# SAT Solving – Implementation

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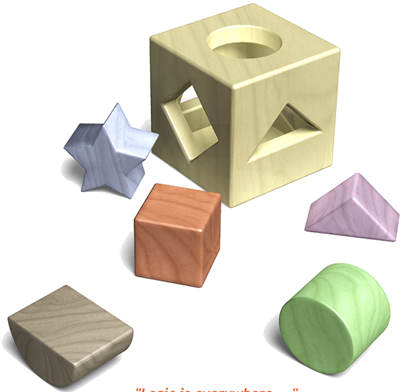
- **Data structures**

- **Algorithms**



*"Logic is everywhere ..."*

# Warm Up

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- **How is an array stored in Java?**

# Revision

- **Used Data Types**

# Clauses and Conjunctive Normal Forms

- **Definition**
  - A clause is a generalized disjunction $[L_1, \ldots, L_n]$, $n \geq 0$, where every $L_i$, $1 \leq i \leq n$, is a literal

# Clauses and Conjunctive Normal Forms

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- **Definition**
  - A formula is in **conjunctive normal form** (**clause form**, **CNF**) **iff** it is of the form $\langle C_1, \ldots, C_m \rangle$, $m \geq 0$, and every $C_j$, $1 \leq j \leq m$, is a clause

- **Agreement**
  - An interpretation for a formula $F$ can be represented as sequence of literals.

# Used Data Structures

- **(Multi-)Sets for clauses and the formula**

- **Sequences for the interpretation**

# Used Algorithms

- **Unit propagation**
- **Clause learning**

# Implementing Interpretations

# How to Implement an Interpretation

- **Given: the input formula $F$**
  - **with the variables $n = |\mathcal{R}_F|$**

- **For a clause, an interpretation $J$ is usually used**
  - **to test $J \models C$ for some clause, or**
  - **to compute $C|_J$.**

- **How to implement an interpretation?**

# Implement an Interpretation as Sequence

`space saving`

**array** S

```
contains (integer a)
  for i in S
    if a = i then return true
  return false

insert (integer a)
  if not contains(a) then
    append a to S

erase (integer a)
  if contains(a) then
    remove a from
```

# Implement an Interpretation as Sequence

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*contains* (integer ***a***)

  **for *i* in *S***
    **if *a* = *i* then return true**
  **return false**

*insert* (integer ***a***)

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`time saving`

**array** S
**array** T with ***n*** elements

*contains* (integer ***a***)

  **return** T[***a***]

*insert* (integer ***a***)

  **if not** contains(***a***) **then**
    append ***a*** to ***S***
    ***T*[*a*] = true**

*erase* (integer ***a***)

  **if** contains(***a***) **then**
    remove ***a*** from ***S***
    ***T*[*a*] = false**

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- **How about in case of an interpretation?**
  - **More particularly in the case of the CDCL algorithm?**

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- Definition
  - A clause $C$ is called a reason clause of a literal $x$ with respect to an interpretation $J$ if there is an interpretation $J'$ with $J = J'J''$ and the reduct $C|_{J'}$ with respect to the interpretation $J'$ is the unit clause $C|_{J'} = (x)$.
- For convenience we introduce a function that maps to the (set of) reason(s): $\texttt{reason}(F, J, x)$.

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# Conflict Literal

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- **Conflict Level**
  - The conflict level of a conflict clause $C$ with respect to an interpretation $J$ is the highest decision level of all the literals $x$ that occur in the clause.

# Implementing Unit Propagation

# Pseudo Code for Unit Propagation

***UP*** (CNF formula $\boldsymbol{F}$, interpretation $\boldsymbol{J}$)

**Input:** A formula $\boldsymbol{F}$ in CNF, an interpretation $\boldsymbol{J}$
**Output:** An extended interpretation $\boldsymbol{J}$

```
P := ()                      // start with empty interpretation
while (x) ∈ F|_JP do         // unit rule
    P := Px                  // extend propagated literals
return (JP)
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- **C source code is still different**

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|     $P := Px$ | // extend propagated literals |
|     $reason(x) = C$ | // set reason |
| **return** $(JP)$ | |

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# Implementing Clause Learning

# Clause Learning

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# Clause Learning

- **Properties of a learned clause**
- **Short**
- **Good backjump distance**
- **Should trigger unit propagation**
  - **Such a clause is called asserting**

# Clause Learning Algorithm

***ConflictAnalysis*** (CNF formula $\boldsymbol{F}$, interpretation $\boldsymbol{J}$, conflict clause $\boldsymbol{C}$)

**Input:** A formula $\boldsymbol{F}$ in CNF, an interpretation $\boldsymbol{J}$, clause $\boldsymbol{C}$
**Output:** A learned clause $\boldsymbol{D}$

```
D := C                                          // start with the conflict
while some condition do                         // depending on the wanted clause
   while J = J′L and ¬L ∉ D do                  // unit rule
      J = J′                                    // remove last literal from J
   if reason(F, J, L) ≠ ∅                       // depending on condition always true
      D := D ⊗ reason(F, J, L)                  // resolve with a reason
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- **Usually, pick first reason (the one stored during UP)**
- **Invariant: $C$ has at least two literals of the conflict level**
- **Invariant: $D$ is always falsified, $D|_J = []$.**

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  - **Decision clause**:
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  - **1st UIP clause (unique implication point):**
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- **1st UIP clause is constructed faster, and usually shorter**
- **Not discussed here: clause minimization**

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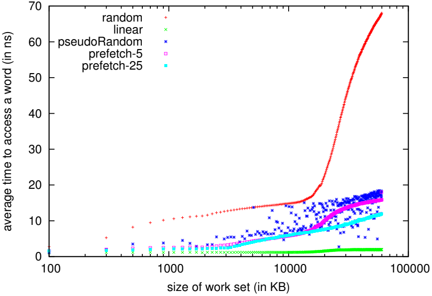
- ***D* is a set of literals**
- ***D* can be represented implicitely by an occurence array and *J***

# Choosing Data Structures

# Data structures

- Things to worry about for efficiency:
  - Number of memory accesses
  - Order of memory locations to be accessed



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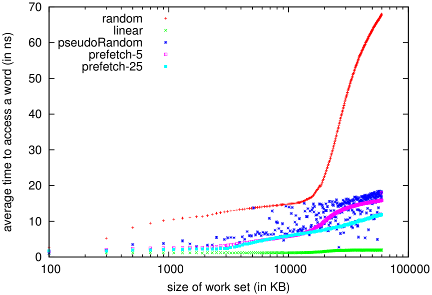


- **pseudoRandom: random cache line, multiple accesses**
- **prefetching: tell the memory where the X-th next access will be**

# Data structures

- Iterate through data structures linearly, use arrays
- Reduce number of memory accesses
  - Blocking Literal in watch list
- Store data about variables together in one block
  - Assignment, reason, decision level