



TECHNISCHE  
UNIVERSITÄT  
DRESDEN

# DEDUCTION SYSTEMS

## Tableau Procedures II

Sebastian Rudolph

# Agenda

- Recap Tableau Calculus
- Tableau with  $\mathcal{ALC}$  TBoxes
- Tableau for  $\mathcal{ALC}$  Knowledge Bases
- Extension by Inverse Roles
- Extension by Functional Roles
- Model Construction with Unravelling
- Summary

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- tableau branch closed if  $G$  contains an atomic contradiction (aka **clash**)
- tableau construction successful if no rules applicable and no contradiction
- $C$  is satisfiable iff there is a successful tableau construction

## Tableau Rules for $\mathcal{ALC}$ Concepts

- $\sqcap$ -rule: For an  $v \in V$  with  $C \sqcap D \in L(v)$  and  $\{C, D\} \not\subseteq L(v)$ , let  $L(v) := L(v) \cup \{C, D\}$ .
- $\sqcup$ -rule: For an  $v \in V$  with  $C \sqcup D \in L(v)$  and  $\{C, D\} \cap L(v) = \emptyset$ , choose  $X \in \{C, D\}$  and let  $L(v) := L(v) \cup \{X\}$ .
- $\exists$ -rule: For an  $v \in V$  with  $\exists r.C \in L(v)$  such that there is no  $r$ -successor  $v'$  of  $v$  with  $C \in L(v')$ , let  $V = V \cup \{v'\}$ ,  $E = E \cup \{(v, v')\}$ ,  $L(v') := \{C\}$  and  $L(v, v') := \{r\}$  for  $v'$  a new node.
- $\forall$ -rule: For  $v, v' \in V$ ,  $v'$   $r$ -neighbor of  $v$ ,  $\forall r.C \in L(v)$  and  $C \notin L(v')$ , let  $L(v') := L(v') \cup \{C\}$ .

# Tableau Algorithm Example

$$C = \exists r.(A \sqcup \exists r.B) \sqcap \exists r.\neg A \sqcap \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))$$

$u$

$$L(u) = \{C\}$$

## Tableau Algorithm Example

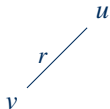
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*u*

$$L(u) = \{C, \exists r.(A \sqcup \exists r.B), \\ \exists r.\neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))\}$$

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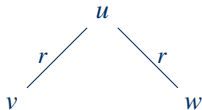


$$L(u) = \{C, \exists r.(A \sqcup \exists r.B), \\ \exists r.\neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))\}$$

$$L(v) = \{A \sqcup \exists r.B\}$$

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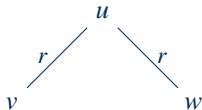
$$L(v) = \{A \sqcup \exists r.B\}$$

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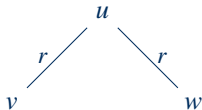
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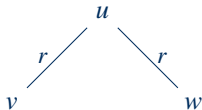
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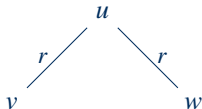
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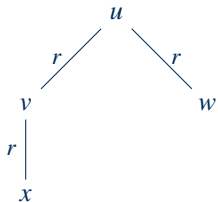
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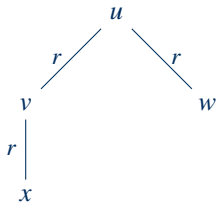
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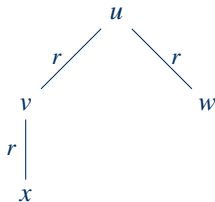
$$L(v) = \{A \sqcup \exists r.B, \neg A, \forall r.(\neg B \sqcup A), \cancel{\exists r.B}\}$$

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$$L(x) = \{B\}$$

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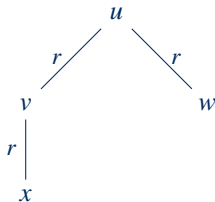
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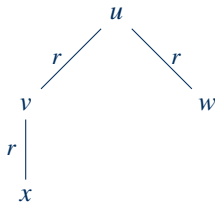
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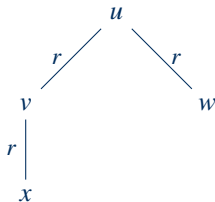
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## Tableau Algorithm Example

the model  $\mathcal{I}$  constructed by the algorithm is the following:

$$\Delta^{\mathcal{I}} = \{u, v, w, x\}$$

$$A^{\mathcal{I}} = \{x\}$$

$$B^{\mathcal{I}} = \{x\}$$

$$r^{\mathcal{I}} = \{\langle u, v \rangle, \langle u, w \rangle, \langle v, x \rangle\}$$

