Computational Logic ∴ Group

Finite Model Theory of the Triguarded Fragment and Related Logics

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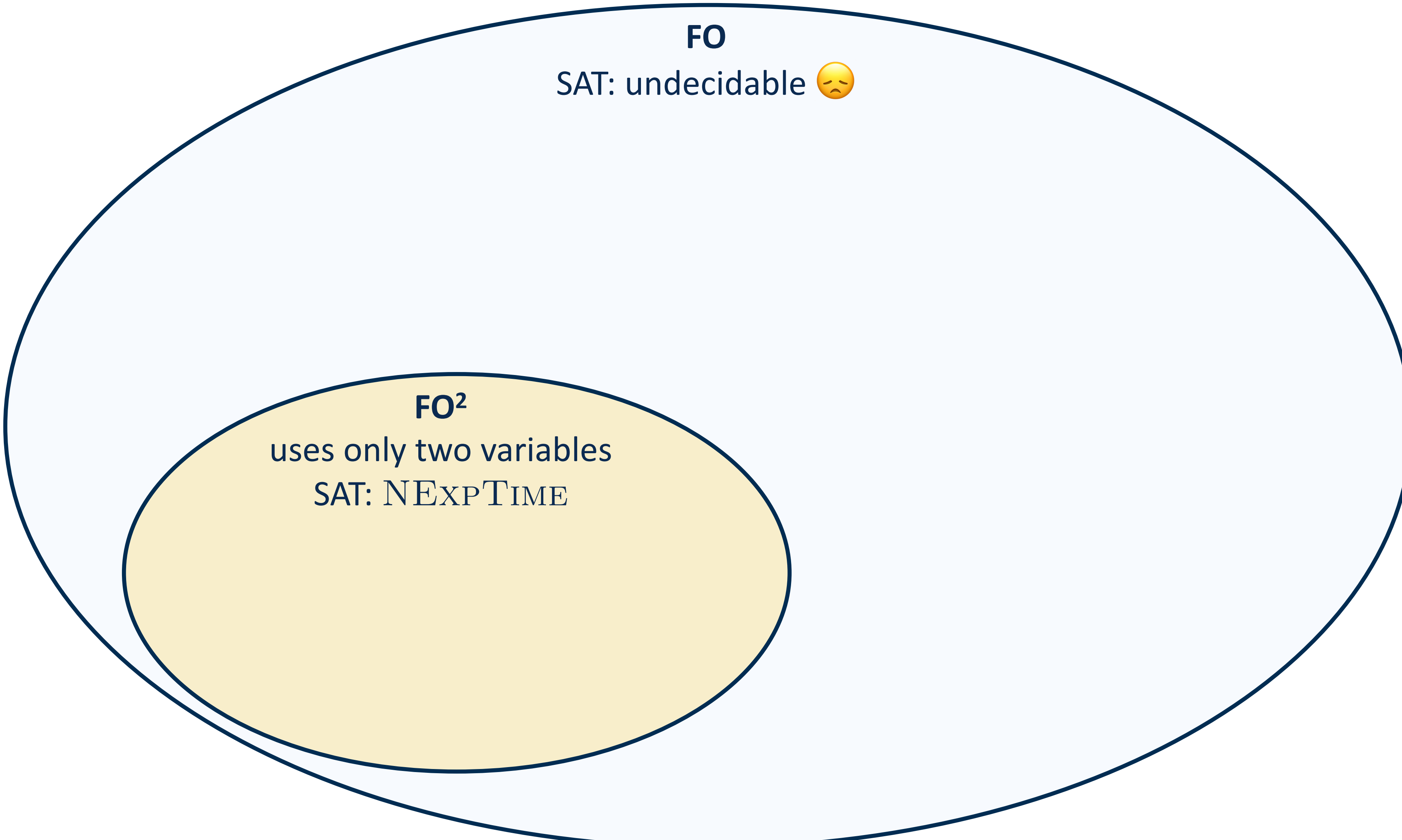
The Triguarded Fragment – Motivation

FO

SAT: undecidable 😞

The Triguarded Fragment – Motivation



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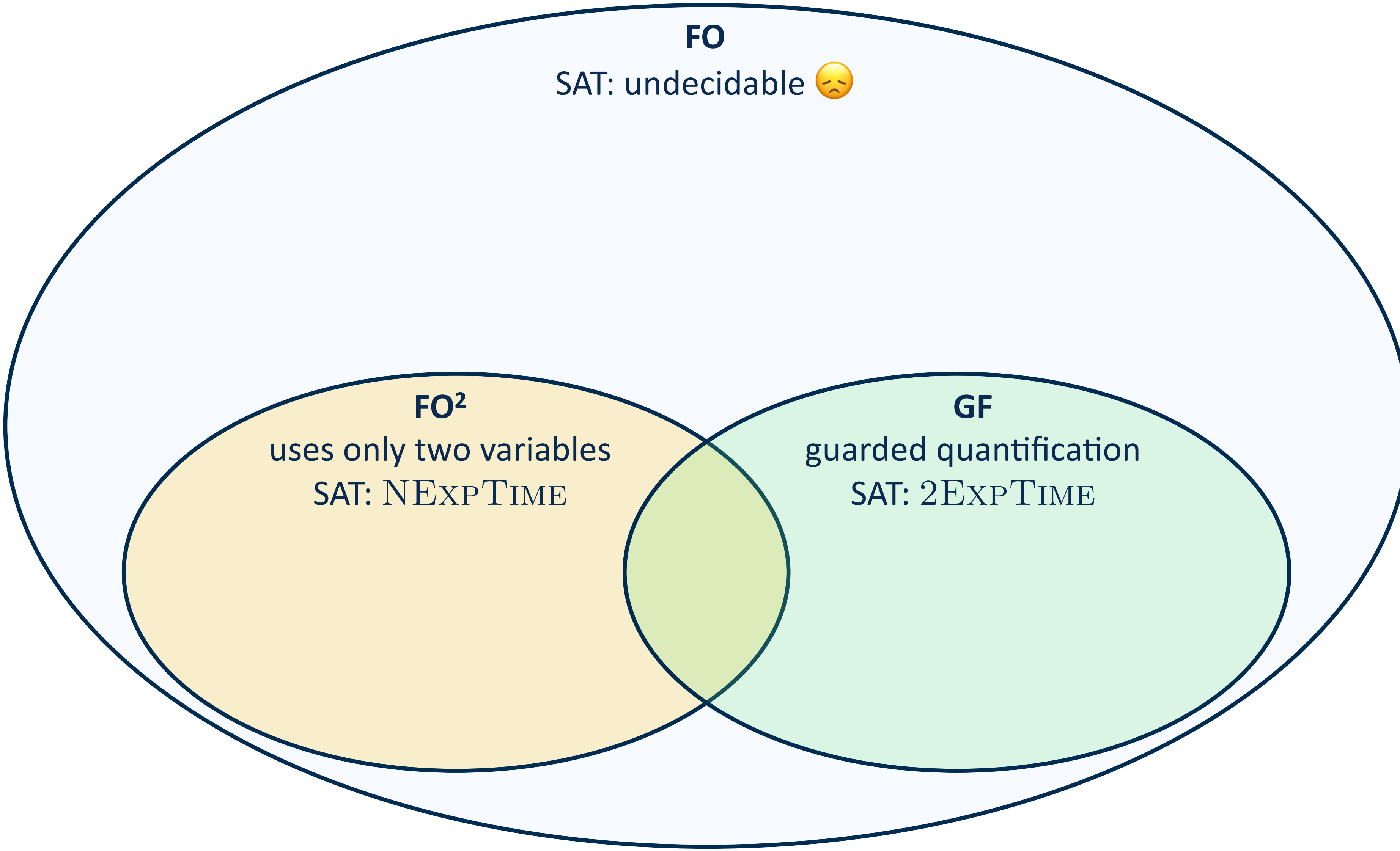

