

Exercise 8: Datalog

Database Theory

2025-06-03

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Exercise 1

o **Exercise.** Consider the example Datalog program from the lecture:

(a) Father(alice,bob)

(b) Mother(alice,carla)

(c) Mother(ewan,carla)

(d) Father(carla,david)

Parent(x,y) \leftarrow Father(x,y) (1)

Parent(x,y) \leftarrow Mother(x,y) (2)

Ancestor(x,y) \leftarrow Parent(x,y) (3)

Ancestor(x,z) \leftarrow Parent(x,y) \wedge Ancestor(y,z) (4)

SameGeneration(x,x) \leftarrow (5)

SameGeneration(x,y) \leftarrow Parent(x,v) \wedge Parent(y,w)
 \wedge SameGeneration(v,w) (6)

1. Give a proof tree for SameGeneration(ewan, alice).
2. Compute the sets $T_P^0, T_P^1, T_P^2, \dots$. When is the fixed point reached?

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2. Compute the sets $T_P^0, T_P^1, T_P^2, \dots$. When is the fixed point reached?

Solution.

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o **Exercise.** Consider the example Datalog program from the lecture:

(a) $\text{Father}(\text{alice}, \text{bob})$

(b) $\text{Mother}(\text{alice}, \text{carla})$

(c) $\text{Mother}(\text{evan}, \text{carla})$

(d) $\text{Father}(\text{carla}, \text{david})$

$\text{Parent}(x, y) \leftarrow \text{Father}(x, y) \quad (1)$

$\text{Parent}(x, y) \leftarrow \text{Mother}(x, y) \quad (2)$

$\text{Ancestor}(x, y) \leftarrow \text{Parent}(x, y) \quad (3)$

$\text{Ancestor}(x, z) \leftarrow \text{Parent}(x, y) \wedge \text{Ancestor}(y, z) \quad (4)$

$\text{SameGeneration}(x, x) \leftarrow \quad (5)$

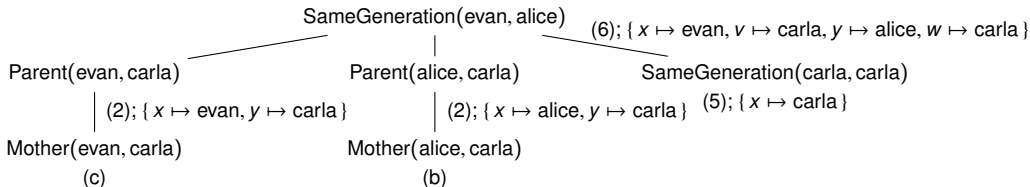
$\text{SameGeneration}(x, y) \leftarrow \text{Parent}(x, v) \wedge \text{Parent}(y, w) \wedge \text{SameGeneration}(v, w) \quad (6)$

1. Give a proof tree for $\text{SameGeneration}(\text{evan}, \text{alice})$.

2. Compute the sets $T_P^0, T_P^1, T_P^2, \dots$. When is the fixed point reached?

Solution.

1.



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1. Give a proof tree for SameGeneration(ewan, alice).

2. Compute the sets $T_P^0, T_P^1, T_P^2, \dots$. When is the fixed point reached?

Solution.

2. $T_P^0 = \emptyset$

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1. Give a proof tree for $\text{SameGeneration}(\text{evan}, \text{alice})$.

2. Compute the sets $T_P^0, T_P^1, T_P^2, \dots$. When is the fixed point reached?

Solution.

2.

$$T_P^0 = \emptyset$$

$$T_P^1 = \{ \text{Father}(\text{alice}, \text{bob}), \text{Mother}(\text{alice}, \text{carla}), \text{Mother}(\text{evan}, \text{carla}), \text{Father}(\text{carla}, \text{david}), \\ \text{SameGeneration}(\text{alice}, \text{alice}), \text{SameGeneration}(\text{bob}, \text{bob}), \text{SameGeneration}(\text{carla}, \text{carla}), \text{SameGeneration}(\text{david}, \text{david}), \text{SameGeneration}(\text{evan}, \text{evan}) \}$$

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2. Compute the sets $T_P^0, T_P^1, T_P^2, \dots$. When is the fixed point reached?