Check that indeed  $C^{\mathcal{I}} = \{u\}$ , given the defined semantics of  $\mathcal{ALC}$

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## Tableau Algorithm for TBoxes

We extend the tableau algorithm to capture  $\mathcal{ALC}$  TBoxes

- a TBox contains axioms (GCIs) of the form  $C \sqsubseteq D$
- assumption: occurrences of  $C \equiv D$  have been replaced by  $C \sqsubseteq D$  and  $D \sqsubseteq C$
- every GCI is equivalent to  $\top \sqsubseteq \neg C \sqcup D$

We can compress the whole TBox into one axiom (we say we “internalize” it):

$$\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$$

is equivalent to:

$$\mathcal{T}' = \{\top \sqsubseteq \prod_{1 \leq i \leq n} \neg C_i \sqcup D_i\}$$

Let  $C_{\mathcal{T}}$  be the concept on the rhs of the GCI in NNF.

## Tableau Algorithm for TBoxes

We extend the rules of the  $\mathcal{ALC}$  tableau algorithm with the rule:

$\mathcal{T}$  rule: For an arbitrary  $v \in V$  with  $C_{\mathcal{T}} \notin L(v)$ ,  
let  $L(v) := L(v) \cup \{C_{\mathcal{T}}\}$ .

**Example:** Let  $\mathcal{T} = A \sqsubseteq \exists r.A$ . Is  $A$  satisfiable given  $\mathcal{T}$ ?

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**solution:** we will recognize cycles (that is, repeating node labellings)

# Tableau Algorithm for TBoxes

## Definition (Blocking)

A node  $v \in V$  **blocks** a node  $v' \in V$  **directly**, if:

- 1  $v'$  is reachable from  $v$ ,
- 2  $L(v') \subseteq L(v)$ ; and
- 3 there is no directly blocking node  $v''$  such that  $v'$  is reachable from  $v''$ .

A node  $v' \in V$  is **blocked** if either

- 1  $v'$  is blocked directly or
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The application of the  $\exists$  rule is restricted to nodes that are **not blocked**.

## Tableau Algorithm with Blocking

**Example:** Let  $\mathcal{T} = A \sqsubseteq \exists r.A$ . Is  $A$  satisfiable w.r.t.  $\mathcal{T}$ ?

we obtain the following contradiction-free tableau:



$$L(v_0) = \{A, C_{\mathcal{T}}, \exists r.A\}$$

$$L(v_1) = \{A, C_{\mathcal{T}}, \exists r.A\}$$

wherein  $v_1$  is **directly** blocked by  $v_0$

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again, the algorithm constructs finite trees

- from a contradiction-free tableau, we can construct a model
- if there is no contradiction-free tableau, there is no model

## From the Tableau to the Model

again, we can construct a finite model from a contradiction-free tableau:

$$\Delta^{\mathcal{I}} = \{v_0\}$$

$$A^{\mathcal{I}} = \Delta^{\mathcal{I}}$$

$$r^{\mathcal{I}} = \{\langle v_0, v_0 \rangle\}$$

- blocked nodes do not represent elements of the model
- when constructing the model, an edge from a node  $v$  to a directly blocked node  $v'$  will be “translated” into an “edge” from  $v$  to the node, that directly blocks  $v'$

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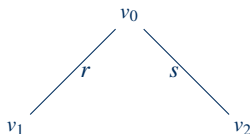
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- ↪ we have the **finite model property**
- ↪ constructed model is not necessarily tree-shaped

## Tableau Algorithm with Blocking II

**Example:** Let  $\mathcal{T} = A \sqsubseteq \exists r.A \sqcap \exists s.B$ . Is  $A$  satisfiable w.r.t.  $\mathcal{T}$ ?