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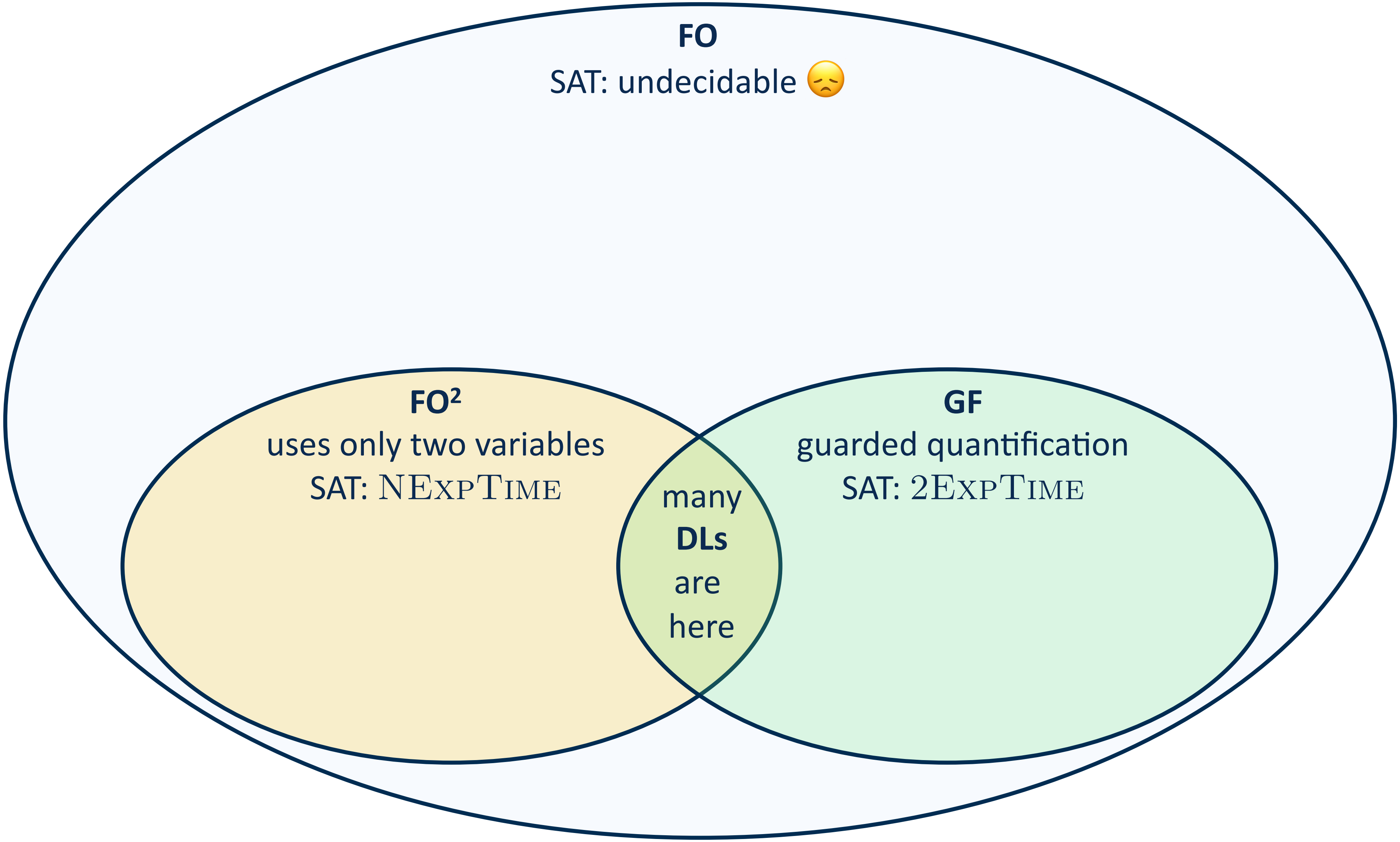

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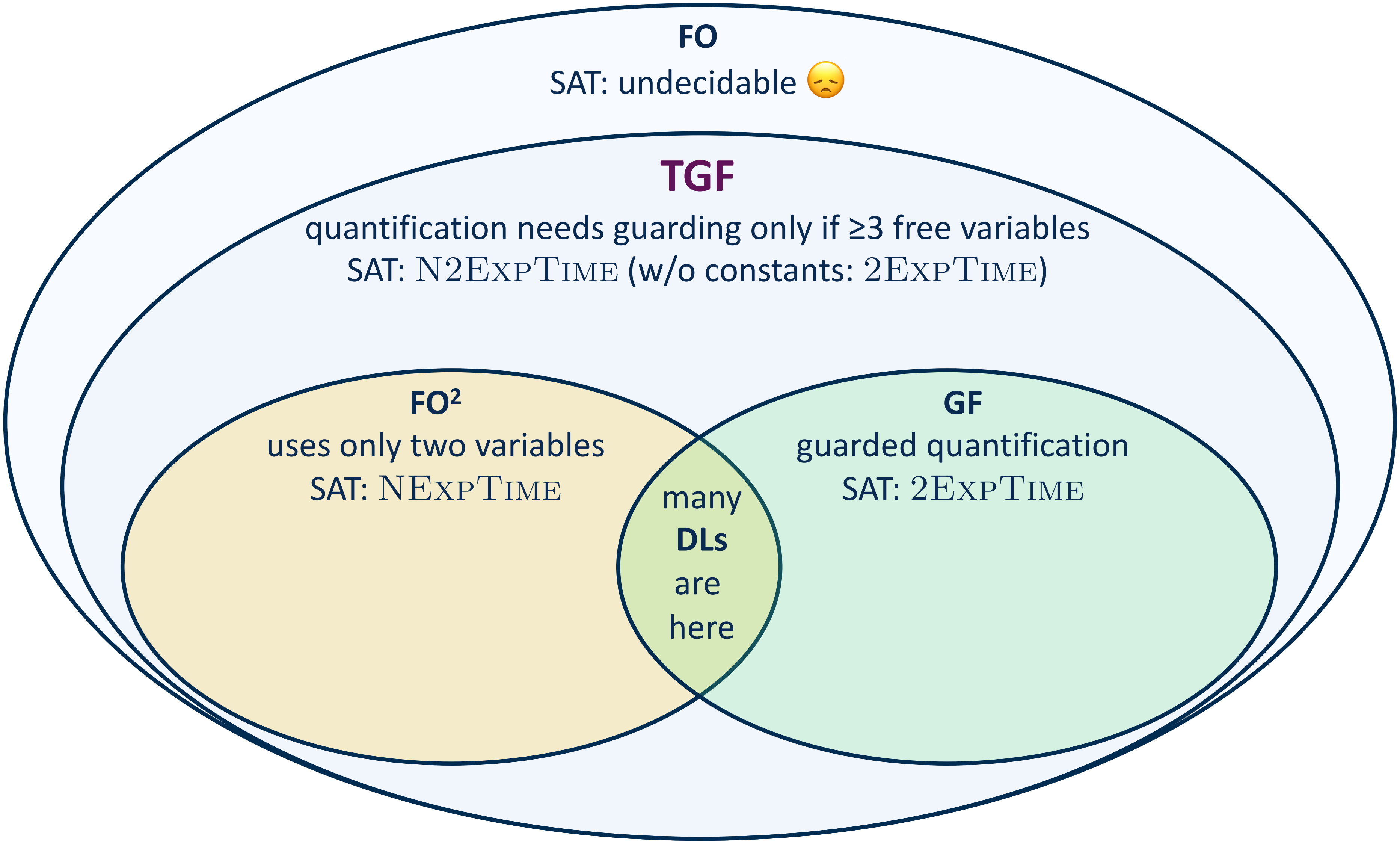

