem (Fagin'1973)

Existential Second Order Logic characterises NP.

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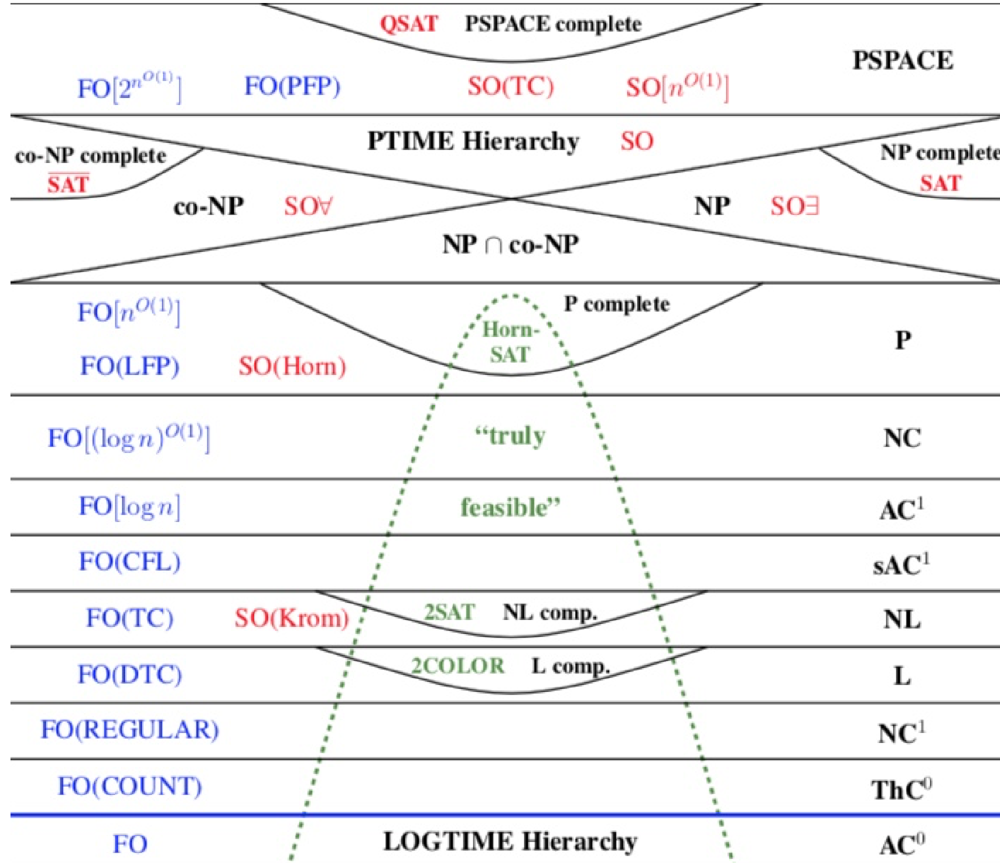

Theorem (Fagin'1973)

Existential Second Order Logic characterises NP.

Is there a logic for $\mathrm{PT{\scriptsize IME}}$?

No idea since 1988.

Motivations IV: why do we care about logic?