We obtain the following contradiction-free tableau:



$$L(v_0) = \{A, C_{\mathcal{T}}, \exists r.A \sqcap \exists s.B, \exists r.A, \exists s.B\}$$

$$L(v_1) = \{A, C_{\mathcal{T}}, \exists r.A \sqcap \exists s.B, \exists r.A, \exists s.B\}$$

$$L(v_2) = \{B, C_{\mathcal{T}}, \neg A\}$$

in which  $v_1$  is again **directly** blocked by  $v_0$



## From the Tableau to a Model II

again, we can construct a finite model from a contradiction-free tableau:

$$\Delta^{\mathcal{I}} = \{v_0, v_2\}$$

$$A^{\mathcal{I}} = \{v_0\}$$

$$B^{\mathcal{I}} = \{v_2\}$$

$$r^{\mathcal{I}} = \{\langle v_0, v_0 \rangle\}$$

$$s^{\mathcal{I}} = \{\langle v_0, v_2 \rangle\}$$

## Tableau Algorithm Example

**Example:** Let  $\mathcal{T} = \{A \sqsubseteq B \sqcap \exists r.C, B \equiv C \sqcup D, C \sqsubseteq \exists r.D\}$ . Is  $A$  satisfiable w.r.t.  $\mathcal{T}$ ?

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Normalization I:

$\mathcal{T}' = \{A \sqsubseteq B, A \sqsubseteq \exists r.C, B \sqsubseteq C \sqcup D, C \sqcup D \sqsubseteq B, C \sqsubseteq \exists r.D\}$

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Normalization II:

$$\mathcal{T}'' = \{A \sqsubseteq B, A \sqsubseteq \exists r.C, B \sqsubseteq C \sqcup D, C \sqsubseteq B, D \sqsubseteq B, C \sqsubseteq \exists r.D\}$$

## Tableau Algorithm Example

**Example:** Let  $\mathcal{T} = \{A \sqsubseteq B \sqcap \exists r.C, B \equiv C \sqcup D, C \sqsubseteq \exists r.D\}$ . Is  $A$  satisfiable w.r.t.  $\mathcal{T}$ ?

Normalization I:

$$\mathcal{T}' = \{A \sqsubseteq B, A \sqsubseteq \exists r.C, B \sqsubseteq C \sqcup D, C \sqcup D \sqsubseteq B, C \sqsubseteq \exists r.D\}$$

Normalization II:

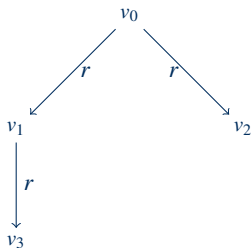
$$\mathcal{T}' = \{A \sqsubseteq B, A \sqsubseteq \exists r.C, B \sqsubseteq C \sqcup D, C \sqsubseteq B, D \sqsubseteq B, C \sqsubseteq \exists r.D\}$$

$$C_{\mathcal{T}} = (\neg A \sqcup B) \sqcap (\neg A \sqcup \exists r.C) \sqcap (\neg B \sqcup C \sqcup D) \sqcap (\neg C \sqcup B) \sqcap (\neg D \sqcup B) \sqcap (\neg C \sqcup \exists r.D)$$

## Tableau Algorithm Example

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we obtain the following contradiction-free tableau:



$$L(v_0) = \{A, C_{\mathcal{T}}, \dots, B, \exists r.C, C, \neg D, \exists r.D\}$$

$$L(v_1) = \{C, C_{\mathcal{T}}, \dots, \neg A, B, \exists r.D\}$$

$$L(v_2) = \{D, C_{\mathcal{T}}, \dots, \neg A, \neg C, B\}$$

$$L(v_3) = \{D, C_{\mathcal{T}}, \dots, \neg A, \neg C, B\}$$

# Agenda

- Recap Tableau Calculus
- Tableau with  $\mathcal{ALC}$  TBoxes
- Tableau for  $\mathcal{ALC}$  Knowledge Bases
- Extension by Inverse Roles
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- Summary

## Treatment of ABoxes

to take an ABox  $\mathcal{A}$  into account, initialize  $G$  such that

- $V$  contains a node  $v_a$  for each individual  $a$  occurring in  $\mathcal{A}$



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the tableau rules can then be applied to this initialized graph

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- 3 replace the term “ $r$ -successor” in the  $\forall$ - and the  $\exists$ -rule with “ $r$ -neighbor”

the  $\exists$ -rule still generates

- an  $r$ -**successor** for a concept  $\exists r.C$  (if no fitting neighbor exists yet)
- an  $r^-$ -**successor** for a concept  $\exists r^-.C$  (if no fitting neighbor exists yet)