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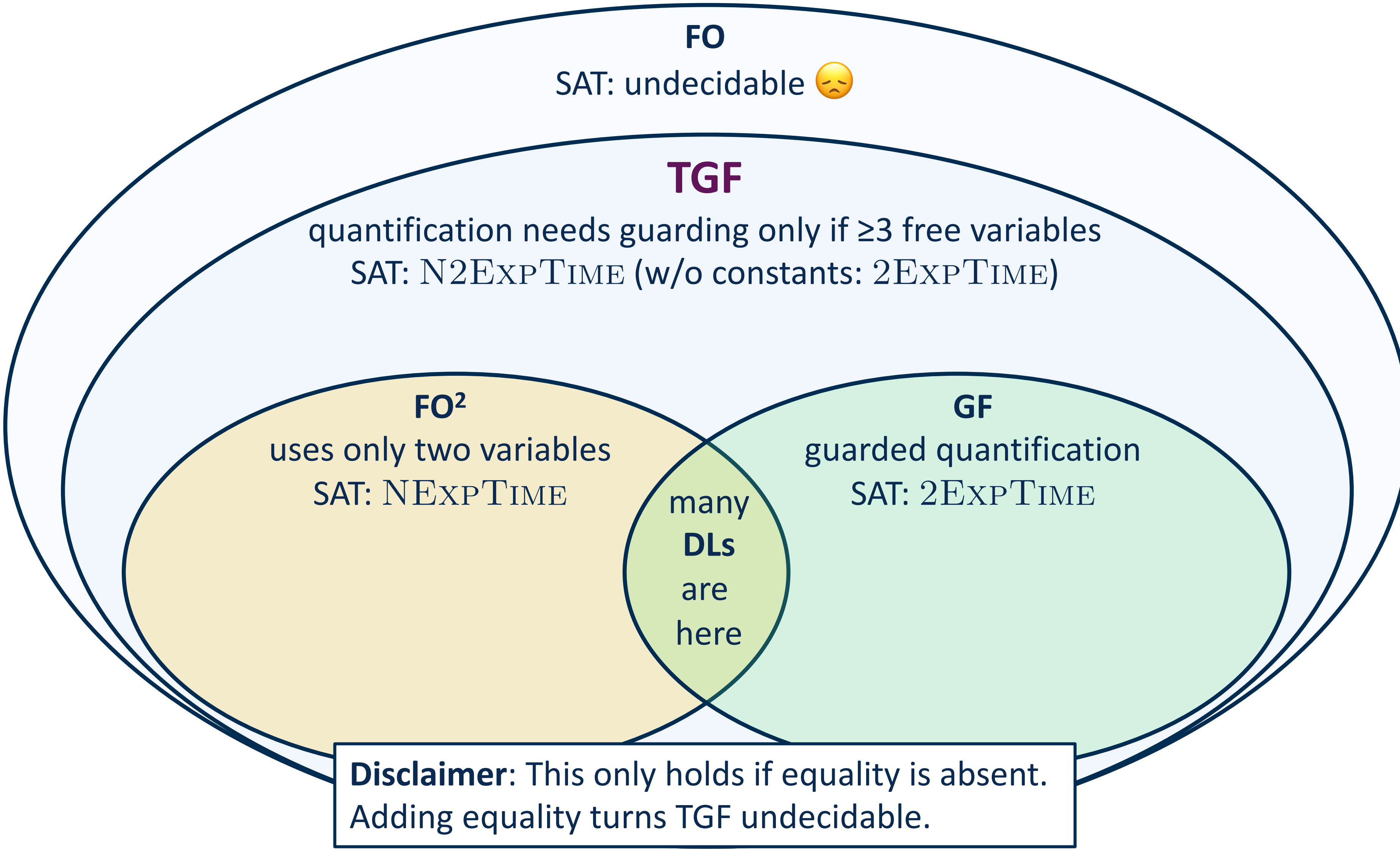

Disclaimer: This only holds if equality is absent.
Adding equality turns TGF undecidable.

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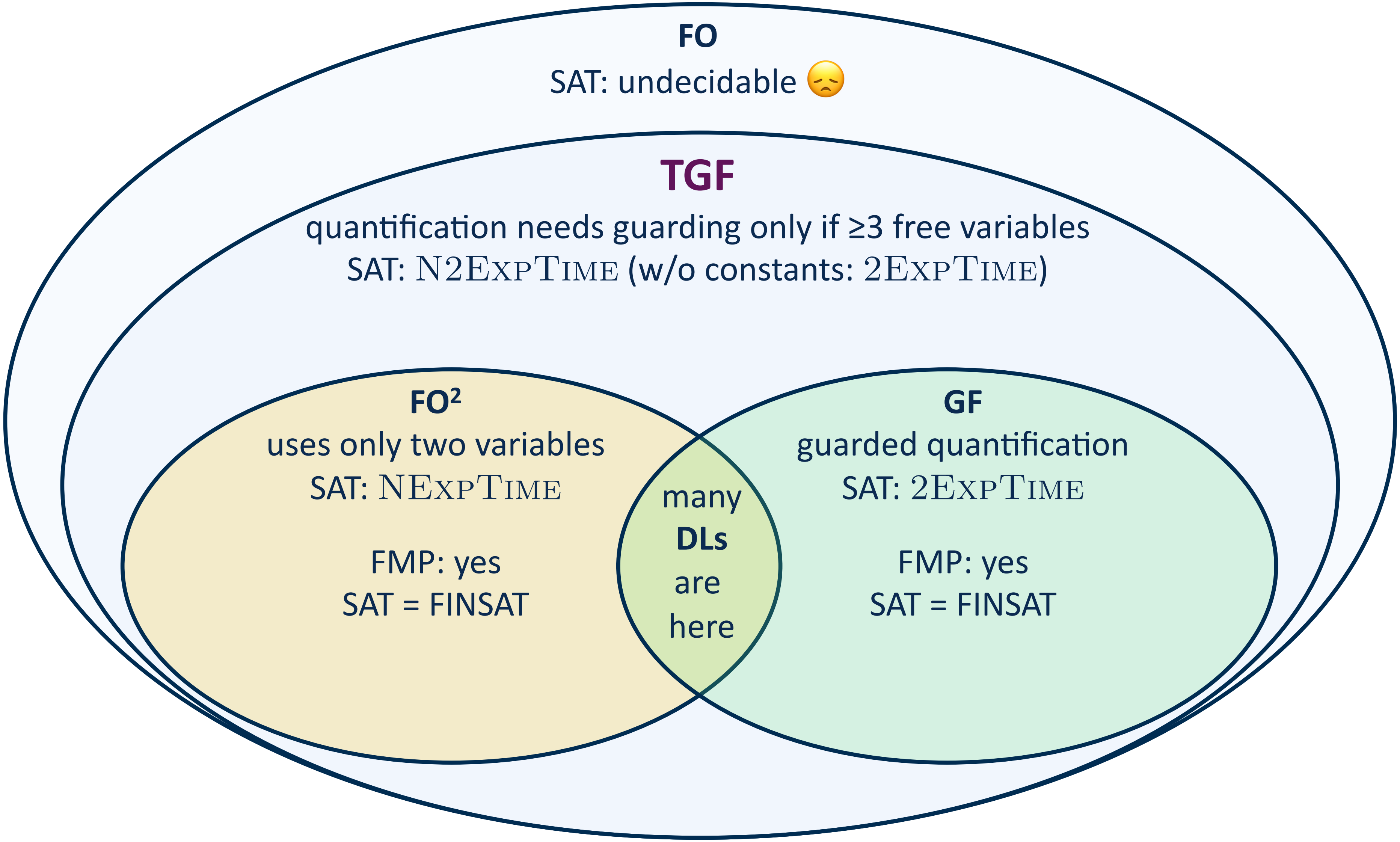

