TECHNISCHE UNIVERSITÄT DRESDEN

Artificial Intelligence, Computational Logic

PROBLEM SOLVING AND SEARCH IN ARTIFICIAL INTELLIGENCE

Lecture 8 Constraint Satisfaction Problems

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DRESDEN concept
Exzellenz aus Wissenschaft und Kultur

Agenda

1. Introduction
2. Uninformed Search versus Informed Search (Best First Search, A* Search, Heuristics)
3. Local Search, Stochastic Hill Climbing, Simulated Annealing
4. Tabu Search
5. Answer-set Programming (ASP)
6. Constraint Satisfaction Problems (CSP)
7. Evolutionary Algorithms/ Genetic Algorithms
8. Structural Decomposition Techniques (Tree/Hypertree Decompositions)

Constraint satisfaction problems (CSPs)

- Standard search problem:
  - state is a “black box”—any old data structure that supports goal test, eval, successor
- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a **formal representation language**
- Allows useful **general-purpose** algorithms with more power than standard search algorithms
- Main idea: eliminate large portions of search space all at once by identifying variable/value combinations that violate constraints

Defining CSPs

Constraint Satisfaction Problem (CSP)

A CSP is defined as a tuple $\mathcal{C} = \langle X, D, C \rangle$, with

- X a set of variables, $\{X_1, \ldots, X_n\}$.
- D a set of domains, $\{D_1, \ldots, D_n\}$, for each variable.
- C a set of constraints that specify allowable combinations of values.

- Each domain D_i consists of a set of allowable values, $\{v_1, \ldots, v_k\}$ for variable X_i.
- Each constraint C_i consists of a pair $\langle scope, rel \rangle$, where $scope$ is a tuple of variables in the constraint, and rel defines the possible values.
- A relation can be
  - an explicit list of all tuples of values satisfying the constraint, or
  - an abstract relation.

Defining CSPs ctd.

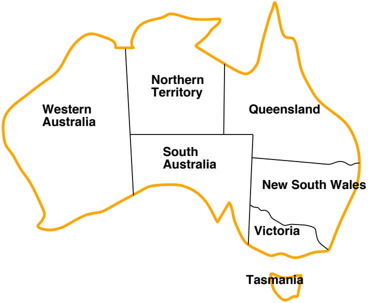
If X_1 and X_2 both have domain $\{A, B\}$, the constraint saying they have different values can be written as:

- $\langle (X_1, X_2), [(A, B), (B, A)] \rangle$, or
- $\langle (X_1, X_2), X_1 \neq X_2 \rangle$.

To solve a CSP, we define a state space and the notion of a solution.

- Each state in a CSP is defined by an assignment of values to some (or all variables), $\{X_i = v_i, X_j = v_j, \ldots\}$.
- An assignment is consistent if it does not violate any constraints.
- A complete assignment has a value assigned to each variable.
- A solution is a consistent, complete assignment.
- A partial assignment is one that assigns values to only some of the variables.

Example: Map-Coloring

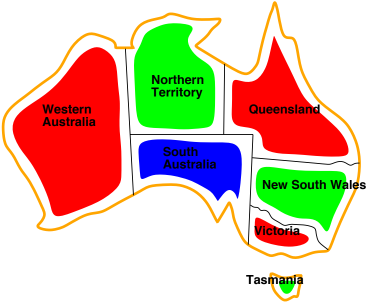


Variables WA, NT, Q, NSW, V, SA, T

Domains $D_i = \{red, green, blue\}$

Constraints: adjacent regions must have different colors e.g., $WA \neq NT$ (if the language allows this), or
$(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \ldots\}$

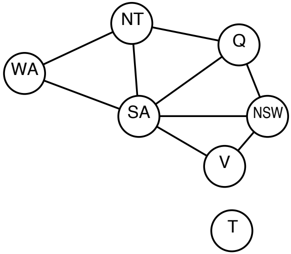
Example: Map-Coloring ctd.