Solution.

2.

$$T_P^0 = \emptyset$$

$$T_P^1 = \{ \text{Father(alice,bob), Mother(alice,carla), Mother(ewan,carla), Father(carla,david),} \\ \text{SameGeneration(alice,alice), SameGeneration(bob,bob), SameGeneration(carla,carla), SameGeneration(david,david), SameGeneration(ewan,ewan)} \}$$

$$T_P^2 = T_P^1 \cup \{ \text{Parent(alice,bob), Parent(alice,carla), Parent(ewan,carla), Parent(carla,david)} \}$$

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Solution.

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$$T_P^0 = \emptyset$$

$$T_P^1 = \{ \text{Father(alice,bob), Mother(alice,carla), Mother(ewan,carla), Father(carla,david),}$$

$$\text{SameGeneration(alice,alice), SameGeneration(bob,bob), SameGeneration(carla,carla), SameGeneration(david,david), SameGeneration(ewan,ewan)} \}$$

$$T_P^2 = T_P^1 \cup \{ \text{Parent(alice,bob), Parent(alice,carla), Parent(ewan,carla), Parent(carla,david)} \}$$

$$T_P^3 = T_P^2 \cup \{ \text{Ancestor(alice,bob), Ancestor(alice,carla), Ancestor(ewan,carla), Ancestor(carla,david),}$$

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$$T_P^2 = T_P^1 \cup \{ \text{Parent(alice,bob), Parent(alice,carla), Parent(ewan,carla), Parent(carla,david)} \}$$

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$$T_P^4 = T_P^3 \cup \{ \text{Ancestor(alice,david), Ancestor(ewan,david)} \} = T_P^5$$

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Solution.

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$$T_P^4 = T_P^3 \cup \{ \text{Ancestor(alice,david), Ancestor(ewan,david)} \} = T_P^5 = T_P^\infty$$

Exercise 2

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e (“edge”). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node n ?”
2. “Are all nodes of the graph reachable from the node n ?”
3. “Does the graph have a directed cycle?”
4. “Does the graph have a path that is labelled by a palindrome?”
(a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node n 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of a labels?”
8. “Which pairs of nodes are connected by a path with the same number of a and b labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

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Solution.

1.

$\text{Reachable}(x, x) \leftarrow$

$\text{Reachable}(x, z) \leftarrow e(x, y, v) \wedge \text{Reachable}(y, z)$

$\text{Ans}(x) \leftarrow \text{Reachable}(n, x)$

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Solution.

2. Not expressible, since Datalog is *monotone*: any query that is true for some set of ground facts I is also true for every set of ground facts $J \supseteq I$, but the query is true on $I = \{e(n, n, a)\}$, but not on $J = I \cup \{e(m, m, b)\}$.

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Solution.

3.

$$\text{Reachable}(x, y) \leftarrow e(x, y, v)$$
$$\text{Reachable}(x, z) \leftarrow e(x, y, v) \wedge \text{Reachable}(y, z)$$
$$\text{Ans}() \leftarrow \text{Reachable}(x, x)$$

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Solution.

4. Paths are allowed to be empty here.

$\text{Reachable}(x, x) \leftarrow$

$\text{Reachable}(x, y) \leftarrow e(x, y, v)$

$\text{Reachable}(x, z) \leftarrow e(x, y, u), \text{Reachable}(y, w), e(w, z, u)$

$\text{Ans}() \leftarrow \text{Reachable}(x, y)$

Exercise 2

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e (“edge”). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

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Solution.

5. Not expressible; consider $I = \{e(n, 1, a), e(1, 2, a)\}$ and $J = I \cup \{e(2, n, a)\}$.

Exercise 2

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e (“edge”). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

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Solution.

