Complexity Theory

Circuit Complexity

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Computational Logic

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Review

Computing with Circuits

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Motivation

Some questions:

- What can complexity theory tell us about parallel computation?
- Are there any meaningful complexity classes below LOGSPACE? Do they contain relevant problems?
- Even if it is hard to find a universal algorithm for solving all instances of a problem, couldn't it still be that there is a simple algorithm for every fixed problem size?

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$\leadsto$ circuit complexity provides some answers

Intuition: use circuits with logical gates to model computation

Boolean Circuits

Definition 17.1

A Boolean circuit is a finite, directed, acyclic graph where

- each node that has no predecessor is an input node
- each node that is not an input node is one of the following types of logical gate:
  - AND with two input wires
  - OR with two input wires
  - NOT with one input wire
- one or more nodes are designated output nodes

The outputs of a Boolean circuit are computed in the obvious way from the inputs.

$\leadsto$ circuits with k inputs and ℓ outputs represent functions $\{0, 1\}^k \to \{0, 1\}^\ell$

We often consider circuits with only one output.

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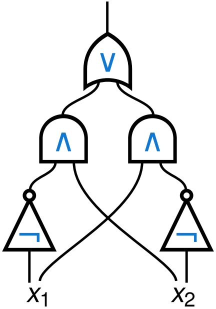
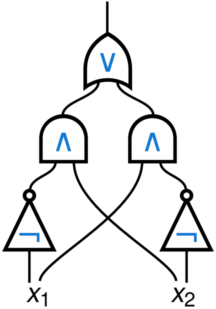
Example 1

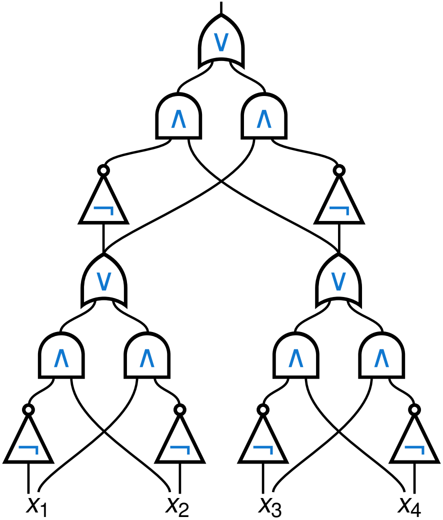


Example 1

XOR function:



Example 2



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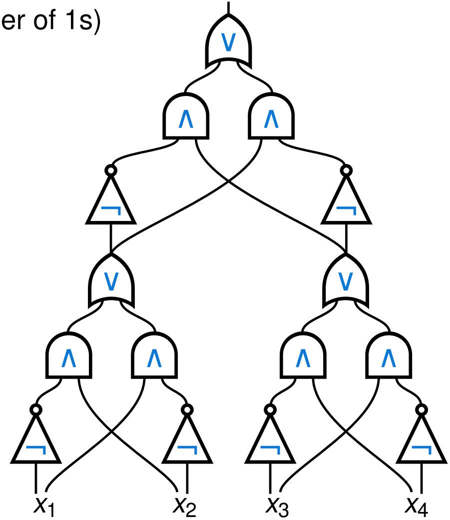
Example 2

Parity function with four inputs:
(true for odd number of 1s)

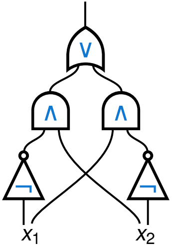


Alternative Ways of Viewing Circuits (1)

Propositional formulae

- propositional formulae are special circuits:
  each non-input node has only one outgoing wire
- each variable corresponds to one input node
- each logical operator corresponds to a gate
- each sub-formula corresponds to a wire

$((\neg x_1 \wedge x_2) \vee (x_1 \wedge \neg x_2))$



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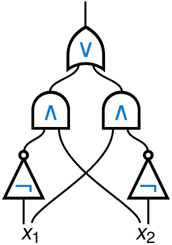
Alternative Ways of Viewing Circuits (2)

Straight-line programs

- are programs without loops and branching (if, goto, for, while, etc.)
- that only have Boolean variables
- and where each line can only be an assignment with a single Boolean operator