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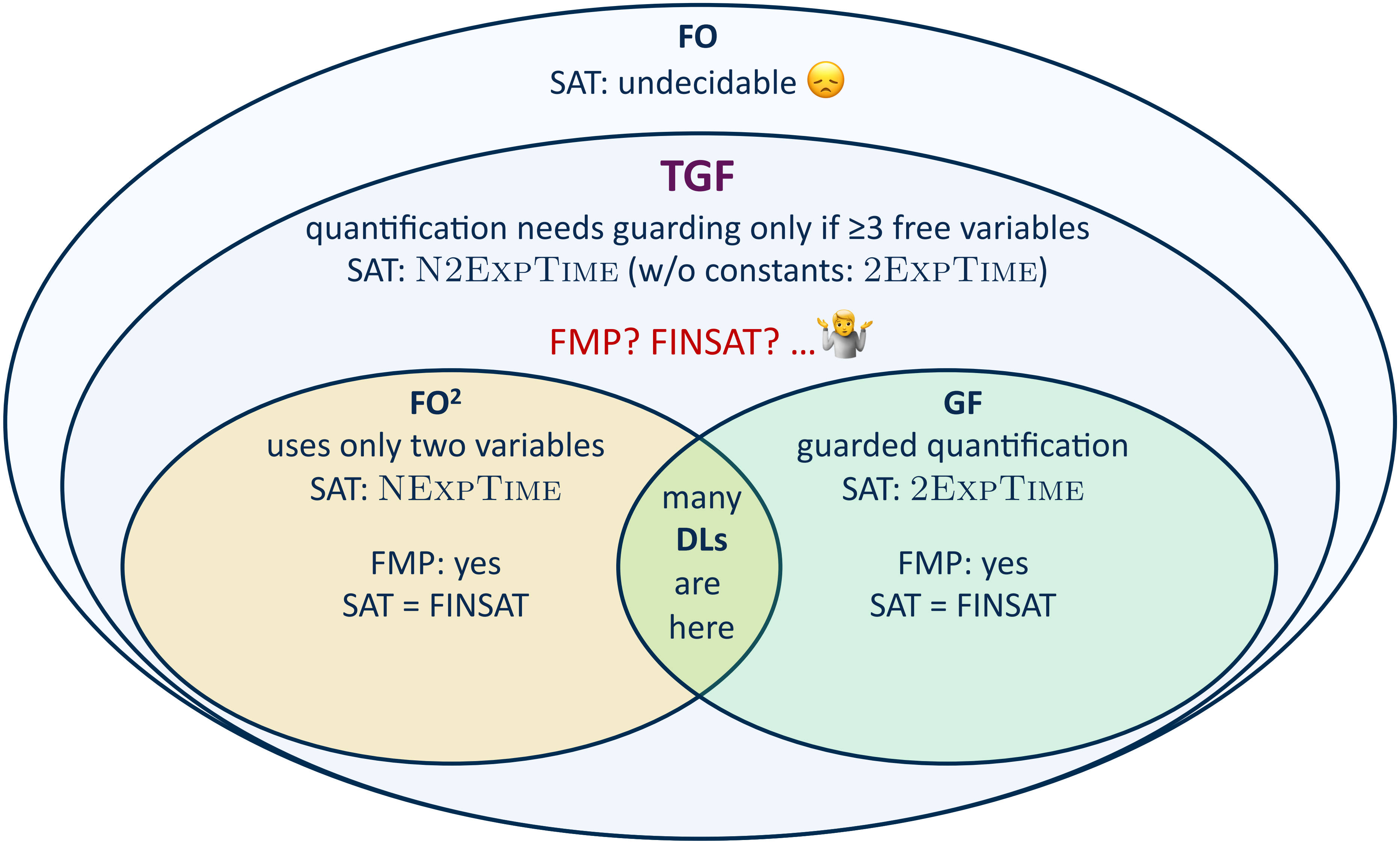

FO2 vs GF vs TGF – Examples

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Sentence	FO^2	GF	TGF
$\forall xy.\ \text{parent\_of}(x, y) \rightarrow \text{person}(x)$	✓	✓	✓
$\forall x.\ \text{person}(x) \rightarrow \exists y.\ \text{parent\_of}(y, x)$	✓	✓	✓
$\forall xy.\ \text{married}(x, y) \rightarrow \exists z.\ \text{witness\_of}(z, x, y)$	✗	✓	✓
$\forall xy.\ \text{elephant}(x) \wedge \text{mouse}(y) \rightarrow \text{bigger\_than}(x, y)$	✓	✗	✓
$\forall xy.\ \text{carb\_acid}(x) \wedge \text{alcohol}(y) \rightarrow \exists z.\ \text{combines\_into}(x, y, z) \wedge \text{ester}(z)$	✗	✗	✓
$\forall xyz.\ \text{bigger\_than}(x, y) \wedge \text{bigger\_than}(y, z) \rightarrow \text{bigger\_than}(x, z)$	✗	✗	✗

Enter: GFU

Borrowed from notion of "universal role" in description logics:

Let U be a distinguished binary predicate symbol.

We call a structure $\mathfrak{A}$ with domain A *U-biquitous*, if $U^{\mathfrak{A}} = A \times A$.

Definition: The logic *GFU*

Syntax: Every GF sentence is a GFU sentence.

Semantics (via Model Theory): $\mathfrak{A}$ is a GFU-model of φ if

1. $\mathfrak{A}$ is a GF-model of φ and
2. $\mathfrak{A}$ is U-biquitous.