# Tableau Example with Inverses

Example: is  $A$  satisfiable w.r.t.  $\mathcal{T}$ ?

$$\mathcal{T} = \{A \equiv \exists r^{-}.A \sqcap (\forall r.(\neg A \sqcup \exists s.B))\}$$



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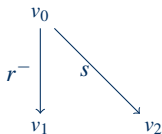
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$$L(v_0) = \{A, C_{\mathcal{T}}, \exists r^- . A, \forall r. (\neg A \sqcup \exists s. B),$$

$$\neg A \sqcup \exists s. B, \exists s. B\}$$

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$v_0$  blocks  $v_1$

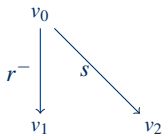
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Is the algorithm thus correct?

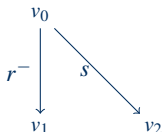
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Is the algorithm thus correct? **No!**

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Example: Is  $C \sqcap \exists s.C$  satisfiable w.r.t.  $\mathcal{T}$ ?

$$\mathcal{T} = \{T \sqsubseteq \forall r^-. (\forall s^-. (\neg C)) \sqcap \exists r.C\}$$

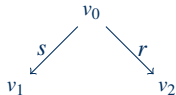
## Tableau Example with Inverses II

Example: Is  $C \sqcap \exists s.C$  satisfiable w.r.t.  $\mathcal{T}$ ?

$$\mathcal{T} = \{ \top \sqsubseteq \forall r^- . (\forall s^- . (\neg C)) \sqcap \exists r . C \}$$
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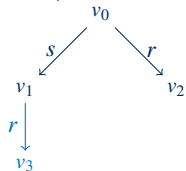
$$L(v_1) = \{ C, C_{\mathcal{T}}, \forall r^- . (\forall s^- . (\neg C)), \exists r . C \}$$

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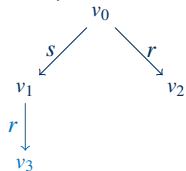
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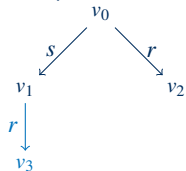
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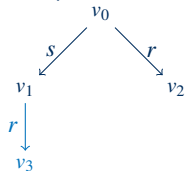
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correctness can be retained by replacing subset blocking with equality blocking  
i.e., replace  $L(v') \subseteq L(v)$  by  $L(v') = L(v)$  in the blocking condition

# Model Construction for Tableau Example with Inverses II

We cannot build a cyclic model as we could up to now !

Example: Is  $C \sqcap \exists s.C$  satisfiable w.r.t.  $\mathcal{T}$ ?



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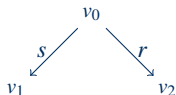
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## Example with Inverses & Equality Blocking

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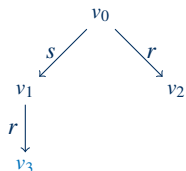
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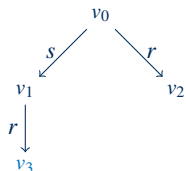
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$v_1$  blocks  $v_3$  but  $\forall$ -rule applicable



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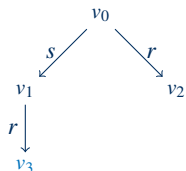
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~~$v_1$  blocks  $v_3$  but  $\forall$  rule applicable~~

Now unsatisfiability is recognized!

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Example: is  $A$  satisfiable w.r.t.  $\mathcal{T}$ ?

Note:  $\top \sqsubseteq \leq 1f$  is equivalent to  $\text{Func}(f)$

$$\mathcal{T} = \{A \sqsubseteq \exists f.B \sqcap \exists f.(\neg B), \top \sqsubseteq \leq 1f\}$$

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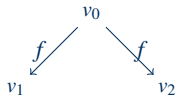
Note:  $\top \sqsubseteq \leq 1f$  is equivalent to  $\text{Func}(f)$

$$\begin{aligned}\mathcal{T} &= \{A \sqsubseteq \exists f.B \sqcap \exists f.(\neg B), \top \sqsubseteq \leq 1f\} \\ C_{\mathcal{T}} &= (\neg A \sqcup (\exists f.B \sqcap \exists f.(\neg B))) \sqcap \leq 1f\end{aligned}$$

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Example: is  $A$  satisfiable w.r.t.  $\mathcal{T}$ ?