6. Not expressible; consider $I = \{e(n, 1, a), e(1, 2, a)\}$ and $J = I \cup \{e(2, n, a)\}$.

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Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e (“edge”). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

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Solution.

7.

$\text{Reachable}(x, y) \leftarrow e(x, y, b)$

$\text{Reachable}(x, z) \leftarrow e(x, y, a), e(y, z, a)$

$\text{Reachable}(x, z) \leftarrow e(x, y, a), \text{Reachable}(y, w), e(w, z, a)$ $\text{Reachable}(x, z) \leftarrow \text{Reachable}(x, y), \text{Reachable}(y, z)$

$\text{Ans}(x, y) \leftarrow \text{Reachable}(x, y)$

Exercise 2

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e (“edge”). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

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Solution.

8.

$\text{Reachable}(x, x) \leftarrow$

$\text{Reachable}(x, z) \leftarrow e(x, y, a), \text{Reachable}(y, w), e(w, z, b)$

$\text{Ans}(x, y) \leftarrow \text{Reachable}(x, y)$

$\text{Reachable}(x, z) \leftarrow \text{Reachable}(x, y), \text{Reachable}(y, z)$

$\text{Reachable}(x, z) \leftarrow e(x, y, b), \text{Reachable}(y, w), e(w, z, a)$

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Solution.

9. Not expressible, since Datalog is *homomorphism-closed*; consider $I = \{e(n, 1, a), e(1, m, a), e(n, 2, a), e(2, m, a)\}$ and $J = \{e(n, 1, a), e(1, m, a)\}$ and the homomorphism $\varphi : I \rightarrow J = \{2 \mapsto 1\}$.

Exercise 3

Exercise. Consider a UCQ of the following form

$$(r_{11}(x) \wedge r_{12}(x)) \vee \dots \vee (r_{\ell 1}(x) \wedge r_{\ell 2}(x))$$

Find a Datalog query that expresses this UCQ. How many rules and how many additional IDB predicates does your solution use (depending on ℓ)?

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Solution.

$$\text{Ans}(x) \leftarrow r_{11}(x), r_{12}(x)$$

$$\text{Ans}(x) \leftarrow r_{21}(x), r_{22}(x)$$

$$\vdots \qquad \vdots \qquad \vdots$$

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This solution uses ℓ rules and one additional IDB predicate.

Exercise 4

Exercise. Consider a Datalog query of the following form:

$$A_1(x) \leftarrow r_{11}(x)$$

...

$$A_\ell(x) \leftarrow r_{\ell 1}(x)$$

$$A_1(x) \leftarrow r_{12}(x)$$

...

$$A_\ell(x) \leftarrow r_{\ell 2}(x)$$

$$\text{Ans}(x) \leftarrow A_1(x), \dots, A_\ell(x)$$

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on ℓ)?

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Solution.

Exercise 4

Exercise. Consider a Datalog query of the following form:

$$\begin{array}{lll}
 A_1(x) \leftarrow r_{11}(x) & \dots & A_\ell(x) \leftarrow r_{\ell 1}(x) \\
 A_1(x) \leftarrow r_{12}(x) & \dots & A_\ell(x) \leftarrow r_{\ell 2}(x) \\
 \text{Ans}(x) \leftarrow A_1(x), \dots, A_\ell(x)
 \end{array}$$

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on ℓ)?

Solution.

$$\begin{aligned}
 \varphi_{11\dots 1}(x) &= r_{11}(x) \wedge r_{21}(x) \wedge \dots \wedge r_{\ell 1}(x) \\
 \varphi_{21\dots 1}(x) &= r_{12}(x) \wedge r_{21}(x) \wedge \dots \wedge r_{\ell 1}(x) \\
 \varphi_{12\dots 1}(x) &= r_{11}(x) \wedge r_{22}(x) \wedge \dots \wedge r_{\ell 1}(x) \\
 \varphi_{22\dots 1}(x) &= r_{12}(x) \wedge r_{22}(x) \wedge \dots \wedge r_{\ell 1}(x) \\
 &\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots \\
 \varphi_{22\dots 2}(x) &= r_{12}(x) \wedge r_{22}(x) \wedge \dots \wedge r_{\ell 2}(x) \\
 \varphi &= \bigvee_{i \in \{11\dots 1, 21\dots 1, \dots, 22\dots 2\}} \varphi_i
 \end{aligned}$$