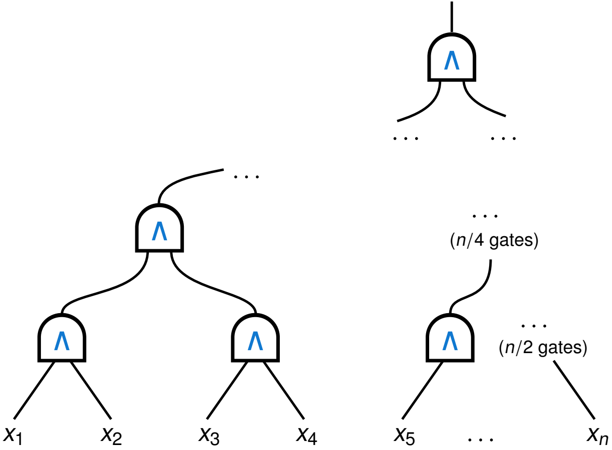
$\rightsquigarrow$ *n*-line programs correspond to *n*-gate circuits




```
01 z1 := ¬x1
02 z2 := ¬x2
03 z3 := z1 ∧ x2
04 z4 := z2 ∧ x1
05 return z3 ∨ z4
```

Example: Generalised AND

The function that tests if all inputs are 1 can be encoded by combining binary AND gates:



- works similarly for OR gates
- number of gates: $n - 1$
- we can use n-way AND and OR (keeping the real size in mind)

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Solving Problems with Circuits

Circuits are not universal: fixed number of inputs!
How can they solve arbitrary problems?

Definition 17.2

A circuit family is an infinite list $\mathcal{C} = C_1, C_2, C_3, \ldots$ where each C_i is a Boolean circuit with i inputs and one output.
We say that $\mathcal{C}$ decides a language $\mathcal{L}$ (over $\{0, 1\}$) if

$$w \in \mathcal{L} \qquad \text{if and only if} \qquad C_n(w) = 1 \text{ for } n = |w|.$$

Example 17.3

The circuits we gave for generalised AND are a circuit family that decides the language $\{1^n \mid n \geq 1\}$.

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Circuit Complexity

To measure difficulty of problems solved by circuits, we can count the number of gates needed:

Definition 17.4

The size of a circuit is its number of gates.

Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. A circuit family C is f-size bounded if each of its circuits C_n is of size at most $f(n)$.

$\textsc{Size}(f(n))$ is the class of all languages that can be decided by an $O(f(n))$-size bounded circuit family.

Example 17.5

Our circuits for generalised AND show that $\{1^n \mid n \geq 1\} \in \textsc{Size}(n)$.

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Examples

Many simple operations can be performed by circuits of polynomial size:

- Boolean functions such as parity (=sum modulo 2), sum modulo n, or majority
- Airhtmetic operations such as addition, subtraction, multiplication, division (taking two fixed-arity binary numbers as inputs)
- Many matrix operations

See exercsie for some more examples

Polynomial Circuits

Polynomial Circuits

A natural class of problems to consider are those that have polynomial circuit families:

Definition 17.6

$\mathrm{P}_{/\mathrm{poly}} = \bigcup_{d \geq 1} \textsc{Size}(n^d)$.

Note: A language is in $\mathrm{P}_{/\mathrm{poly}}$ if it is solved by some polynomial-sized circuit family. There may not be a way to compute (or even finitely represent) this family.

How does $\mathrm{P}_{/\mathrm{poly}}$ relate to other classes?

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Quadratic Circuits for Deterministic Time

Theorem 17.7

For $f(n) \geq n$, we have $\mathrm{DT{\scriptsize IME}}(f) \subseteq \mathrm{S{\scriptsize IZE}}(f^2)$.

Proof sketch (see also Sipser, Theorem 9.30).