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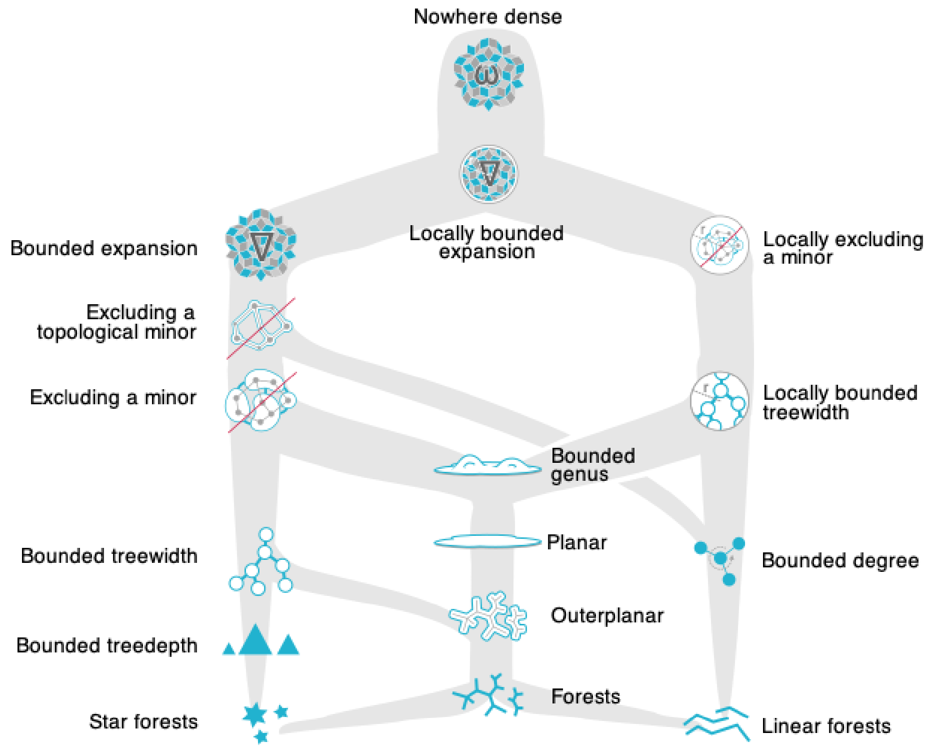

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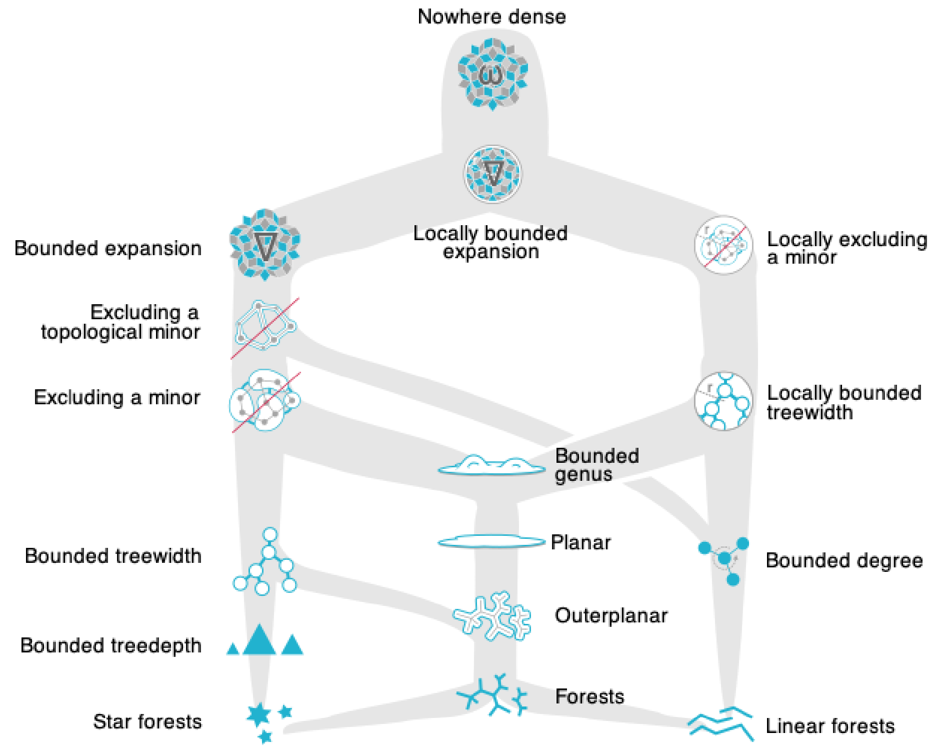
Theorem (Courcelle 1990)

$\mathcal{C}$:= graphs of bounded-treewidth.



