Exercise 7: Query Optimisation and First-Order Query Expressivity

Database Theory

2025-05-27

Lukas Gerlach, Maximilian Marx, Markus Krötzsch

Exercise 1

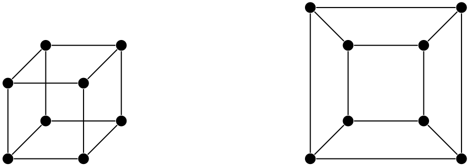
Exercise. For the following pairs of structures, find the maximal r such that $\mathcal{I} \sim_r \mathcal{J}$:

(i)

(ii)

(iii)

(iv)



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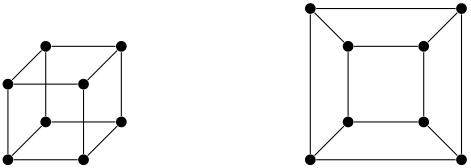

Solution.

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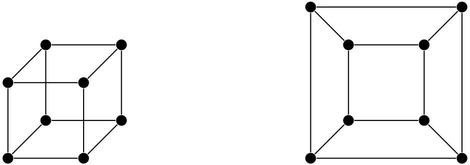

Solution.

(i) $r \leq 1$,

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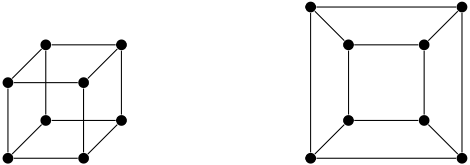

Solution.

(i) $r \leq 1$,

(ii) $r \leq 2$,

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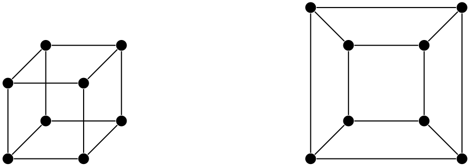

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(i) $r \leq 1$,

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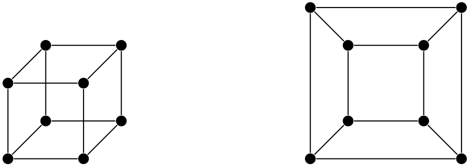

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(i) $r \leq 1$,

(ii) $r \leq 2$,

(iii) $r = 0$, and

(iv) $r \geq 0$.

Exercise 2

Exercise. A *linear order* is a relational structure with one binary relational symbol $\leq$ that is interpreted as a reflexive, asymmetric, transitive and total relation over the domain. Up to renaming of domain elements there is exactly one linear order for every finite domain, which can be depicted as a chain of elements. We denote the linear order of size n by $\mathcal{L}_n$. For example:

$\mathcal{L}_6 : 1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6$ and $\mathcal{L}_7 : 1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6 \leq 7$

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The following are equivalent:

- $\mathcal{L}_m \sim_r \mathcal{L}_n$, *and*
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Solution.

1. $r \leq 2$.
2. $n \geq 2^r - 1 \implies r \leq \lfloor \log_2(n+1) \rfloor$.

Exercise 3

Exercise. A graph is *planar* if it can be drawn on the plane without intersections of edges. For example, the following graph A is planar, while graph B is not:

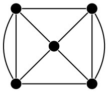

Figure: A

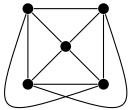

Figure: B

1. Can the graphs A and B be distinguished by a FO query?
2. Show that planarity is not FO-definable by using locality.

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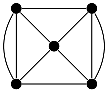

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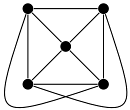

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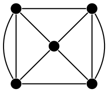

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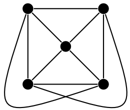

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1. This query matches B but not A:

$$\exists x, y, z, w, v.\ \mathrm{E}(x, y) \wedge \mathrm{E}(y, z) \wedge \mathrm{E}(z, w) \wedge \mathrm{E}(w, x) \wedge \mathrm{E}(x, v) \wedge \mathrm{E}(y, v) \wedge \mathrm{E}(z, v) \wedge \mathrm{E}(w, v) \wedge \mathrm{E}(x, z) \wedge \mathrm{E}(y, w)$$

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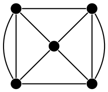

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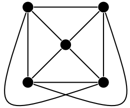

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2. For φ with quantifier rank r, consider counterexamples of size $d = 3^r$: