Maximally Permissive Coordination Supervisory Control – Towards Necessary and Sufficient Conditions

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Abstract—In this paper, we further develop the coordination control framework for discrete-event systems with both complete and partial observation. A new weaker sufficient condition for the computation of the supremal conditionally controllable sublanguage is presented. This result is then used for the computation of the supremal conditionally controllable and conditionally normal sublanguage. The paper further generalizes the previous study by considering general, non-prefix-closed languages.

I. INTRODUCTION

Large scale discrete-event systems (DES) are often formed in a compositional way as a synchronous or asynchronous composition of smaller components, typically automata (or 1-safe Petri nets that can be viewed as products of automata). Supervisory control theory was proposed in [10] for automata as a formal approach that aims to solve the safety issue and nonblockingness.

A major issue is the computational complexity of the centralized supervisory control design, because the global system has an exponential number of states in the number of components. Therefore, a modular supervisory control of DES based on a compositional (local) control synthesis has been introduced and developed by many authors. Structural conditions have been derived for the local control synthesis to equal the global control synthesis in the case of both local and global specification languages.

Specifications are mostly defined over the global alphabet, which means that the global specifications are more relevant than the local specifications. However, several restrictive conditions have to be imposed on the modular plant such as mutual controllability (and normality) of local plant languages for maximal permissiveness of modular control, and other conditions are required for nonblockingness.

For that reason, a coordination control approach was proposed for modular DES in [8] and further developed in [6]. Coordination control can be seen as a reasonable trade-off between a purely modular control synthesis, which is in some cases unrealistic, and a global control synthesis, which is naturally prohibitive for high complexity reasons. The concept of a coordinator is useful for both safety and nonblockingness. The complete supervisor then consists of the coordinator, its supervisor, and the local supervisors for the subsystems. In [8], necessary and sufficient conditions are formulated for nonblockingness and safety, and a sufficient condition is formulated for the maximally permissive control synthesis satisfying a global specification using a coordinator. Later, in [6], a procedure for a distributive computation of the supremal conditionally controllable sublanguage of a given specification has been proposed. We have extended coordination control for non-prefix-closed specification languages in [7] and for partial observations in [4].

In this paper, we first propose a new sufficient condition for a distributive computation of the supremal conditionally controllable sublanguages. We show that it generalizes (is weaker than) both conditions we have introduced earlier in [7] and [6]. Then we revise (simplify) the concepts of conditional observability and conditional normality and present new sufficient conditions for a distributive computation of the supremal conditionally controllable and conditionally normal sublanguage.

The paper is organized as follows. The next section recalls the basic concepts from the algebraic language theory that are needed in this paper. Our coordination control framework is briefly recalled in Section III. In Section IV new results in coordination control with complete observations are presented: a new, weaker, sufficient condition for distributed computation of supremal conditionally controllable sublanguages. Section V is dedicated to coordination control with partial observations, where the main concepts are simplified. Concluding remarks are in Section VI.

II. PRELIMINARIES

We now briefly recall the elements of supervisory control theory. The reader is referred to [1] for more details. Let \( \Sigma \) be a finite nonempty set of events, and let \( \Sigma^* \) denote the set of all finite words (strings) over \( \Sigma \). The empty word is denoted by \( \epsilon \). Let \( |\Sigma| \) denote the cardinality of \( \Sigma \).

A generator is a quintuple \( G = (Q, \Sigma, f, q_0, Q_m) \), where \( Q \) is a finite nonempty set of states, \( \Sigma \) is an event set, \( f : Q \times \Sigma \to Q \) is a partial transition function, \( q_0 \in Q \) is the initial state, and \( Q_m \subseteq Q \) is the set of marked states. In the usual way, the transition function \( f \) can be extended to the domain \( Q \times \Sigma^* \) by induction. The behavior of \( G \) is described in terms of languages. The language generated by \( G \) is the set \( L(G) = \{ s \in \Sigma^* | f(q_0,s) \in Q \} \) and the language marked by \( G \) is the set \( L_m(G) = \{ s \in \Sigma^* | f(q_0,s) \in Q_m \} \subseteq L(G) \).