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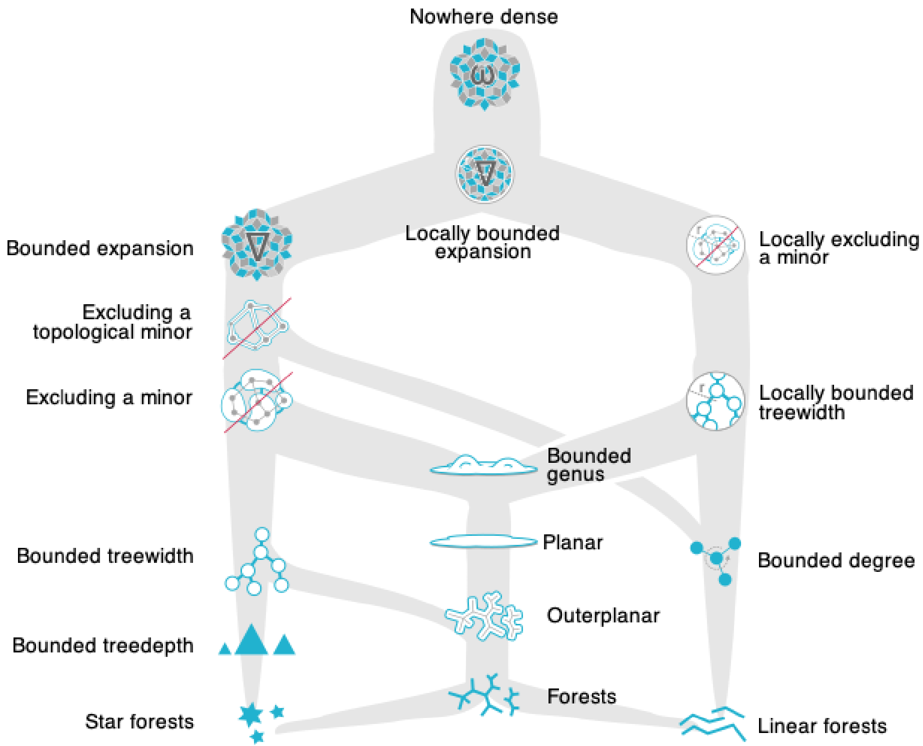

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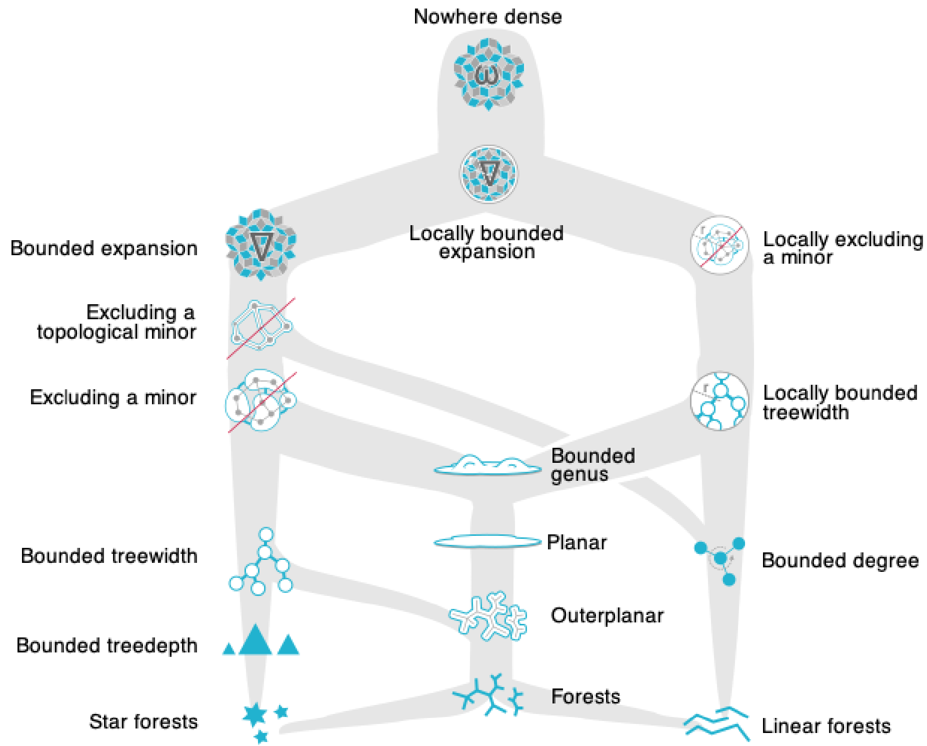