Observation: For every TGF sentence φ (not using the symbol U), one can polynomially compute a GFU sentence φ' such that

$$\mathfrak{A} \text{ is a model of } \varphi \Leftrightarrow \mathfrak{A}[U \mapsto A \times A] \text{ is a GFU-model of } \varphi'.$$

Example:

$$\forall xy.\ \mathbf{U}(x,y) \rightarrow (\ \text{carb\_acid}(x) \wedge \text{alcohol}(y) \rightarrow \exists z.\ \text{combines\_into}(x, y, z) \wedge \text{ester}(z)\ )$$

→ We will work with GFU instead of the original TGF.

TGF and Finite Models

So far, proofs related to TGF / GFU required infinite models.

FMP proofs neither for FO^2 nor for GF generalize easily to TGF / GFU.

However, it turns out, we can use machinery for finite-model-construction for GF.

(For clarity, we focus on the case w/o constants.)

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Given satisfiable GFU-sentence φ, assume infinite U-biquitous model $\mathfrak{A}$.
Construct finite U-biquitous model $\mathfrak{A}'$ of φ as follows:

1. Strengthen φ into φ^* (still guarded) s.t. $\mathfrak{A}$ remains model.
   φ^* enforces enough U-connections, even when interpreted non-U-biquitously.

2. Use FMP of GF to obtain finite (yet non-U-biquitous) model $\mathfrak{C}$ of φ^*.

3. Obtain $\mathfrak{A}_0$ as $125 \cdot |C|^2$-fold disjoint union of $\mathfrak{C}$ with itself, (still model of φ^*).

4. **U-saturation**: Obtain $\mathfrak{A}_1$, $\mathfrak{A}_2$ … by iteratively picking a pair of yet U-unconnected elements and connecting them, using an appropriate pair of connected elements as template (hence maintaining φ^*-modelhood).

5. As the number of elements remains constant, the procedure terminates and yields a U-biquitous $\mathfrak{A}_n = \mathfrak{A}'$.

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1 Strengthen φ into φ^* (still guarded) s.t. $\mathfrak{A}$ remains model.
φ^* enforces enough U-connections, even when interpreted non-U-biquitously.

Construct GF-sentence φ^* by conjunctively adding to φ statements enforcing that

- exactly all 1-types from $\mathfrak{A}$ are realized,
- for any two 1-types from $\mathfrak{A}$, there are U-connected representatives, and
- U holds between any two elements co-occurring in any other relation.

$$\forall x\Big(\bigvee_{\alpha\in\boldsymbol{\alpha}} \alpha(x)\Big)$$

$$\bigwedge_{\alpha,\alpha'\in\boldsymbol{\alpha}} \exists xy\big(\alpha(x)\wedge\alpha'(y)\wedge \mathsf{U}(x,y)\wedge \mathsf{U}(y,x)\big)$$

$$\bigwedge_{P\in\sigma} \forall\bar{x}\Big(P(\bar{x}) \Rightarrow \bigwedge_{1\leq i,j\leq|\bar{x}|} \mathsf{U}(x_i,x_j)\Big)$$

FMP for TGF – Roadmap

2 Use FMP of GF to obtain finite (yet non-U-biquitous) small model $\mathfrak{C}$ of $\boldsymbol{\varphi}^*$.

Based on:

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QUERYING THE GUARDED FRAGMENT *

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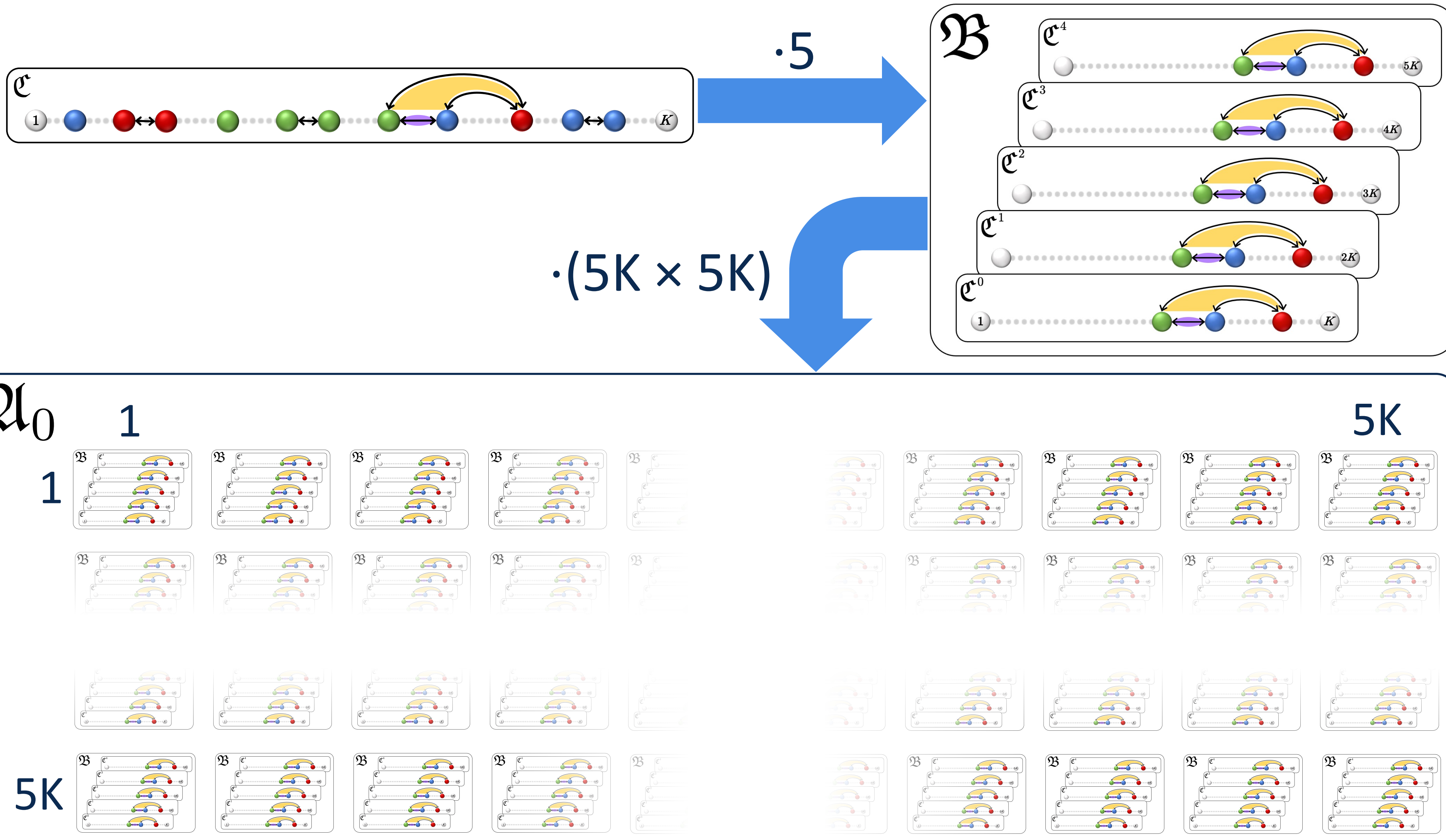
Straightforward for finiteness, but not when it comes to size:

- small-model-property in above paper guarantees 2Exp upper bound of the model wrt. length of GF sentence.
- Yet: $\boldsymbol{\varphi}^*$ has exponential length wrt. $\boldsymbol{\varphi}$.
- by careful analysis of $\boldsymbol{\varphi}^*$'s structure and the proofs in above paper, we can still ensure: size of $\mathfrak{C}$ is (only) double exponentially bounded by length of $\boldsymbol{\varphi}$.