A (regular) language \( L \) over an event set \( \Sigma \) is a set \( L \subseteq \Sigma^* \) such that there exists a generator \( G \) with \( L_m(G) = L \). The
prefix closure of $L$ is the set $\bar{L} = \{w \in \Sigma^* \mid \text{there exists } u \in \Sigma^* \text{ such that } wu \in L\}$; $L$ is prefix-closed if $L = \bar{L}$.

A (natural) projection $P: \Sigma^* \rightarrow \Sigma_o^*$, for some $\Sigma_o \subseteq \Sigma$, is a homomorphism defined so that $P(a) = \varepsilon$, for $a \in \Sigma \setminus \Sigma_o$, and $P(a) = a$, for $a \in \Sigma_o$. The inverse image of $P$, denoted by $P^{-1} : \Sigma_o^* \rightarrow \Sigma^*$, is defined as $P^{-1}(a) = \{w \in \Sigma^* \mid P(w) = a\}$. The definitions can naturally be extended to languages. The projection of a generator $G$ is a generator $P(G)$ whose behavior satisfies $L(P(G)) = P(L(G))$ and $L_m(P(G)) = P(L_m(G))$.

A generator $G$ is a structure $(G, \Sigma_c, P, \Gamma)$, where $G$ is a generator over $\Sigma$, $\Sigma_c \subseteq \Sigma$ is the set of controllable events, $\Sigma_u \subseteq \Sigma$ is the set of uncontrollable events, $\Sigma_u = \Sigma \setminus \Sigma_o$ is the set of uncontrollable events, $\Gamma = \{\gamma \subseteq \Sigma \mid \Sigma_u \subseteq \gamma\}$ is the set of control patterns. A supervisor for the controlled generator $(G, \Sigma_c, P, \Gamma)$ is a map $S : P(L(G)) \rightarrow \Gamma$. A closed-loop system associated with the controlled generator $(G, \Sigma_c, P, \Gamma)$ and the supervisor $S$ is defined as the smallest language $L(S/G) \subseteq \Sigma^*$ such that (i) $e \in E(L(S/G))$ and (ii) if $s \in L(S/G)$, $s \in L(G)$, and $a \in S(P(s))$, then $sa \in L(S/G)$. The marked behavior of the closed-loop system is defined as $L_m(S/G) = L(S/G) \cap L_m(G)$.

Let $G$ be a generator over $\Sigma$, and let $K \subseteq L_m(G)$ be a specification. The aim of supervisory control theory is to find a nonblocking supervisor $S$ such that $L_m(S/G) = K$. The nonblockingness means that $L_m(S/G) = L(S/G)$, hence $L(G/S) = K$. It is known that such a supervisor exists if and only if $K$ is (i) controllable with respect to $L(G)$ and $\Sigma_o$, that is $\mathcal{R}_{\Sigma_o} \cap L \subseteq K$, (ii) $L_m(G/S)$-closed, that is $K = K \cap L_m(G/S)$, and (iii) observable with respect to $L(G)$, $\Sigma_u$, and $\Sigma_c$, that is for all $s \in L(G)$ and $\sigma \subseteq \Sigma_c$ and $\sigma \subseteq \Sigma(G)$ imply that $P^{-1}[P(s)]\sigma \cap \Sigma = \emptyset$, where $P : \Sigma^* \rightarrow \Sigma_o^*$, cf. [1].

The synchronous product (parallel composition) of languages $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_2^*$ is defined by $L_1 \cap L_2 = L_1 \cap L_2 = L_1 \cap L_2$, where $P : \Sigma^* \rightarrow \Sigma_1^*$, for $i = 1, 2$, are projections to local event sets. In terms of generators, see [1] for more details, it is known that $L(G_1|G_2) = L(G_1) \cap L(G_2)$ and $L_m(G_1|G_2) = L_m(G_1) \cap L_m(G_2)$.

III. Coordination Control Framework

A language $K \subseteq (\Sigma_1 \cup \Sigma_2)^*$ is conditionally decomposable with respect to event sets $\Sigma_1$ and $\Sigma_2$, and $\Sigma_3$, where $\Sigma_1 \cap \Sigma_2 \subseteq \Sigma_3$, if $K = P_{1+k}(K) \cap P_{2+k}(K)$, where $P_{i+k} : (\Sigma_i \cup \Sigma_2)^* \rightarrow (\Sigma_i \cup \Sigma_3)^*$ is a projection, for $i = 1, 2$. Note that $\Sigma_3$ can always be extended so that the language $K$ becomes conditionally decomposable. A polynomial algorithm how to compute an extension can be found in [5]. However, to find the minimal extension is NP-hard [7].