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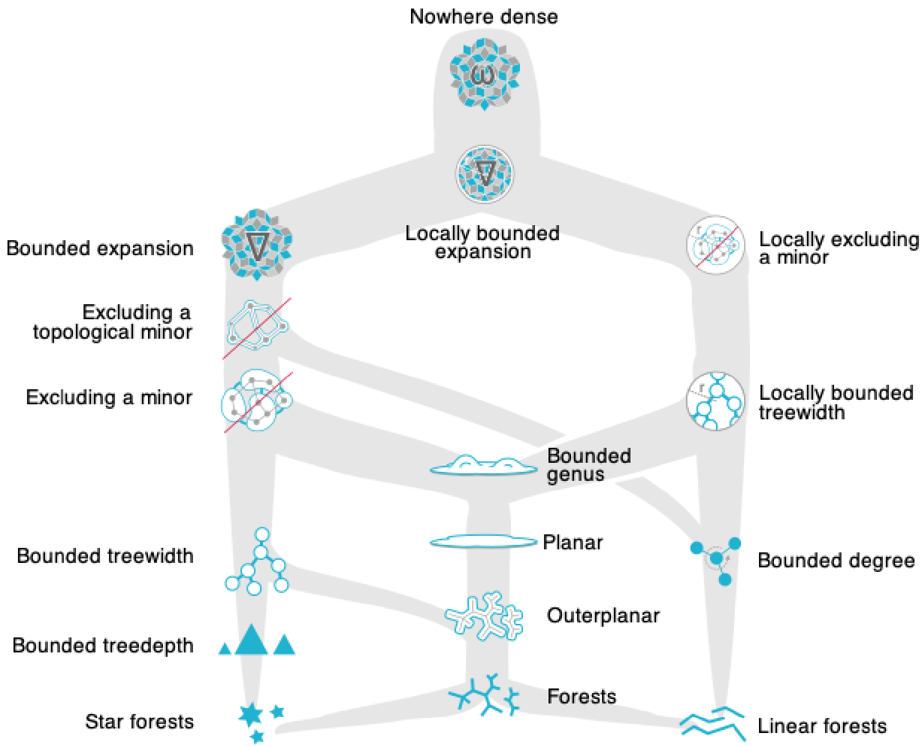

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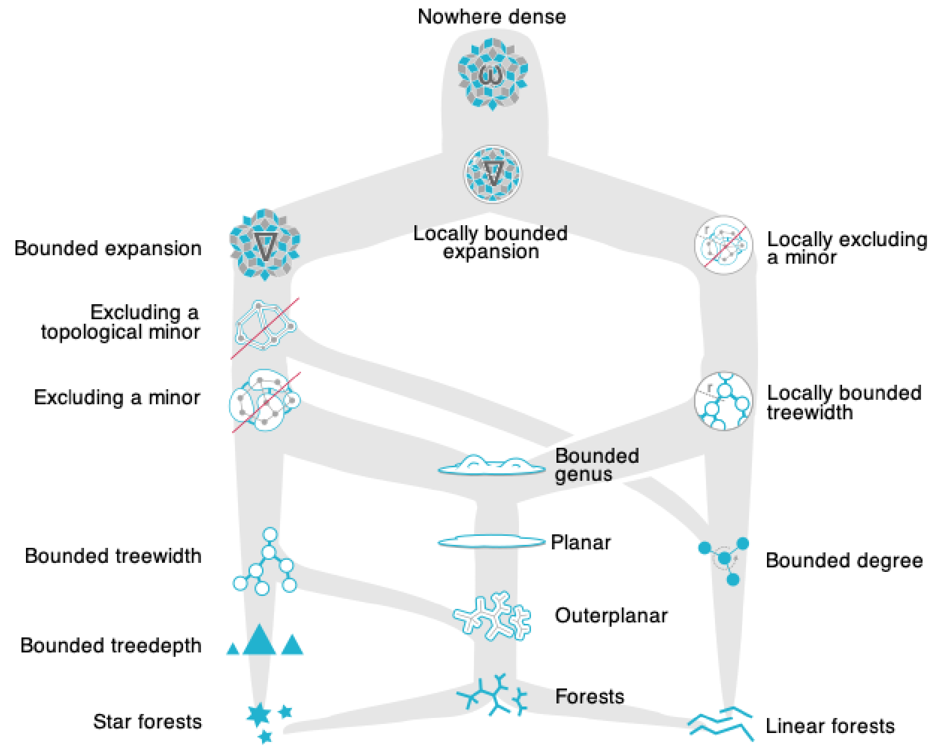
Theorem (Grohe, Kreutzer, Siebertz 2014)