FMP for TGF – Roadmap

3 Let K=|C|. Get $\mathfrak{A}_0$ as $125{\cdot}K^2$-fold disjoint union of $\mathfrak{C}$ with itself, (still model of φ^*).



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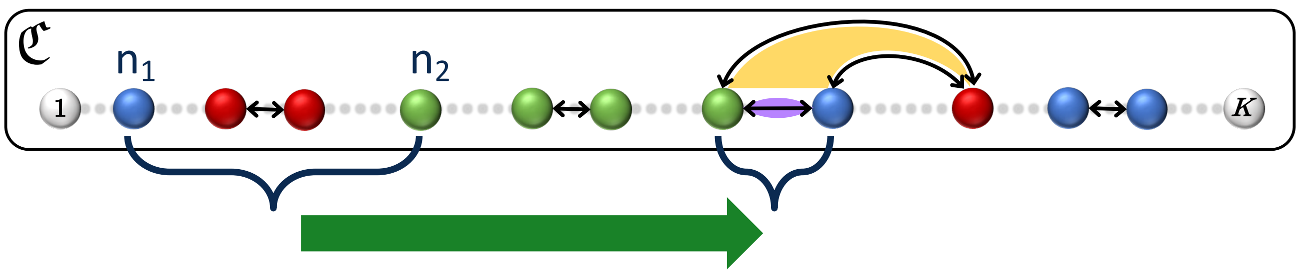
4 **U-saturation**: Obtain $\mathfrak{A}_1, \mathfrak{A}_2$ … by iteratively picking a pair of yet U-unconnected elements and connecting them, using an appropriate pair of connected elements as template (hence maintaining φ^*-modelhood).

Note: φ^* ensures, for any two elements n_1 and n_2 existence of a "1-type-compatible representative U-connected pair":



This representative pair will be called "entry pair" for n_1 and n_2

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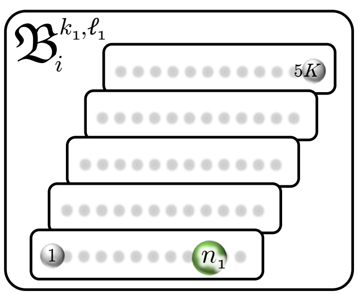

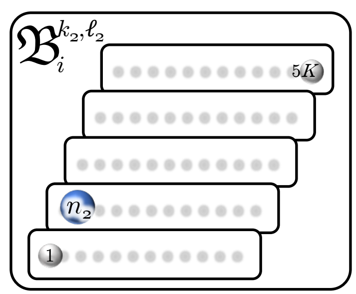

FMP for TGF – Roadmap

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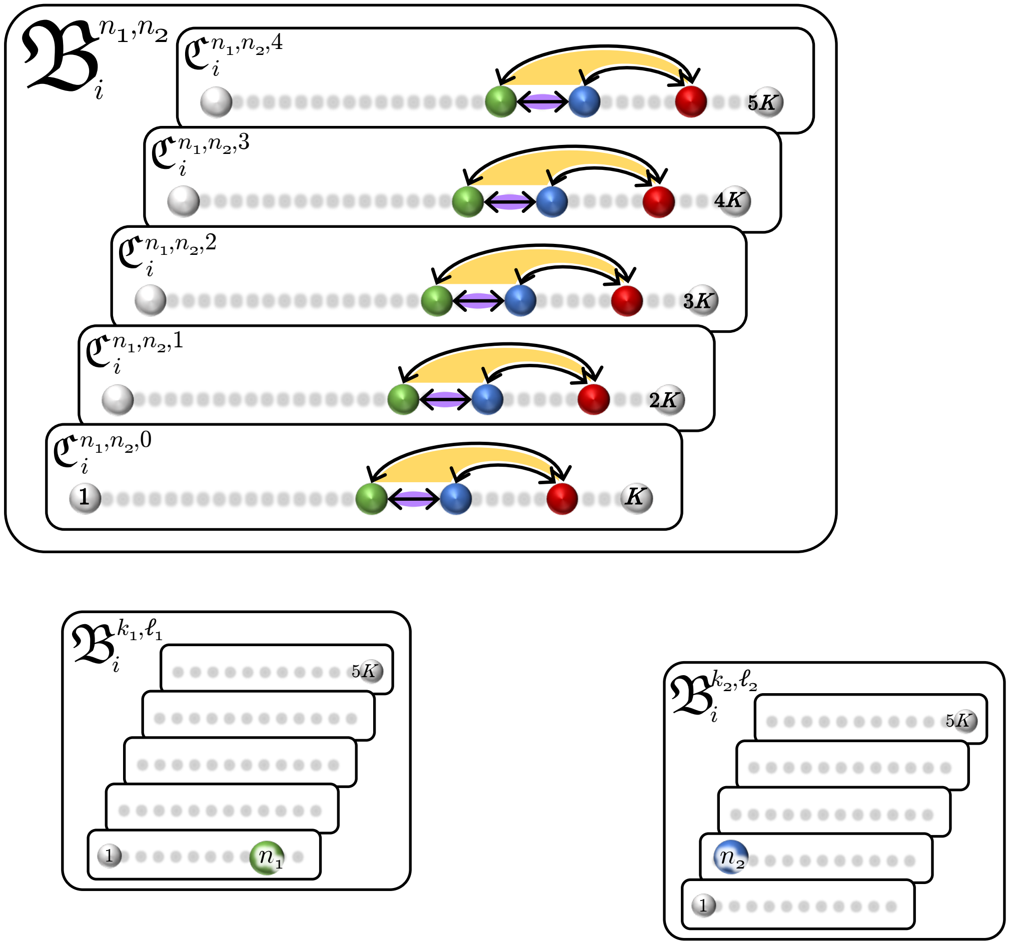