Now we recall the coordination control problem that is further developed in this paper.

Problem 1: Consider generators $G_1$ and $G_2$ over $\Sigma_1$ and $\Sigma_2$, respectively, and a generator $G_k$ (called a coordinator) over $\Sigma_k$ with $\Sigma_1 \cap \Sigma_2 \subseteq \Sigma_k$. Assume that a specification $K \subseteq L_m(G_1|G_2)(G_k)$ and its prefix-closure $\overline{K}$ are conditionally decomposable with respect to event sets $\Sigma_1$, $\Sigma_2$, and $\Sigma_3$. The aim of coordination control is to determine nonblocking supervisors $S_1$, $S_2$, and $S_k$ for respective generators such that $L_m(S_k/G_k) \subseteq P_k(K) \cap L_m(S_k/G_k) \subseteq P_1+k(K)$, for $i = 1, 2$, and

$\quad L_m(S_1|G_1 \cap (S_k/G_k)) \cap L_m(S_2|G_2 \cap (S_k/G_k)) \cap K$.

Recall that one way how to construct a coordinator is to set $G_k = P_k(G_1) \cap P_k(G_2)$, cf. [6], [7].

IV. Coordination Control with Complete Observations

Conditional controllability introduced in [8] and further studied in [3], [4], [6], [7] plays the central role in coordination control. In what follows, we use the notation $\Sigma_{i,u} = \Sigma \cap \Sigma_u$ to denote the set of uncontrollable events of the event set $\Sigma_i$.

Definition 2 (Conditional controllability): Let $G_1$ and $G_2$ be generators over $\Sigma_1$ and $\Sigma_2$, respectively, and let $G_k$ be a coordinator over $\Sigma_k$. A language $K \subseteq L_m(G_1|G_2)G_k$ is conditionally controllable with respect to generators $G_1$, $G_2$, $G_k$ and uncontrollable event sets $\Sigma_{1,u}$, $\Sigma_{2,u}$, $\Sigma_{k,u}$ if

1) $P_k(K)$ is controllable with respect to $L(G_k)$ and $\Sigma_{k,u}$,
2) $P_{1+k}(K)$ is controllable with respect to $L(G_1)$ and $P_1+k(K)$ and $\Sigma_{1,u}$,
3) $P_{2+k}(K)$ is controllable with respect to $L(G_2)$ and $P_k(K)$ and $\Sigma_{2,u}$,

where $\Sigma_{i+k,u} = (\Sigma_{i,u} \cap \Sigma_k) \cap \Sigma_{i,u}$, for $i = 1, 2$. The supremal conditionally controllable sublanguage always exists and equals the union of all conditionally controllable sublanguages [7]. Let $supC(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}))$ denote the supremal conditionally controllable sublanguage of $K$ with respect to $L = L(G_1|G_2)$ and sets of uncontrollable events $\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}$.

The problem is now reduced to determining how to calculate the supremal conditionally-controllable sublanguage. Consider the setting of Problem 1 and define the languages

\[
\begin{align*}
\supC_k &= supC(P_k(K), L(G_k), \Sigma_{k,u}) \\
\supC_{1+k} &= supC(P_{1+k}(K), L(G_1), \supC_k, \Sigma_{1+k,u}) \\
\supC_{2+k} &= supC(P_{2+k}(K), L(G_2), \supC_k, \Sigma_{2+k,u})
\end{align*}
\]

where $supC(K, L, \Sigma_u)$ denotes the supremal controllable sublanguage of $K$ with respect to $L$ and $\Sigma_u$, see [1] for more details and algorithms.

We have shown that $P_k(\supC_{i+k}) \subseteq \supC_k$ always holds, for $i = 1, 2$, and that if the converse inclusion holds, we can compute the supremal conditionally-controllable sublanguage in a distributed way.

Theorem 3 (cf. [7]): Consider the setting of Problem 1 and languages defined in (1). If $\supC_k \subseteq P_k(\supC_{i+k})$, for $i = 1, 2$, then

$\supC_{1+k} \parallel \supC_{2+k} = supC(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}))$,

where $L = L(G_1|G_2)$.
We can now further improve this result by introducing a weaker condition for nonconflicting supervisors. Recall that two languages \( L_1 \) and \( L_2 \) are nonconflicting if \( L_1 \parallel L_2 = \emptyset \).

**Theorem 4:** Consider the setting of Problem 1 and languages defined in \( \Pi \). Assume that \( \sup C_{1+k} \) and \( \sup C_{2+k} \) are nonconflicting. If \( P_k(\sup C_{1+k}) \cap P_k(\sup C_{2+k}) \) is controllable with respect to \( L(G_k) \) and \( \Sigma_{k,u} \), then

\[
\sup C_{1+k} \parallel \sup C_{2+k} = \sup C_{1+k} \cup \sup C_{2+k} \]

where \( L = L(G_k) \parallel G_k \).

**Proof:** Let \( \sup C = \sup C(K,L,\{\Sigma_{1,u},\Sigma_{2,u},\Sigma_{k,u}\}) \) and \( M = \sup C_{1+k} \parallel \sup C_{2+k} \). To prove \( M \subseteq \sup C \), we show that \( M \subseteq \sup C_{1+k} \parallel P_k(\sup C_{2+k}) \) (by conditional decomposability) is conditionally controllable with respect to \( G_1, G_2, G_k \) and \( \Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u} \). However, \( P_k(M) = P_k(\sup C_{1+k}) \cap P_k(\sup C_{2+k}) \) (by Lemma 23) is controllable with respect to \( L(G_k) \) and \( \Sigma_{k,u} \) by the assumption. Furthermore, \( P_k(1+k)(M) = \sup C_{1+k} \parallel P_k(\sup C_{2+k}) \) implies that \( P_k(1+k)(\sup C_{1+k}) \parallel P_k(\sup C_{2+k}) = \sup C_{1+k} \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) = \sup C_{1+k} \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) \) is controllable with respect to \( L(G_k) \parallel \sup C \) \( \parallel P_k(M) = L(G_k) \parallel P_k(M) \) by Lemma 21 (because nonconflictingness of \( \sup C_{1+k} \) and \( \sup C_{2+k} \) implies nonconflictingness of \( \sup C_{1+k} \) and \( P_k(\sup C_{1+k}) \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) \) and by the fact that \( P_k(\sup C_{1+k}) \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) = P_k(\sup C_{1+k}) \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) \parallel P_k(\sup C_{2+k}) \) is controllable with respect to \( L(G_k) \parallel \sup C \). The other case is analogous, hence \( \sup C \subseteq M \) and the proof is complete.

Note that the controllability condition of Theorem 4 is weaker than to require that \( \sup C_k \subseteq P_k(\sup C_{1+k}) \), for \( i = 1, 2 \).

**Proposition 5:** If \( \sup C_{1+k} \subseteq P_k(\sup C_{1+k}) \), for \( i = 1, 2 \), then \( P_k(\sup C_{1+k}) \cap P_k(\sup C_{2+k}) \) is controllable with respect to \( L(G_k) \) and \( \Sigma_{k,u} \).

**Proof:** This is obvious, because due to the converse inclusion being always true we have that \( P_k(\sup C_{1+k}) = \sup C_k \), for \( i = 1, 2 \). Hence, \( P_k(\sup C_{1+k}) \cap P_k(\sup C_{2+k}) = \sup C_k \) is controllable with respect to \( L(G_k) \) and \( \Sigma_{k,u} \) by definition of \( \sup C \).

Using the example from [7] we can now show that there are languages such that \( \sup C_k \not\subseteq P_k(\sup C_{1+k}) \), but such that \( P_k(\sup C_{1+k}) \cap P_k(\sup C_{2+k}) \) is controllable with respect to \( L(G_k) \) and \( \Sigma_{k,u} \).