$\mathcal{O}(|\varphi|^{1+\varepsilon})$ for $\mathcal{C}$:= nowhere-dense graphs.



4–5. Recap from BSc studies & Gödel's Completeness theorem

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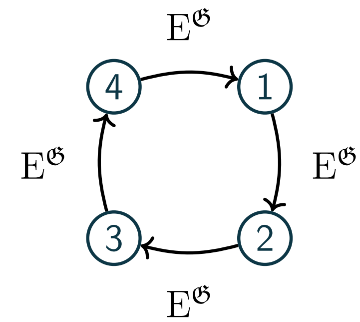

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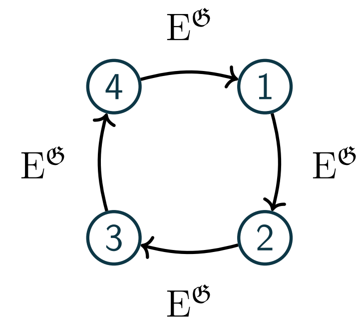

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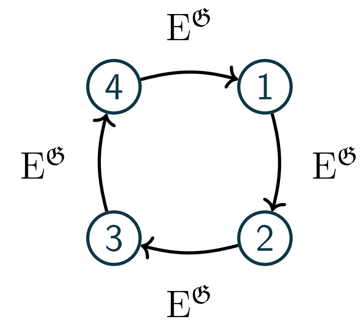

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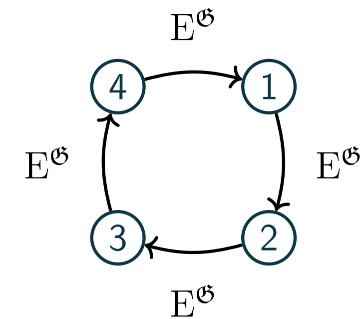

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Let $\mathfrak{A}, \mathfrak{B}$ be σ-structures. A homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is $\mathfrak{h} : A \to B$ satisfying:

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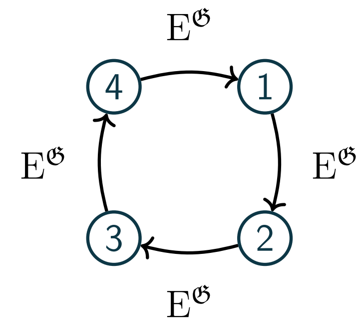

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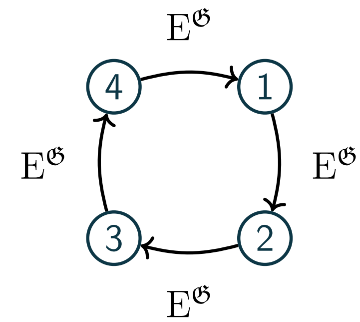

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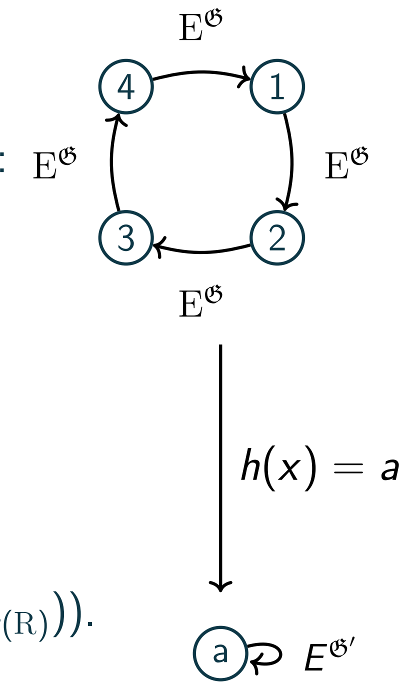