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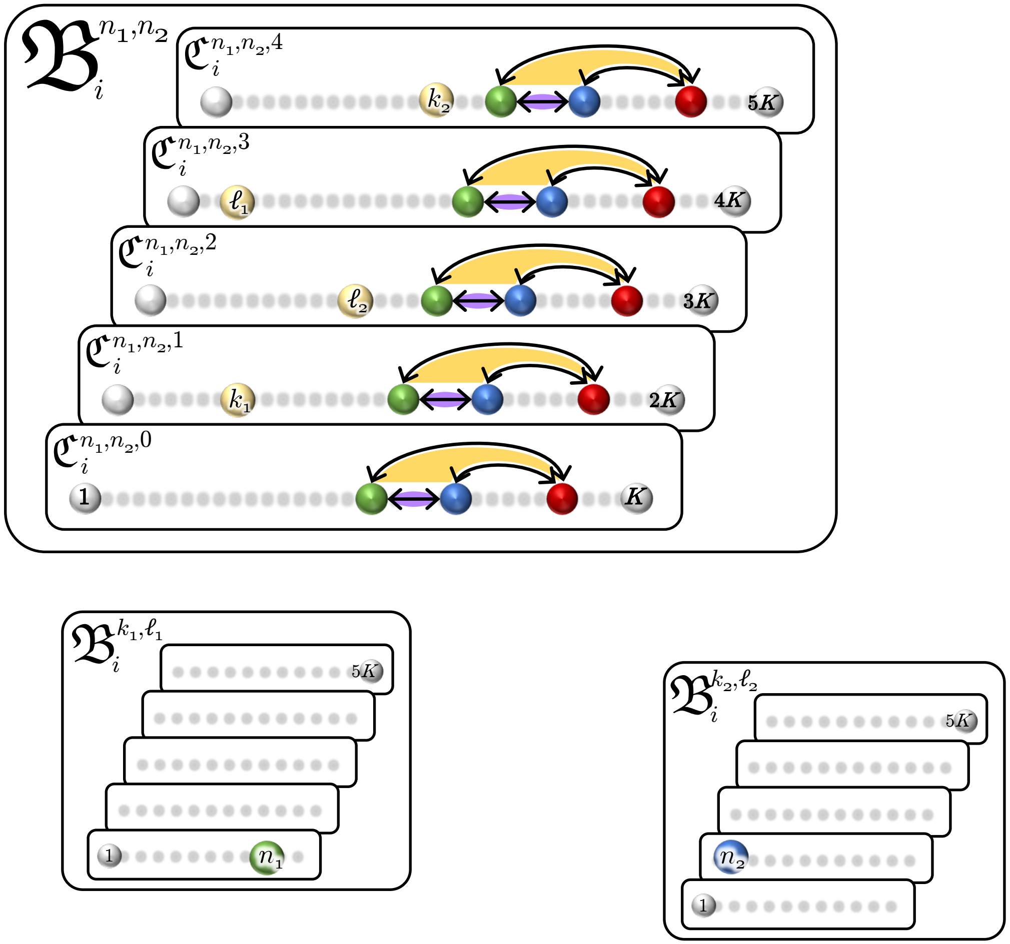

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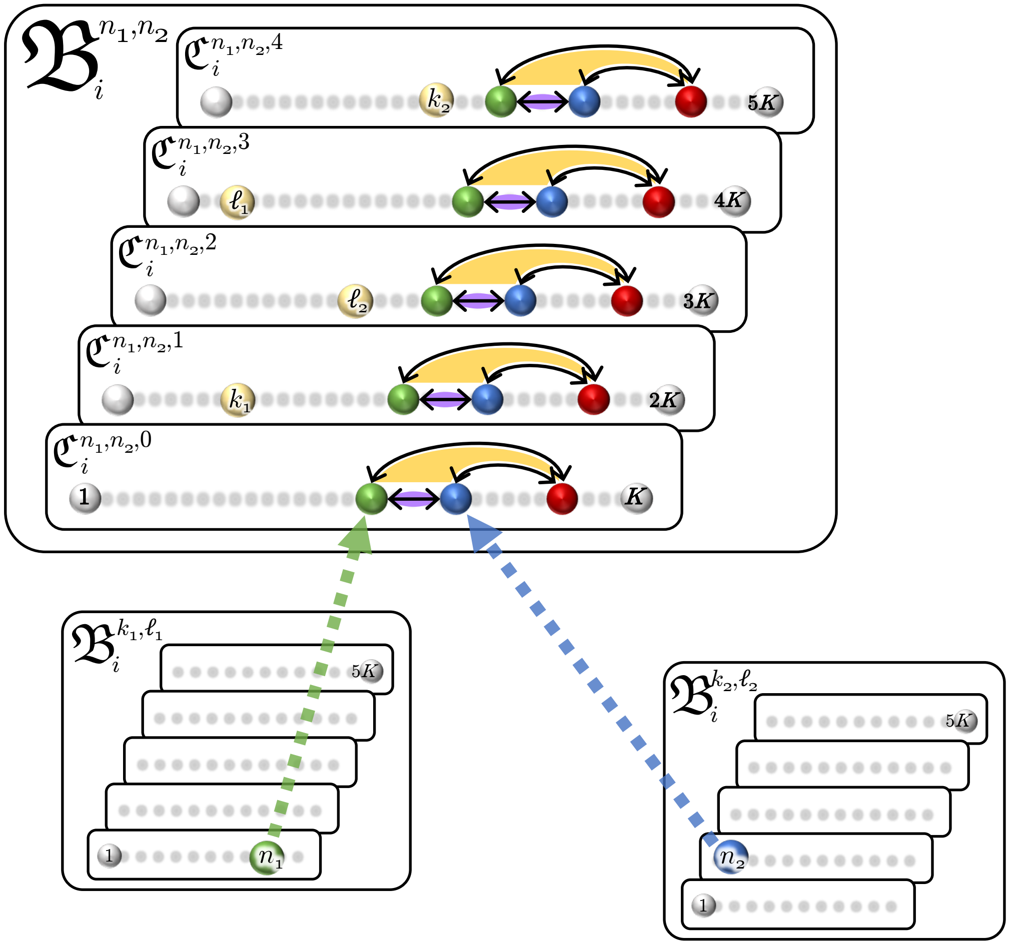

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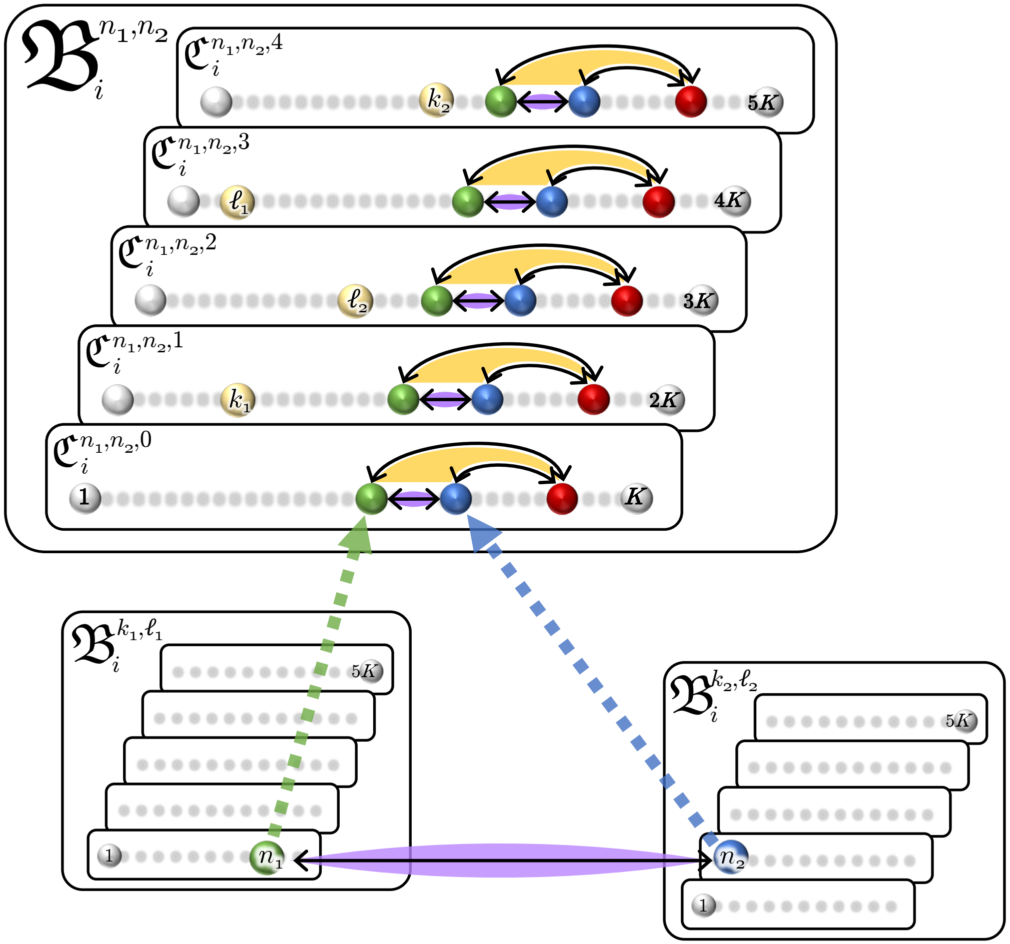

FMP for TGF – Roadmap

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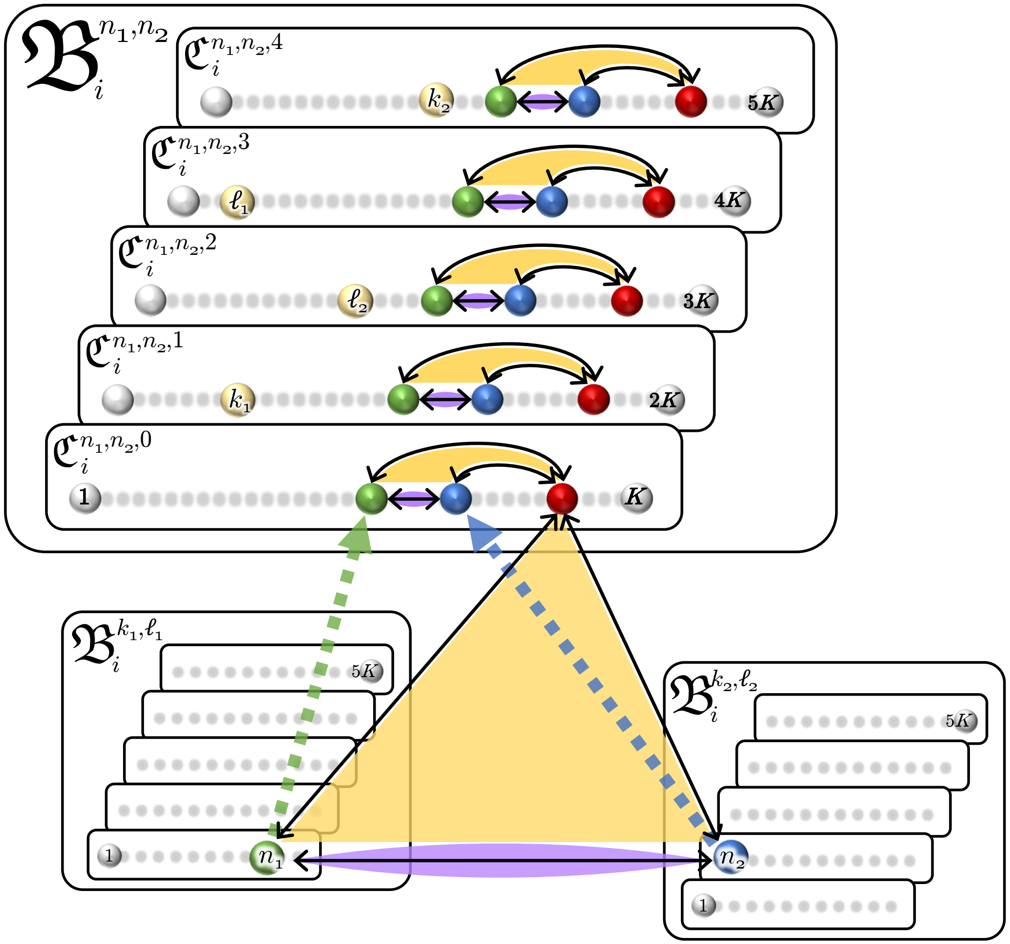

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5 As the number of elements remains constant, the procedure terminates and yields a U-biquitous $\mathfrak{A}_n = \mathfrak{A}'$.

"Small" model property:

- discussed before: size of $\mathfrak{C}$ doubly exponentially bounded by length of φ
- only polynomial blowup from $\mathfrak{C}$ to $\mathfrak{A}_0$: $|\mathfrak{A}_0| = 125 \cdot |\mathfrak{C}|^3$
- no change in size from $\mathfrak{A}_0$ to $\mathfrak{A}_n = \mathfrak{A}'$ (domain stays the same)
- thus: size of $\mathfrak{A}'$ doubly exponentially bounded by length of φ

Adding constants requires slight adaptation, but nothing serious.

We obtain:

Theorem: *Every satisfiable TGF sentence φ (with or without constants) has a finite model, the size of which is bounded doubly exponentially by the length of φ. Consequently, satisfiability and finite satisfiability of TGF sentences coincide.*

Transitivity?

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Sentence	FO^2	GF	TGF
$\forall xy.\ \text{parent\_of}(x, y) \to \text{person}(x)$	✓	✓	✓
$\forall x.\ \text{person}(x) \to \exists y.\ \text{parent\_of}(y, x)$	✓	✓	✓
$\forall xy.\ \text{married}(x, y) \to \exists z.\ \text{witness\_of}(z, x, y)$	✗	✓	✓
$\forall xy.\ \text{elephant}(x) \wedge \text{mouse}(y) \to \text{bigger\_than}(x, y)$	✓	✗	✓
$\forall xy.\ \text{carb\_acid}(x) \wedge \text{alcohol}(y) \to \exists z.\ \text{combines\_into}(x, y, z) \wedge \text{ester}(z)$	✗	✗	✓
$\forall xyz.\ \text{bigger\_than}(x, y) \wedge \text{bigger\_than}(y, z) \to \text{bigger\_than}(x, z)$	✗	✗	✗

But what about transitivity? Transitive relations are important for logical modelling!

- **Bad news:** Adding "built-in" transitive relations to FO^2 or GF turns SAT undecidable.
- **Good news:** "Built-in" transitive relations in GF or TGF are OK, when they only appear as guards: GF+TG, TGF+TG.
- So far, results only for constant-free case (SAT: 2ExpTime).
- But: constructed models are generally infinite…

FINSAT of (T)GF + TG

Question: We know GF+TG and TGF+TG do not have FMP. What about FINSAT?

Using some similar and some different ideas regarding parameter analysis and model surgery yields:

Theorem: *The finite satisfiability problem for (T)GF+TG without constants is* 2ExpTime*-complete. Every finitely satisfiable (T)GF+TG formula has a model of size bounded doubly exponentially in its length.*

Note: GF+TG supports equality while TGF+TG doesn't.

Conclusion

Conclusion

Remark:

- results remain intact when allowing equality statements of the form x = c
- properly increases expressivity (allows expressing "nominals" known from DLs)

ToDo:

- adding constants to (T)GF + TG
- conjecture: resulting fragments still decidable
- lower bound for TGF+TG comes from TGF (N2ExpTime), i.e., harder than constant-free case (under standard assumptions)