- We can represent the DTIME computation as in the proof of Theorem 15.5: as a list of configurations encoded as words

$$* \; \sigma_1 \; \cdots \; \sigma_{i-1} \; \langle q, \sigma_i \rangle \; \sigma_{i+1} \; \cdots \; \sigma_m \; *$$

  of symbols from the set $\Omega = \{*\} \cup \Gamma \cup (Q \times \Gamma)$. $\rightsquigarrow$ tableau with $O(f^2)$ cells.
- We can describe each cell with a list of bits (wires in a circuit).
- We can compute one configuration from its predecessor by $O(f)$ circuits (idea: compute the value of each cell from its three upper neighbours as in Theorem 15.5)
- Acceptance can be checked by assuming that the TM returns to a unique configuration position/state when accepting

□

From Polynomial Time to Polynomial Size

From $\mathrm{DT{\small IME}}(f) \subseteq \mathrm{S{\small IZE}}(f^2)$ we get:

Corollary 17.8

$\mathrm{P} \subseteq \mathrm{P}_{/\mathrm{poly}}$.

This sugggests another way of approaching the P vs. NP question:

If any language in NP is not in $\mathrm{P}_{/\mathrm{poly}}$, then $\mathrm{P} \neq \mathrm{NP}$.
(but nobody has found any such language yet)

CIRCUIT-SAT

Input: A Boolean Circuit C with one output.

Problem: Is there any input for which C returns 1?

Theorem 17.9

CIRCUIT-SAT *is* NP*-complete.*

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Theorem 17.9

CIRCUIT-SAT *is* NP*-complete.*

Proof.

Inclusion in NP is easy (just guess the input).

For NP-hardness, we use that NP problems are those with a P-verifier:

- The DTM simulation of Theorem 17.7 can be used to implement a verifier (input: $(w\#c)$ in binary)
- We can hard-wire the w-inputs to use a fixed word instead (remaining inputs: c)
- The circuit is satisfiable iff there is a certificate for which the verifier accepts w

□

A New Proof for Cook-Levin

Theorem 17.10

3SAT *is* NP*-complete.*

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Proof.

Membership in NP is again easy (as before).

For NP-hardness, we express the circuit that was used to implement the verifier in Theorem 17.9 as propositional logic formula in 3-CNF:

- Create a propositional variable X for every wire in the circuit
- Add clauses to relate input wires to output wires, e.g., for AND gate with inputs X_1 and X_2 and output X_3, we encode $(X_1 \wedge X_2) \leftrightarrow X_3$ as:

$$(\neg X_1 \vee \neg X_2 \vee X_3) \wedge (X_1 \vee \neg X_3) \wedge (X_2 \vee \neg X_3)$$

- Fixed number of clauses per gate = linear size increase
- Add a clause (X) for the output wire X.

Is $\mathrm{P} = \mathrm{P}_{/\mathrm{poly}}$?

We showed $\mathrm{P} \subseteq \mathrm{P}_{/\mathrm{poly}}$. Does the converse also hold?

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No!

Theorem 17.11

$\mathrm{P}_{/\mathrm{poly}}$ *contains undecidable problems.*

Proof.

We define the unary Halting problem as the (undecidable) language:

$$\textsc{UHalt} := \{1^n \mid \text{the binary encoding of } n \text{ encodes a pair } \langle \mathcal{M}, w\rangle \text{ where } \mathcal{M} \text{ is a TM that halts on word } w\}$$

For a number $1^n \in \textsc{UHalt}$, let C_n be the circuit that computes a generalised AND of all inputs. For all other numbers, let C_n be a circuit that always returns 0. The circuit family $C_1, C_2, C_3, \ldots$ accepts $\textsc{UHalt}$. □

Uniform Circuit Families

$\mathrm{P}_{/\mathrm{poly}}$ too powerful, since we do not require the circuits to be computable. We can add this:

Definition 17.12

A circuit family $C_1, C_2, C_3, \ldots$ is log-space-uniform if there is a log-space computable function that maps words 1^n to (an encoding of) C_n.

(We could also define similar notions of uniformity for other complexity classes.)

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(We could also define similar notions of uniformity for other complexity classes.)

Theorem 17.13

The class of all languages that are accepted by a log-space-uniform circuit family of polynomial size is exactly P*.*

Proof sketch.

A detailed analysis shows that out that our earlier reduction of P DTMs to circuits is log-space-uniform. Conversely, a polynomial-time procedure can be obtained by first computing a suitable circuit (in log-space) and then evaluating it (in polynomial time). □

Turing Machines That Take Advice

One can also describe $\mathrm{P}_{/\mathrm{poly}}$ using TMs that take "advice":

Definition 17.14

Consider a function $a : \mathbb{N} \to \mathbb{N}$. A language $\mathcal{L}$ is accepted by a Turing Machine $\mathcal{M}$ with *a* bits of advice if there is a sequence of advice strings $\alpha_0, \alpha_1, \alpha_2, \ldots$ of length $|\alpha_i| = a(i)$ and $\mathcal{M}$ accepts inputs of the form $(w\#a_{|w|})$ if ad only if $w \in \mathcal{L}$.