Note:  $\top \sqsubseteq \leq 1f$  is equivalent to  $\text{Func}(f)$



$$\mathcal{T} = \{A \sqsubseteq \exists f.B \sqcap \exists f.(\neg B), \top \sqsubseteq \leq 1f\}$$

$$C_{\mathcal{T}} = (\neg A \sqcup (\exists f.B \sqcap \exists f.(\neg B))) \sqcap \leq 1f$$

$$L(v_0) = \{A, C_{\mathcal{T}}, \dots, \exists f.B, \exists f.(\neg B), \leq 1f\}$$

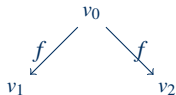
$$L(v_1) = \{B, C_{\mathcal{T}}, \dots, \neg A, \leq 1f\}$$

$$L(v_2) = \{\neg B, C_{\mathcal{T}}, \dots, \neg A, \leq 1f\}$$

## Tableau with Functional Roles

Example: is  $A$  satisfiable w.r.t.  $\mathcal{T}$ ?

Note:  $\top \sqsubseteq \leq 1f$  is equivalent to  $\text{Func}(f)$



$$\mathcal{T} = \{A \sqsubseteq \exists f.B \sqcap \exists f.(\neg B), \top \sqsubseteq \leq 1f\}$$

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$$L(v_1) = \{B, C_{\mathcal{T}}, \dots, \neg A, \leq 1f\}$$

$$L(v_2) = \{\neg B, C_{\mathcal{T}}, \dots, \neg A, \leq 1f\}$$

functionality requires  $v_1 = v_2$ !

$\rightsquigarrow$  we need a new tableau rule for treating functional roles

# Tableau Rules for $\mathcal{ALCIF}$ Concepts and TBoxes

- $\sqcap$ -rule: For an  $v \in V$  with  $C \sqcap D \in L(v)$  and  $\{C, D\} \not\subseteq L(v)$ , let  $L(v) := L(v) \cup \{C, D\}$ .
- $\sqcup$ -rule: For an  $v \in V$  with  $C \sqcup D \in L(v)$  and  $\{C, D\} \cap L(v) = \emptyset$ , choose  $X \in \{C, D\}$  and let  $L(v) := L(v) \cup \{X\}$ .
- $\exists$ -rule: For a non-blocked  $v \in V$  with  $\exists r.C \in L(v)$  such that there is no  $r$ -neighbor  $v'$  of  $v$  with  $C \in L(v')$ , let  $V = V \cup \{v'\}$ ,  $E = E \cup \{(v, v')\}$ ,  $L(v') := \{C\}$  and  $L(v, v') := \{r\}$  for  $v'$  a new node.
- $\forall$ -rule: For  $v, v' \in V$ ,  $v'$   $r$ -neighbor of  $v$ ,  $\forall r.C \in L(v)$  and  $C \notin L(v')$ , let  $L(v') := L(v') \cup \{C\}$ .
- $\leq 1$ -rule: For a functional role  $f$  and a  $v \in V$  with two  $f$ -neighbors  $v_1$  and  $v_2$ , execute  $\text{merge}(v_1, v_2)$ .
- $\mathcal{T}$ -rule: For a  $v \in V$  with  $C_{\mathcal{T}} \notin L(v)$ , let  $L(v) := L(v) \cup \{C_{\mathcal{T}}\}$ .



# Merging Nodes

we define  $\text{merge}(v_1, v_2)$  as follows:

- if  $v_1$  is an ancestor of  $v_2$ ,  
let  $v_i = v_1$  and  $v_o = v_2$ ;
- otherwise let  $v_i = v_2$  and  $v_o = v_1$ .

let  $L(v_i) = L(v_i) \cup L(v_o)$  and execute  $\text{prune}(v_o)$ .

where  $\text{prune}(v_o)$  is defined as:

- $V_o = \{v \mid v \text{ belongs to the subtree with root } v_o\}$ ,
- let  $V = V \setminus V_o$  and  $E = E \setminus \{\langle v, v_o \rangle \mid v_o \in V_o, \langle v, v_o \rangle \in E\}$ .