**Example 6:** Let \( G_1 \) and \( G_2 \) be generators as shown in Fig. 4 and let \( K \) be the language of the generator shown in Fig. 2. Let \( \Sigma_c = \{a_1, a_2, c\} \) and \( \Sigma_u = \{a_1, a_2, c, u\} \). Let the coordinator \( G_k = P_k(G_1) \parallel P_k(G_2) \). Then \( K \) is conditionally decomposable, \( \sup C_k = \{a_1 a_2, a_2 a_1\} \), \( \sup C_{1+k} = \{a_2 a_1 u_1\} \), \( \sup C_{2+k} = \{a_1 a_2 u_2\} \), and \( \sup C_k \not\subseteq P_k(\sup C_{1+k}) \). However, \( P_k(\sup C_{1+k}) \cap P_k(\sup C_{2+k}) = \{\varepsilon\} \) is controllable with respect to \( L(G_k) \) and \( \Sigma_{k,u} \).

**Example 7:** Let \( G_1 \) and \( G_2 \) be generators as shown in Fig. 3 and let \( K \) be the language of the generator shown in Fig. 4. Let \( \Sigma_c = \{a_1, c_1, c_2\} \) and \( \Sigma_u = \{a, b\} \). Let the coordinator \( G_k = P_k(G_1) \parallel P_k(G_2) \). Then the language \( K \) is conditionally decomposable, \( \sup C_k = \{b\} \), \( \sup C_{1+k} = \{c_1 b\} \), \( \sup C_{2+k} = \{\varepsilon\} \), and \( P_k(\sup C_{1+k}) \cap P_k(\sup C_{2+k}) = \{\varepsilon\} \) is not controllable with respect to \( L(G_k) = \{a b, b\} \) and \( \Sigma_{k,u} = \{b\} \).

**Fig. 1.** Generators \( G_1 \) and \( G_2 \).

**Fig. 2.** Specification \( K \).

**Fig. 3.** Generators \( G_1 \) and \( G_2 \).

**Fig. 4.** Specification \( K \).
Recall that it is still an open problem how to compute the supremal conditionally observable sublanguage for a general, non-prefix-closed language.

The following conditions were required in [6] to prove the main result for prefix-closed languages. We weaken the result here and show that the previous condition is a weaker condition than the one required in [6].

The projection \( P : \Sigma^* \to \Sigma_0^* \), where \( \Sigma_0 \subseteq \Sigma \), is an \( L \)-observer for \( L \subseteq \Sigma^* \) if, for all \( t \in P(L) \) and \( s \in T \), \( P(s) \) is a prefix of \( t \) implies that there exists \( u \in \Sigma^* \) such that \( su \in L \) and \( P(su) = t \).

The projection \( P : \Sigma^* \to \Sigma_0^* \) is output projection consistent (OCC) for \( L \subseteq \Sigma^* \) if for every \( s \in L \) of the form \( s = \sigma_1 \ldots \sigma_k \) or \( s = s' \sigma_0 \sigma_1 \ldots \sigma_k \), \( k \geq 1 \), where \( s' \in \Sigma^* \), \( \sigma_0, \sigma_k \in \Sigma_k \), and \( \sigma_i \in \Sigma \setminus \Sigma_k \), for \( i = 1, 2, \ldots, k - 1 \), if \( \sigma_k \in \Sigma_k \), then \( \sigma_i \in \Sigma_u \), for all \( i = 1, 2, \ldots, k - 1 \).

The OCC condition can be replaced by a weaker condition called local control consistency (LCC) discussed in [12], [11], see [7]. Let \( L \) be a prefix-closed language over \( \Sigma \), and let \( \Sigma_0 \) be a subset of \( \Sigma \). The projection \( P : \Sigma^* \to \Sigma_0^* \) is locally control consistent (LCC) with respect to a word \( s \in L \) if for all events \( \sigma_i \in \Sigma_0 \cap \Sigma_0 \) such that \( P(s) \sigma_i \in P(L) \), it holds that either there does not exist any word \( u \in (\Sigma \setminus \Sigma_0)^* \) such that \( u \sigma_i \in L \), or there exists a word \( u \in (\Sigma_0 \setminus \Sigma_0)^* \) such that \( u \sigma_i \in L \). The projection \( P \) is LCC with respect to \( L \) if \( P \) is LCC for all words of \( L \).