Solutions are assignments satisfying all constraints, e.g.,
$\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$

Constraint Graph

Binary CSP: each constraint relates at most two variables
Constraint graph: nodes are variables, arcs show constraints



General-purpose CSP algorithms use the graph structure
to speed up search. E.g., Tasmania is an independent sub-problem!

Varieties of CSPs

Discrete variables

- finite domains; size $d \implies O(d^n)$ complete assignments
  - e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
- infinite domains (integers, strings, etc.)
  - e.g., job scheduling, variables are start/end days for each job
  - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
  - linear constraints solvable, nonlinear undecidable

Continuous variables

- e.g., start/end times for Hubble Telescope observations
- linear constraints solvable in poly time by LP methods

Varieties of constraints

Unary constraints involve a single variable,
e.g., $SA \neq green$

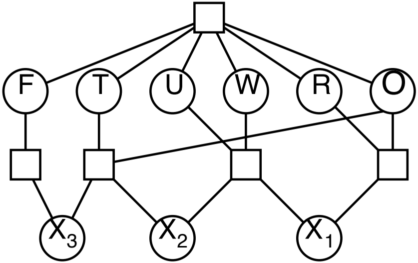
Binary constraints involve pairs of variables,
e.g., $SA \neq WA$

Higher-order constraints involve 3 or more variables,
e.g., cryptarithmetic column constraints

Preferences (soft constraints), e.g., red is better than $green$
often representable by a cost for each variable assignment
$\rightarrow$ constrained optimization problems

Example: Cryptarithmetic

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$



Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints alldiff(F, T, U, W, R, O), $O + O = R + 10 \cdot X_1$, etc.

Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Hardware configuration
- Spreadsheets
- Transportation scheduling
- Factory scheduling
- Floorplanning

Notice that many real-world problems involve real-valued variables

Constraint Propagation: Inference in CSPs

In regular state-space search, an algorithm can only perform search. In CSPs there is a choice

- an algorithm can search (choose a new variable assignment form several possibilities), or
- do a specific type of inference called constraint propagation:
  - using the constraints to reduce the number of legal values for a variable
  - this can reduce the legal values for another variable,
  - and so on.

Constraint propagation may be

- intertwined with search, or
- done as a pre-processing step (could solve the whole problem; no search is required).

Constraint Propagation

The key idea is local consistency.

- Treat each variable as a node in a graph.
- Each binary constraint represents an arc.
- Enforcing local consistency in each part of the graph eliminates inconsistent values throughout the graph.

Different types of local consistency:

- Node consistency
- Arc consistency
- Path consistency

Node Consistency

Node consistency

A variable X is node-consistent if all values in the domain of X satisfy the unary constraints of X. A CSP is node-consistent if every variable is node consistent.

Example

South Australia dislikes green.

- Variable SA starts with $\{red, green, blue\}$,
- make it node consistent by eliminating $green$,
- reduced domain of SA is $\{red, blue\}$.

Arc Consistency

Arc consistency

A variable is arc-consistent if every value in its domain satisfies the variable's binary constraints. X_i is arc-consistent wrt. X_j if for every value in D_i there is some value in D_j that satisfies the binary constraint on the arc (X_i, X_j). A CSP is arc-consistent if every variable is arc-consistent with every other variable.

Example

Consider the constraint $Y = X^2$, where the domain of both X and Y is the set of digits. We can write the constraint explicitly as

$$\langle (X, Y), \{(0,0), (1,1), (2,4), (3,9)\} \rangle.$$

To make X arc-consistent wrt. Y, we reduce X's domain to $\{0, 1, 2, 3\}$. We also reduce Y's domain to $\{0, 1, 4, 9\}$ and the CSP is arc-consistent.

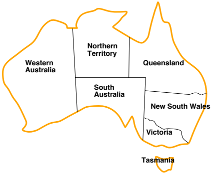
Path Consistency

- Arc consistency can reduce domains of variables and sometimes find a solution (or failure).
- But for other networks, arc consistency fails to make enough inferences.
- Example of map coloring of Australia with two colors.