# Tableau with Functional Roles

Example: Is  $\exists f.A$  satisfiable w.r.t.  $\mathcal{T}$ ?

$$\mathcal{T} = \{A \sqsubseteq \exists f.A, \top \sqsubseteq \leq 1f^-\}$$

## Tableau with Functional Roles

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# Tableau with Functional Roles

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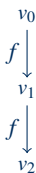
$$L(v_0) = \{\exists f.A, C_{\mathcal{T}}, \neg A, \leq 1f^-\}$$

$$L(v_1) = \{A, C_{\mathcal{T}}, \exists f.A, \leq 1f^-\}$$

$$L(v_2) = \{A, C_{\mathcal{T}}, \exists f.A, \leq 1f^-\}$$

# Tableau with Functional Roles

Example: Is  $\exists f.A$  satisfiable w.r.t.  $\mathcal{T}$ ?



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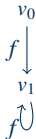
$$C_{\mathcal{T}} = (\neg A \sqcup \exists f.A) \sqcap \leq 1f^-$$

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$$L(v_2) = \{A, C_{\mathcal{T}}, \exists f.A, \leq 1f^-\}$$

$v_1$  blocks  $v_2$ , but cyclic model construction does not work (functionality violated)!



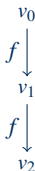
# Agenda

- Recap Tableau Calculus
- Tableau with  $\mathcal{ALC}$  TBoxes
- Tableau for  $\mathcal{ALC}$  Knowledge Bases
- Extension by Inverse Roles
- Extension by Functional Roles
- Model Construction with Unravelling
- Summary

# Unravelling

goal: we build an infinite model

How? Every blocked node is replaced by a subtree whose root is the corresponding blocking node.



$$L(v_0) = \{\exists f.A, C_{\mathcal{T}}, \neg A, \leq 1f^{-}\}$$

$$L(v_1) = \{A, C_{\mathcal{T}}, \exists f.A, \leq 1f^{-}\}$$

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$v_1$  blocks  $v_2$

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$v_1$  blocks  $v_2$

## Blocking: Inverse and Functional Roles

Example: Is  $\neg C \sqcap \exists f^{-}.D$  satisfiable w.r.t.  $\mathcal{T}$ ?

$$\mathcal{T} = \{D \sqsubseteq C \sqcap \exists f.(\neg C) \sqcap \exists f^{-}.D, \top \sqsubseteq \leq 1f\}$$

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$$C_{\mathcal{T}} = (\neg D \sqcup (C \sqcap \exists f.(\neg C) \sqcap \exists f^{-}.D)) \sqcap \leq 1f$$

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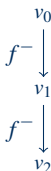
$$C_{\mathcal{T}} = (\neg D \sqcup (C \sqcap \exists f. (\neg C) \sqcap \exists f^- . D)) \sqcap \leq 1f$$

$$L(v_0) = \{\neg C, \exists f^- . D, C_{\mathcal{T}}, \dots, \neg D, \leq 1f\}$$

$$L(v_1) = \{D, C_{\mathcal{T}}, \dots, C, \exists f. (\neg C), \exists f^- . D, \leq 1f\}$$

$$L(v_2) = \{D, C_{\mathcal{T}}, \dots, C, \exists f. (\neg C), \exists f^- . D, \leq 1f\}$$

$v_1$  blocks  $v_2$  (same label)



## Blocking: Inverse and Functional Roles

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$$\mathcal{T} = \{D \sqsubseteq C \sqcap \exists f. (\neg C) \sqcap \exists f^- . D, \top \sqsubseteq \leq 1f\}$$

$$C_{\mathcal{T}} = (\neg D \sqcup (C \sqcap \exists f. (\neg C) \sqcap \exists f^- . D)) \sqcap \leq 1f$$



$$L(v_0) = \{\neg C, \exists f^- . D, C_{\mathcal{T}}, \dots, \neg D, \leq 1f\}$$

$$L(v_1) = \{D, C_{\mathcal{T}}, \dots, C, \exists f. (\neg C), \exists f^- . D, \leq 1f\}$$

$$L(v'_1) = \{D, C_{\mathcal{T}}, \dots, C, \exists f. (\neg C), \exists f^- . D, \leq 1f\}$$

$v_1$  blocks  $v_2$  (same label) but

$$L(v''_1) = \{D, C_{\mathcal{T}}, \dots, C, \exists f. (\neg C), \exists f^- . D, \leq 1f\}$$

but we cannot build a model any more (neither cyclic nor infinite)!