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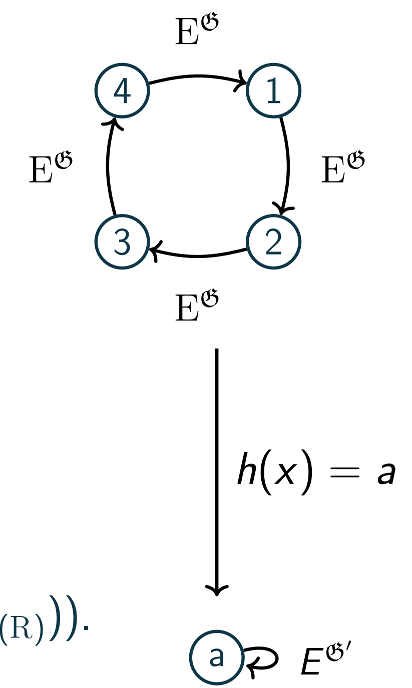

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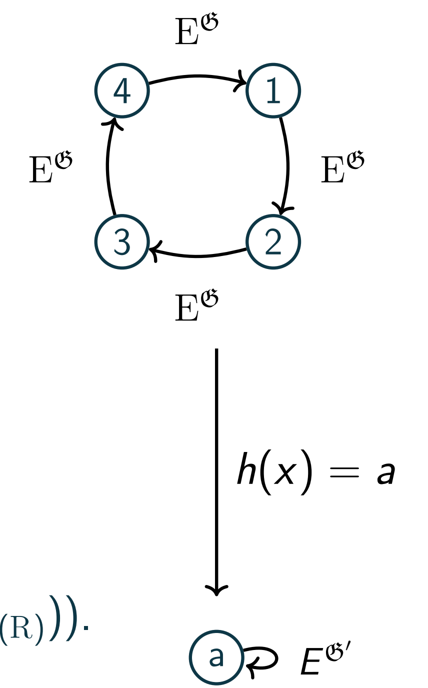

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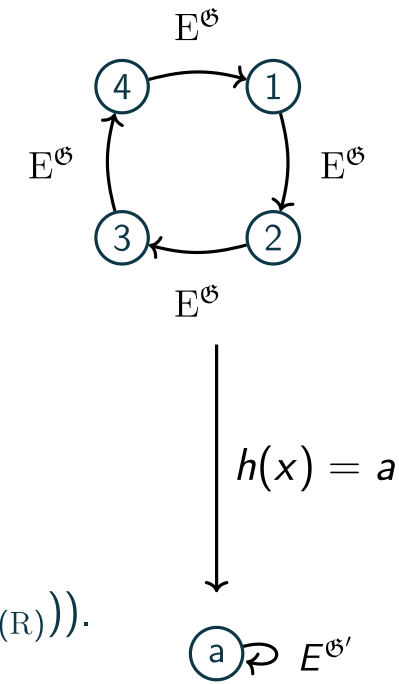

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SAT for FO is Recursively Enumerable

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4. The actual lecture

The G"odel's Compactness Theorem

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1. If $\mathcal{T} \models \varphi$ then there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ such that $\mathcal{T}_0 \models \varphi$.

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1. If $\mathcal{T} \models \varphi$ then there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ such that $\mathcal{T}_0 \models \varphi$.

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Showing

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"$\models \; = \; \vdash$"

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Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable.

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$\mathcal{T}$ unSAT iff $\mathcal{T} \models \bot$

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Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \bot$.

By (1) there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ s.t. $\mathcal{T}_0 \models \bot$.

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Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \bot$. By (1) there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ s.t. $\mathcal{T}_0 \models \bot$. A contradiction!