$\mathrm{P}_{/\mathrm{poly}}$ is equivalent to the class of problems that can be solved by a $\mathrm{PT}\textsc{ime}$ TM that takes a polynomial amount of "advice".

(This is where the notation $\mathrm{P}_{/\mathrm{poly}}$ comes from.)

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$\mathrm{P}_{/\mathrm{poly}}$ and NP

We showed $\mathrm{P} \subseteq \mathrm{P}_{/\mathrm{poly}}$. Does $\mathrm{NP} \subseteq \mathrm{P}_{/\mathrm{poly}}$ also hold?

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Nobody knows

Theorem 17.15 (Karp-Lipton Theorem)

If $\mathrm{NP} \subseteq \mathrm{P}_{/\mathrm{poly}}$ *then* $\mathrm{PH} = \Sigma_2^p$.

Proof sketch (see Arora/Barak Theorem 6.19).

- if $\mathrm{NP} \subseteq \mathrm{P}_{/\mathrm{poly}}$ then there is a polysize circuit family solving `Sat`
- Using this, one can argue that there is also a polysize circuit family that computes the lexicographically first" satisfying assignment (k output bits for k variables)
- A Π_2-QBF formula $\forall \mathbf{X}.\exists \mathbf{Y}.\varphi$ is true if, for all values of $\mathbf{X}$, $\varphi[\mathbf{X}]$ is satisfiable.
- In Σ_2^{P}, we can: (1) guess the polysize circuit for SAT, (2) check for all values of $\mathbf{X}$ if its output is really a satisfying assignment (to verify the guess)
- This solves Π_2^{P}-hard problems in Σ_2^{P}
- But then the Polynomial Hierarchy collapses at Σ_2^{P}, as claimed.

$\square$

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$\mathrm{P}_{/\mathrm{poly}}$ and $\textsc{ExpTime}$

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Theorem 17.16 (Meyer's Theorem)

If $\textsc{ExpTime} \subseteq \mathrm{P}_{/\mathrm{poly}}$ *then* $\textsc{ExpTime} = \mathrm{PH} = \Sigma_2^p$.

See [Arora/Barak, Theorem 6.20] for a proof sketch.

Corollary 17.17

If $\textsc{ExpTime} \subseteq \mathrm{P}_{/\mathrm{poly}}$ *then* $\mathrm{P} \neq \mathrm{NP}$.

Proof.

If $\textsc{ExpTime} \subseteq \mathrm{P}_{/\mathrm{poly}}$ then $\textsc{ExpTime} = \Sigma_2^p$ (Meyer's Theorem).
By the Time Hierarchy Theorem, $\mathrm{P} \neq \textsc{ExpTime}$, so $\mathrm{P} \neq \Sigma_2^p$.
So the Polynomial Hierarchy doesn't collapse completely, and $\mathrm{P} \neq \mathrm{NP}$. □

How Big a Circuit Could We Need?

We should not be surprised that $\mathrm{P}_{/\mathrm{poly}}$ is so powerful:
exponential circuit families are already enough to accept any language

Exercise: show that every Boolean function over n variables can be expressed by a circuit of size $\leq n2^n$.

It turns out that these exponential circuits are really needed:

Theorem 17.18 (Shannon 1949 (!))

For every n, there is a function $\{0,1\}^n \to \{0,1\}$ that cannot be computed by any circuit of size $2^n/(10n)$.

In fact, one can even show: almost every Boolean function requires circuits of size $> 2^n/(10n)$ – and is therefore not in $\mathrm{P}_{/\mathrm{poly}}$

Is any of these functions in NP? Or at least in $\textsc{Exp}$? Or at least in $\textsc{NExp}$?

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Is any of these functions in NP? Or at least in $\mathrm{E}\textsc{xp}$? Or at least in $\mathrm{NE}\textsc{xp}$?
Nobody knows