## Pairwise Blocking

A node  $x$  with predecessor  $x'$  blocks a node  $y$  with predecessor  $y'$  directly, if:

- 1  $y$  is reachable from  $x$ ,
- 2  $L(x) = L(y)$ ,  $L(x') = L(y')$  and  $L(x', x) = L(y', y)$ ; and
- 3 there is no directly blocked node  $z$  such that  $y$  is reachable from  $z$ .

A node  $y \in V$  is blocked if either

- 1  $y$  is directly blocked or
- 2 there is a directly blocked node  $x$ , such that  $y$  can be reached from  $x$ .

# Pairwise Blocking: Inverses and Functional Roles

Example: Is  $\neg C \sqcap \exists f^{-}.D$  satisfiable w.r.t.  $\mathcal{T}$ ?

$$\mathcal{T} = \{D \sqsubseteq C \sqcap \exists f.(\neg C) \sqcap \exists f^{-}.D, \top \sqsubseteq \leq 1f\}$$

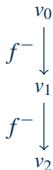
$$C_{\mathcal{T}} = (\neg D \sqcup (C \sqcap \exists f.(\neg C) \sqcap \exists f^{-}.D)) \sqcap \leq 1f$$

$$L(v_0) = \{\neg C, \exists f^{-}.D, C_{\mathcal{T}}, \dots, \neg D, \leq 1f\}$$

$$L(v_1) = \{D, C_{\mathcal{T}}, \dots, C, \exists f.(\neg C), \exists f^{-}.D, \leq 1f\}$$

$$L(v_2) = \{D, C_{\mathcal{T}}, \dots, C, \exists f.(\neg C), \exists f^{-}.D, \leq 1f\}$$

$v_1$  cannot block  $v_2$  pairwise





# Pairwise Blocking: Inverses and Functional Roles

Example: Is  $\neg C \sqcap \exists f^- . D$  satisfiable w.r.t.  $\mathcal{T}$ ?

$$\mathcal{T} = \{D \sqsubseteq C \sqcap \exists f^- . (\neg C) \sqcap \exists f^- . D, \top \sqsubseteq \leq 1f\}$$

$$C_{\mathcal{T}} = (\neg D \sqcup (C \sqcap \exists f^- . (\neg C) \sqcap \exists f^- . D)) \sqcap \leq 1f$$



$$L(v_0) = \{\neg C, \exists f^- . D, C_{\mathcal{T}}, \dots, \neg D, \leq 1f\}$$

$$L(v_1) = \{D, C_{\mathcal{T}}, \dots, C, \exists f^- . (\neg C), \exists f^- . D, \leq 1f\}$$

$$L(v_2) = \{D, C_{\mathcal{T}}, \dots, C, \exists f^- . (\neg C), \exists f^- . D, \leq 1f\}$$

$v_1$  cannot block  $v_2$  pairwise

$$L(v_3) = \{\neg C, C_{\mathcal{T}}, \dots, \neg D, \leq 1f\}$$

# Pairwise Blocking: Inverses and Functional Roles

Example: Is  $\neg C \sqcap \exists f^- . D$  satisfiable w.r.t.  $\mathcal{T}$ ?

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# Pairwise Blocking: Inverses and Functional Roles

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$$C_{\mathcal{T}} = (\neg D \sqcup (C \sqcap \exists f. (\neg C) \sqcap \exists f^- . D)) \sqcap \leq 1f$$



$$L(v_0) = \{\neg C, \exists f^- . D, C_{\mathcal{T}}, \dots, \neg D, \leq 1f\}$$

$$L(v_1) = \{D, C_{\mathcal{T}}, \dots, C, \exists f. (\neg C), \exists f^- . D, \leq 1f\}$$

$$L(v_2) = \{D, C_{\mathcal{T}}, \dots, C, \exists f. (\neg C), \exists f^- . D, \leq 1f\}$$

$v_1$  cannot block  $v_2$  pairwise

$$L(v_3) = \{\neg C, C_{\mathcal{T}}, \dots, \neg D, \leq 1f\}$$

$v_3$  is merged into  $v_1$

$$L(v_1) = L(v_1) \cup L(v_3) \supseteq \{\neg D, D\}$$

now the contradiction can be detected

# Agenda

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## Summary

- we now have a tableau algorithm for  $\mathcal{ALCIF}$  knowledge bases
  - treat the ABox like for  $\mathcal{ALC}$
  - number restrictions can be handled similar to functional roles
- termination through cycle detection
  - becomes harder the more expressive the logic gets