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$$\begin{array}{lll} A_1(x) \leftarrow r_{11}(x) & \dots & A_\ell(x) \leftarrow r_{\ell 1}(x) \\ A_1(x) \leftarrow r_{12}(x) & \dots & A_\ell(x) \leftarrow r_{\ell 2}(x) \\ \text{Ans}(x) \leftarrow A_1(x), \dots, A_\ell(x) \end{array}$$

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on ℓ)?

Solution.

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This solution uses 2^ℓ CQs.

Exercise 5

Exercise. Show that T_P^∞ is the least fixed point of the T_P operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of T_P must contain every fact in T_P^∞ .

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Solution.

1. ▶ We first show that for all $i \geq 0$, we have $T_P^i \subseteq T_P^{i+1}$: Clearly $T_P^0 = \emptyset \subseteq T_P^1 = T_P(\emptyset)$.

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 - ▶ Consider some fixed point F of T_P . We show $T_P^i \subseteq F$ for all $i \geq 0$.
 - ▶ Clearly, $T_P^0 = \emptyset \subseteq F$.

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 - ▶ Consider some fixed point F of T_P . We show $T_P^i \subseteq F$ for all $i \geq 0$.
 - ▶ Clearly, $T_P^0 = \emptyset \subseteq F$.
 - ▶ Assume that $T_P^i \subseteq F$ for some $i \geq 0$. Then $T_P^{i+1} = T_P(T_P^i) \subseteq T_P(F) = F$, by monotonicity and since F is a fixed point.

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1.
 - ▶ We first show that for all $i \geq 0$, we have $T_P^i \subseteq T_P^{i+1}$: Clearly $T_P^0 = \emptyset \subseteq T_P^1 = T_P(\emptyset)$.
 - ▶ Assume that $T_P^i \subseteq T_P^{i+1}$ for some $i \geq 0$, and consider a ground fact $H \in T_P^{i+1} = T_P(T_P^i)$. Then there is some ground rule $H \leftarrow B_1, \dots, B_n \in \text{ground}(P)$ with $B_1, \dots, B_n \in T_P^i$. Since $T_P^i \subseteq T_P^{i+1}$, we have $B_1, \dots, B_n \in T_P^{i+1}$, and hence $H \in T_P^{i+2} = T_P(T_P^{i+1})$.
 - ▶ Thus, we have $T_P^i \subseteq T_P^{i+k}$ for all $i, k \geq 0$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.
 - ▶ Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \notin T_P^\infty$.
 - ▶ Then there is a ground rule $H \leftarrow B_1, \dots, B_n \in \text{ground}(P)$ with $B_1, \dots, B_n \in T_P^\infty$.
 - ▶ Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are i_1, \dots, i_n with $B_{ij} \in T_P^{i_j}$, and thus $B_1, \dots, B_n \in T_P^k$ with $k = \max\{i_1, \dots, i_n\}$.
 - ▶ But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \notin T_P^\infty$.
2.
 - ▶ First, note that T_P is clearly *monotone*, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.
 - ▶ Consider some fixed point F of T_P . We show $T_P^i \subseteq F$ for all $i \geq 0$.
 - ▶ Clearly, $T_P^0 = \emptyset \subseteq F$.
 - ▶ Assume that $T_P^i \subseteq F$ for some $i \geq 0$. Then $T_P^{i+1} = T_P(T_P^i) \subseteq T_P(F) = F$, by monotonicity and since F is a fixed point.
 - ▶ But then $T_P^i \subseteq F$ for all $i \geq 0$, and hence also $T_P^\infty = \bigcup_{i \geq 0} T_P^i \subseteq F$.