Path consistency

A two-variable set $\{X_i, X_j\}$ is path-consistent wrt. a third variable X_m if, for every assignment $\{X_i = a, X_j = b\}$ consistent with constraints on $\{X_i, X_j\}$, there is an assignment to X_m that satisfies the constraints on $\{X_i, X_m\}$ and $\{X_m, X_j\}$.

Example: Path Consistency



Consider two-coloring of Australia. We make $\{WA, SA\}$ path consistent wrt. NT.

- Start by enumerating the consistent assignments to the set.
  - $\{WA = red, SA = blue\}$
  - $\{WA = blue, SA = red\}$
- With both assignments NT can be neither red nor $blue$.
- Eliminate both assignments.
- Thus, there is no solution to the problem.

Standard search formulation (incremental)

- Let's start with the straightforward, dumb approach, then fix it
- States are defined by the values assigned so far

Initial state: the empty assignment, $\{\}$

Successor function: assign a value to an unassigned variable that does not conflict with current assignment.
$\Longrightarrow$ fail if no legal assignments (not fixable!)

Goal test: the current assignment is complete

1. This is the same for all CSPs! ☺
2. Every solution appears at depth n with n variables
   $\Longrightarrow$ use depth-first search
3. Path is irrelevant, so can also use complete-state formulation
4. $b = (n - \ell)d$ at depth ℓ, hence $n!d^n$ leaves!!!! ☹

Backtracking search

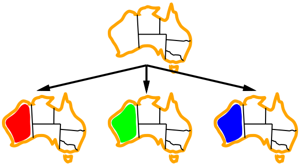
- Variable assignments are commutative, i.e.,
  [$WA = red$ then $NT = green$] same as [$NT = green$ then $WA = red$]
- Only need to consider assignments to a single variable at each node
  $\Longrightarrow$ $b = d$ and there are d^n leaves
- Depth-first search for CSPs with single-variable assignments is called backtracking search
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for $n \approx 25$

Backtracking search

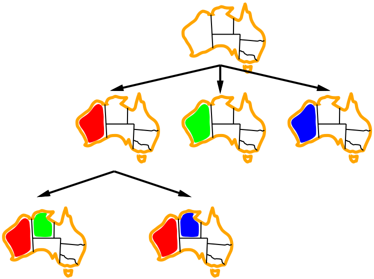
```
function Backtracking-Search(csp) returns solution/failure
   return Recursive-Backtracking({ }, csp)

function Recursive-Backtracking(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← Select-Unassigned-Variable(Variables[csp], assignment, csp)
   for each value in Order-Domain-Values(var, assignment, csp) do
       if value is consistent with assignment given Constraints[csp] then
           add {var = value} to assignment
           result ← Recursive-Backtracking(assignment, csp)
           if result ≠ failure then return result
           remove {var = value} from assignment
   return failure
```

Backtracking example

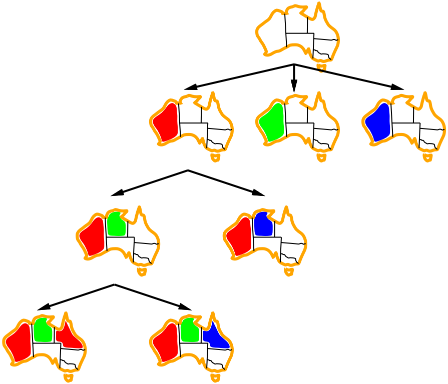
Backtracking example



Backtracking example



Backtracking example



Improving backtracking efficiency

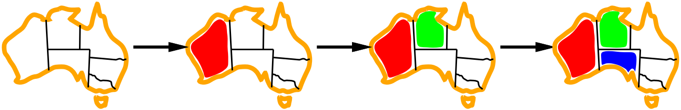
General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

Minimum remaining values

Minimum remaining values (MRV):

- choose the variable with the fewest legal values

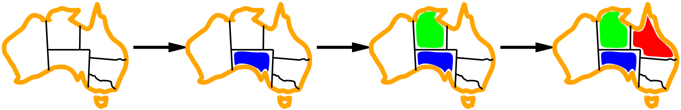


Degree heuristic

Tie-breaker among MRV variables

Degree heuristic:

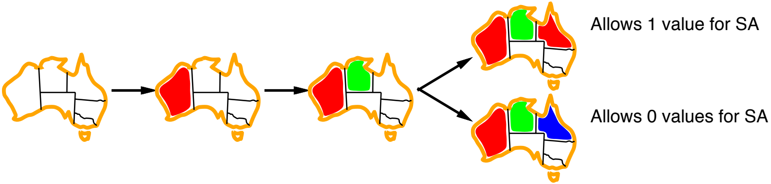
- choose the variable with the most constraints on remaining variables



Least constraining value

Given a variable, choose the least constraining value:

- the one that rules out the fewest values in the remaining variables

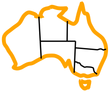


Combining these heuristics makes 1000 queens feasible

Forward checking

Idea:

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values



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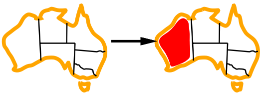

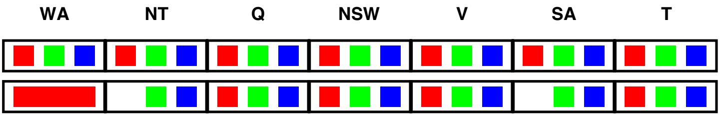

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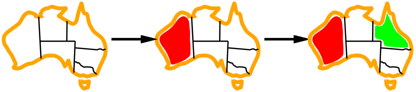

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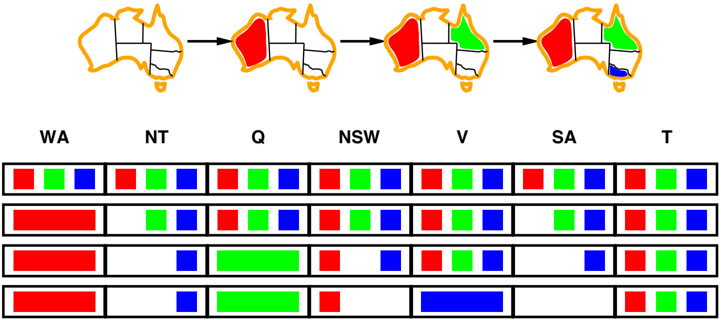
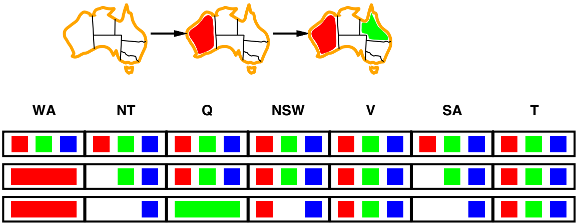


Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



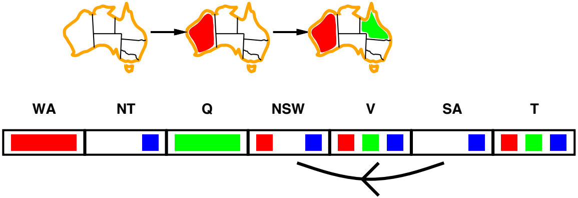
- *NT* and *SA* cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally

Arc consistency

Simplest form of propagation makes each arc consistent

$X \to Y$ is consistent iff

for **every** value x of X there is **some** allowed y



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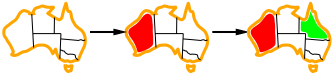

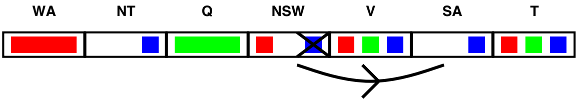

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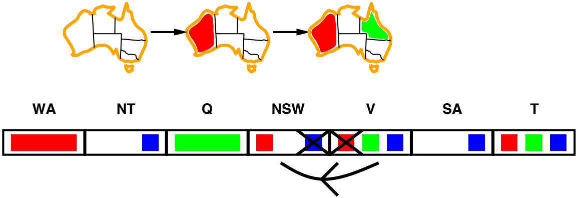

