Exercise 6.1. Find the fault in the following proof of $P \neq NP$.

Assume that $P = NP$. Then $SAT \in P$ and thus there exists a $k \in \mathbb{N}$ such that $SAT \in DTime(n^k)$. Because every language in $NP$ is polynomial-time reducible to $SAT$ we have $NP \subseteq DTime(n^k)$. It follows that $P \subseteq DTime(n^k)$. But by the Time Hierarchy Theorem there exist languages in $DTime(n^{k+1})$ that are not in $DTime(n^k)$, contradicting $P \subseteq DTime(n^k)$. Therefore, $P \neq NP$.

Exercise 6.2. Show the following.

1. $\text{Time}(2^n) = \text{Time}(2^{n+1})$
2. $\text{Time}(2^n) \subset \text{Time}(2^{2n})$
3. $\text{NTIME}(n) \subset \text{PSPACE}$

Exercise 6.3. Show that there exists a function that is not time-constructible.

Exercise 6.4. Consider the function $\text{pad} : \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \#^*$ defined as $\text{pad}(s, \ell) = s\#^j$, where $j = \max(0, \ell - |s|)$. In other words, $\text{pad}(s, \ell)$ adds enough copies of $\#$ to the end of $s$ so that the length is at least $\ell$.

For some language $A \subseteq \Sigma^*$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ define $\text{pad}(A, f) = \{ \text{pad}(s, f(|s|)) \mid s \in A \}$.

1. Show that if $A \in \text{DTIME}(n^6)$, then $\text{pad}(A, n^2) \in \text{DTIME}(n^3)$.
2. Show that if $\text{NEXPTime} \neq \text{EXPTime}$, then $P \neq NP$.
3. Show for every $A \subseteq \Sigma^*$ and every $k \in \mathbb{N}$ that $A \in P$ if and only if $\text{pad}(A, n^k) \in P$.
4. Show that $P \neq \text{DSPACE}(n)$.
5. Show that $NP \neq \text{DSPACE}(n)$. 