Complexity Theory

Alternation

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Computational Logic

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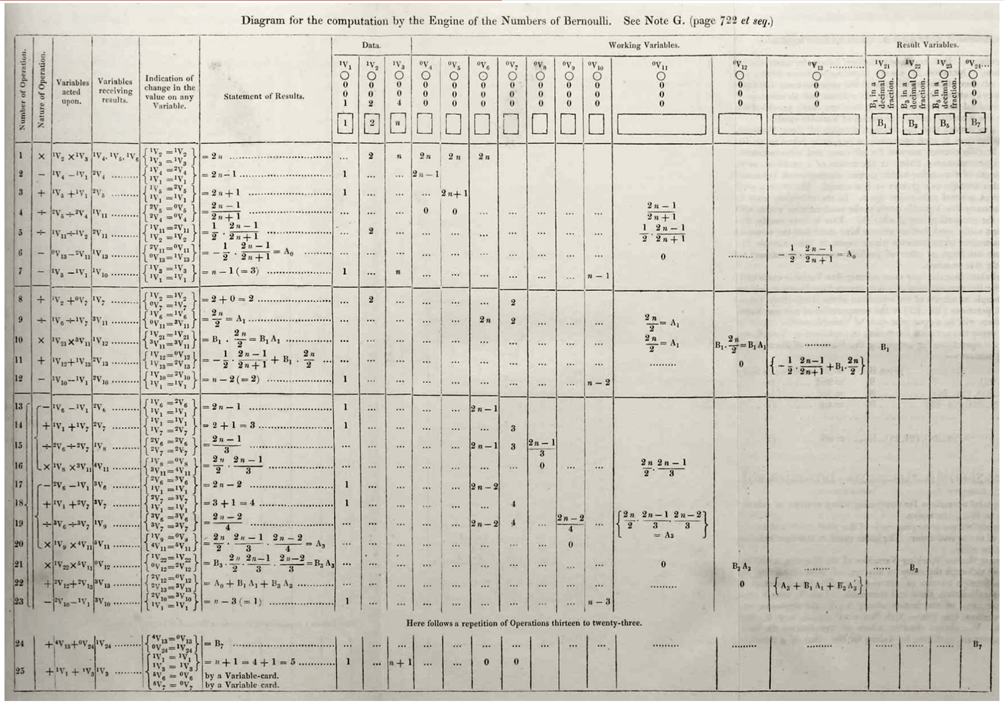

(early computation path written by Ada Lovelace)

Extended New Year's Review: Lectures 15–19

Alternation

Alternating Computations

Non-deterministic TMs:

- Accept if there is an accepting run.
- Used to define classes like NP

Complements of non-deterministic classes:

- Accept if all runs are accepting.
- Used to define classes like coNP

We have seen that existential and universal modes can also alternate:

- Players take turns in games
- Quantifiers may alternate in QBF

Is there a suitable Turing Machine model to capture this?

Alternating Turing Machines

Definition 14.1

An alternating Turing machine (ATM) $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ is a Turing machine with a non-deterministic transition function $\delta\colon Q \times \Gamma \to \mathfrak{P}(Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\})$ whose set of states is partitioned into existential and universal states:

$Q_\exists$: set of existential states $\qquad Q_\forall$: set of universal states

- Configurations of ATMs are the same as for (N)TMs: tape(s) + state + head position
- A configuration can be universal or existential, depending on whether its state is universal or existential
- Possible transitions between configurations are defined as for NTMs

Alternating Turing Machines: Acceptance

Acceptance is defined recursively:

Definition 14.2

A configuration C of an ATM $\mathcal{M}$ is accepting if one of the following is true:

- C is existential and some successor configuration of C is accepting.
- C is universal and all successor configurations of C are accepting.

$\mathcal{M}$ accepts a word w if the start configuration on w is accepting.

Note: configurations with no successor are the base case, since we have:

- An existential configuration without any successor configurations is rejecting.
- A universal configuration without any successor configurations is accepting.

Hence we don't need to specify accepting or rejecting states explicitly.

Nondeterminism and Parallelism

ATMs can be seen as a generalisation of non-deterministic TMs:

An NTM is an ATM where all states are existential (besides the single accepting state, which is always universal according to our definition).

ATMs can be seen as a model of parallel computation:

In every step, fork the current process to create sub-processes that explore each possible transition in parallel

- for universal states, combine the results of sub-processes with AND
- for existential states, combine the results of sub-processes with OR

Alternative view: an ATM accepts if its computation tree, considered as an AND-OR tree, evaluates to TRUE

Example: Alternating Algorithm for MinFormula

MinFormula

Input: A propositional formula φ.

Problem: Is φ the shortest formula that is satisfied by the same assignments as φ?

MinFormula can be solved by an alternating algorithm:

```
MinFormula(formula φ) :
  universally choose ψ := formula shorter than φ
  exist. guess I := assignment for variables in φ
  if φ^I = ψ^I :
    return FALSE
  else :
    return TRUE
```


Example: Alternating Algorithm for Geography

```
AltGeography(directed graph G, start node s) :
  Visited := {s} // visited nodes
  cur := s       // current node
  while TRUE :
    // existential move:
    if all successors of cur are in Visited:
      return FALSE
    existentially guess cur := unvisited successor of cur
    Visited := Visited ∪ {cur}
    // universal move:
    if all successors of cur are in Visited:
      return TRUE
    universally choose cur := unvisited successor of cur
    Visited := Visited ∪ {cur}
```


Time and Space Bounded ATMs

As before, time and space bounds apply to any computation path in the computation tree.

Definition 14.3

Let $\mathcal{M}$ be an alternating Turing machine and let $f : \mathbb{N} \to \mathbb{R}^+$ be a function.

- $\mathcal{M}$ is f-time bounded if it halts on every input $w \in \Sigma^*$ and on every computation path after $\leq f(|w|)$ steps.
- $\mathcal{M}$ is f-space bounded if it halts on every input $w \in \Sigma^*$ and on every computation path using $\leq f(|w|)$ cells on its tapes.

  (Here we typically assume that Turing machines have a separate input tape that we do not count in measuring space complexity.)

Defining Alternating Complexity Classes

Definition 14.4

Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function.

- $\text{ATime}(f(n))$ is the class of all languages $\mathcal{L}$ for which there is an $O(f(n))$-time bounded alternating Turing machine deciding $\mathcal{L}$, for some $k \geq 1$.
- $\text{ASpace}(f(n))$ is the class of all languages $\mathcal{L}$ for which there is an $O(f(n))$-space bounded alternating Turing machine deciding $\mathcal{L}$.

Common Alternating Complexity Classes

$$\mathrm{AP} = \mathrm{APT{\small IME}} = \bigcup_{d \geq 1} \mathrm{AT{\small IME}}(n^d)$$ alternating polynomial time

$$\mathrm{AE{\small XP}} = \mathrm{AE{\small XP}T{\small IME}} = \bigcup_{d \geq 1} \mathrm{AT{\small IME}}(2^{n^d})$$ alternating exponential time

$$\mathrm{A2E{\small XP}} = \mathrm{A2E{\small XP}T{\small IME}} = \bigcup_{d \geq 1} \mathrm{AT{\small IME}}(2^{2^{n^d}})$$ alt. double-exponential time

$$\mathrm{AL} = \mathrm{AL{\small OG}S{\small PACE}} = \mathrm{AS{\small PACE}}(\log n)$$ alternating logarithmic space

$$\mathrm{APS{\small PACE}} = \bigcup_{d \geq 1} \mathrm{AS{\small PACE}}(n^d)$$ alternating polynomial space

$$\mathrm{AE{\small XP}S{\small PACE}} = \bigcup_{d \geq 1} \mathrm{AS{\small PACE}}(2^{n^d})$$ alternating exponential space

Example: GEOGRAPHY $\in$ APTIME

Alternating Complexity Classes: Basic Properties

Nondeterminism: ATMs can do everything that the corresponding NTMs can do, e.g., $\mathrm{NP} \subseteq \mathrm{APT{\small IME}}$

Reductions: Polynomial many-one reductions can be used to show membership in many alternating complexity classes, e.g., if $\mathcal{L} \in \mathrm{APT{\small IME}}$ and $\mathcal{L}' \leq_p \mathcal{L}$ then $\mathcal{L}' \in \mathrm{APT{\small IME}}$.

In particular: $\mathrm{PS{\small PACE}} \subseteq \mathrm{APT{\small IME}}$ (since GEOGRAPHY $\in \mathrm{APT{\small IME}}$)

Complementation: ATMs are easily complemented:

- Let $\mathcal{M}$ be an ATM accepting language $\mathcal{L}(\mathcal{M})$
- Let $\mathcal{M}'$ be obtained from $\mathcal{M}$ by swapping existential and universal states
- Then $\mathcal{L}(\mathcal{M}') = \overline{\mathcal{L}(\mathcal{M})}$

For alternating algorithms this means: (1) negate all return values, (2) swap universal and existential branching points

Example: Complement of MINFORMULA

Original algorithm:

```
MinFormula(formula φ) :
  universally choose ψ := formula shorter than φ
  exist. guess I := assignment for variables in φ
  if φ^I = ψ^I :
    return FALSE
  else :
    return TRUE
```

Complemented algorithm:

```
ComplMinFormula(formula φ) :
  existentially guess ψ := formula shorter than φ
  univ. choose I := assignment for variables in φ
  if φ^I = ψ^I :
    return TRUE
  else :
    return FALSE
```