- If X loses a value, neighbors of X need to be rechecked

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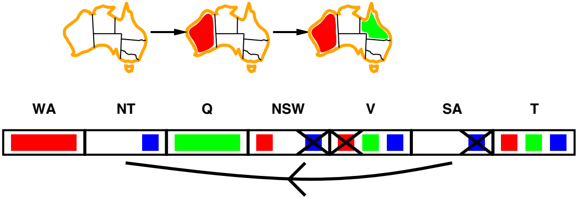


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a pre-processor or after each assignment

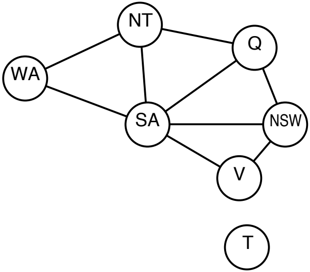
Arc consistency algorithm

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (X_i, X_j) ← Remove-First(queue)
      if Remove-Inconsistent-Values(X_i, X_j) then
         for each X_k in Neighbors[X_i] do
            add (X_k, X_i) to queue
```

```
function Remove-Inconsistent-Values(X_i, X_j) returns true iff succeeds
   removed ← false
   for each x in Domain[X_i] do
      if no value y in Domain[X_j] allows (x,y) to satisfy the constraint X_i ↔ X_j
         then delete x from Domain[X_i];  removed ← true
   return removed
```

Problem structure

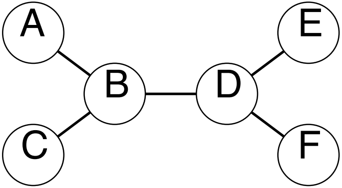


- Tasmania and mainland are independent sub-problems
- Identifiable as connected components of constraint graph

Problem structure ctd.

- Suppose each subproblem has c variables out of n total
- Worst-case solution cost is $n/c \cdot d^c$, **linear** in n
- E.g., $n = 80$, $d = 2$, $c = 20$
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $4 \cdot 2^{20}$ = 0.4 seconds at 10 million nodes/sec

Tree-structured CSPs



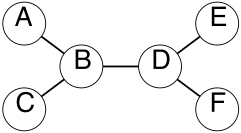
Theorem

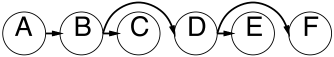
If the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time.

- Compare to general CSPs, where worst-case time is $O(d^n)$
- This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.

Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering

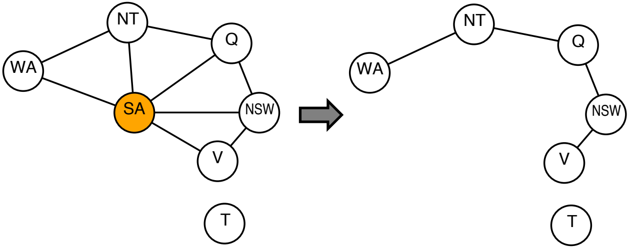




2. For j from n down to 2, apply RemoveInconsistent$(Parent(X_j), X_j)$
3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

Nearly tree-structured CSPs

- Conditioning: instantiate a variable, prune its neighbors' domains



- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size $c \implies$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

Iterative algorithms for CSPs

- Hill-climbing, simulated annealing typically work with “complete” states, i.e., all variables assigned
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators **reassign** variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hillclimb with $h(n)$ = total number of violated constraints

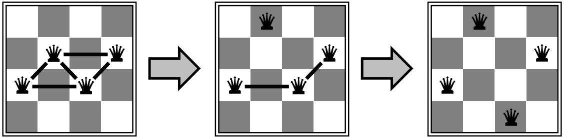
Example: 4-Queens

States: 4 queens in 4 columns ($4^4 = 256$ states)

Operators: move queen in column

Goal test: no attacks

Evaluation: $h(n)$ = number of attacks



h = 5 **h = 2** **h = 0**

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- The CSP representation allows analysis of problem structure
- Tree-structured CSPs can be solved in linear time
- Iterative min-conflicts is usually effective in practice

References

Stuart J. Russell and Peter Norvig.
Artificial Intelligence - A Modern Approach (3. edition). Pearson Education, 2010.