Employing compactness I: Reachability in $\{E\}$-structures

Employing compactness I: Reachability in $\{\mathrm{E}\}$-structures

The general proof scheme to show that the property $\mathcal{P}$ is not FO-definable.

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Ad absurdum suppose that φ defines $\mathcal{P}$.

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So there is no formula saying that between any two nodes there is a directed $\{E\}$-path.

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$$\varphi_0^{\mathsf{reach}(\mathsf{a},\mathsf{b})} := \mathsf{a} = \mathsf{b},\ \varphi_1^{\mathsf{reach}(\mathsf{a},\mathsf{b})} := \mathrm{E}(\mathsf{a},\mathsf{b}),\ \varphi_k^{\mathsf{reach}(\mathsf{a},\mathsf{b})} := \exists x_1 \ldots \exists x_{k-1}\ \mathrm{E}(\mathsf{a}, x_1) \wedge \wedge_{i=1}^{k-2} \mathrm{E}(x_i, x_{i+1}) \wedge \mathrm{E}(x_{k-1}, \mathsf{b})$$

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Assume that there is such φ, and let $\mathcal{T}$ be

$$\mathcal{T} := \{\varphi\} \cup \{\neg\varphi_k^{\mathsf{reach(a,b)}} \mid k \geq 0\}.$$

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Since a and b are disconnected, $\mathcal{T}$ is unSAT.

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Employing compactness I: Reachability in $\{\mathrm{E}\}$-structures

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Ad absurdum suppose that φ defines $\mathcal{P}$. $\rightsquigarrow$ Manufacture a theory $\mathcal{T}$ containing φ. $\rightsquigarrow$

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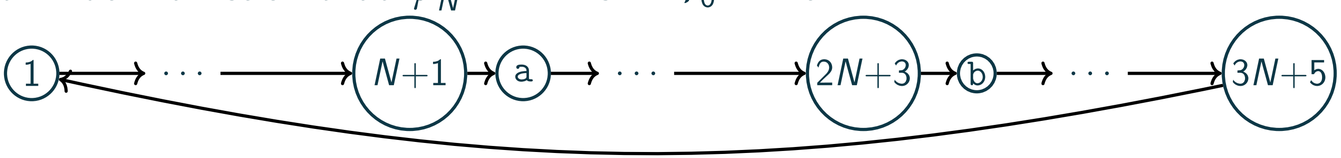

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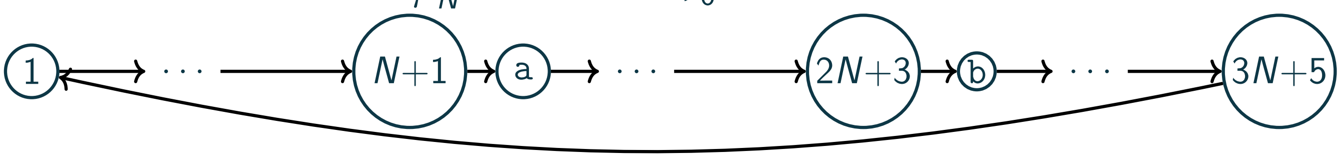

$\models \mathcal{T}_0$. A contradiction!

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No info about the finite models!

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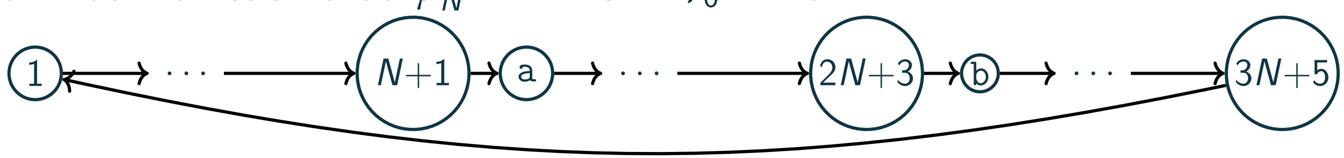

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Suppose that such a φ_{even} exists. Consider two theories $\mathcal{T}_1$ and $\mathcal{T}_2$:

Exploit ∞!

Let λ_k say "there are $\geq k$ elem.".

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