**Theorem 8 ([7]):** Consider the setting of Problem 1 with a prefix-closed specification \( K \). Consider the languages defined in (1) and assume that \( \sup C_{1+k} \) and \( \sup C_{2+k} \) are noncontrolling. Let \( P_k^{1+k} \) be an \( (P_k^{1+k})^{-1}L(G_1) \)-observer and OCC (resp. LCC) for \( (P_k^{1+k})^{-1}L(G_1) \), for \( i = 1, 2 \). Then

\[
\sup C_{1+k} \cap \sup C_{2+k} = \sup C(K, L, (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)),
\]

where \( L = L(G_1) \cap L(G_2) \). □

We can now prove that the assumptions of the previous theorem are stronger than the assumptions of Theorem 3.

This is shown in the following lemma and corollary, and summarized in Theorem 11.

**Lemma 9:** Consider the setting of Problem 1 and the languages defined in (1). Assume that \( \sup C_{1+k} \) and \( \sup C_{2+k} \) are noncontrolling, and let the projection \( P_k^{1+k} : (\Sigma_1 \cup \Sigma_2) \to \Sigma^* \) be an \( (P_k^{1+k})^{-1}L(G_1) \)-observer and OCC (resp. LCC) for \( (P_k^{1+k})^{-1}L(G_1) \), for \( i = 1, 2 \). Then \( P_k^{1+k} \cap \sup C_{1+k} \cap \sup C_{2+k} \) is controllable with respect to \( P_k^{1+k} \cap L(G_1) \) and \( L(G_2) \) and \( \Sigma_{1+k} \).

**Proof:** Since \( \Sigma_1 \cap \Sigma_2 = \Sigma_k \), Lemma 23 implies that \( P_k^{1+k}(\sup C_{1+k}) \cap P_k^{2+k}(\sup C_{2+k}) = P_k^{1+k}(\sup C_{1+k} \cap \sup C_{2+k}) \). By Lemma 23, because \( P_k^{1+k} = \text{id} \) is an \( L(G_1) \)-observer, \( P_k \) is an \( (P_k^{1+k})^{-1}L(G_1) \)-observer. Assume that \( t \in P_k(\sup C_{1+k} \cap \sup C_{2+k}) \), \( u \in \Sigma_{1+k} \), and \( t = P_k(t) \). Then \( P_k(t) = P_k(t) \) and \( P_k(t) = P_k(t) \). Then there exists \( s \in \sup C_{1+k} \cap \sup C_{2+k} \subseteq \Sigma_k \) such that \( P_k(s) = t \). By the OCC property, there exists \( v \in \Sigma_u \) such that \( sv \in L \) and \( P_k(sv) = t \), that is, \( v = v_1 u \) with \( P_k(v_1 u) = u \). By the OCC property, \( v_1 \in \Sigma_u \), and by controllability of \( \sup C_{1+k} \), \( s \in \sup C_{1+k} \cap \sup C_{2+k} \), hence \( t \in P_k(\sup C_{1+k} \cap \sup C_{2+k}) \).

Similarly for LCC; from \( sv = sv_1 u \in L \), by the LCC property, there exists \( v_2 \in (\Sigma_u \setminus \Sigma_k)^* \) such that \( sv_2 u \in L \), and by controllability of \( \sup C_{1+k} \), \( i = 1, 2 \), \( sv_2 u \in \sup C_{1+k} \cap \sup C_{2+k} \), hence \( t \in P_k(\sup C_{1+k} \cap \sup C_{2+k}) \). □

Note that if \( L(G_k) \subseteq P_k(L(G_1)) \cap P_k(L(G_2)) \), which is actually the way we usually define the coordinator (since we usually define \( G_k = P_k(G_1) \cap P_k(G_2) \)), we get the following corollary.

**Corollary 10:** Consider the setting of Problem 1 with \( L(G_k) \subseteq P_k(L(G_1)) \cap P_k(L(G_2)) \) and the languages defined in (1). Assume that \( \sup C_{1+k} \) and \( \sup C_{2+k} \) are noncontrolling. Let \( P_k^{1+k} : (\Sigma_1 \cup \Sigma_2)^* \to \Sigma_k^* \) be an \( (P_k^{1+k})^{-1}L(G_1) \)-observer and OCC (resp. LCC) for \( (P_k^{1+k})^{-1}L(G_1) \), for \( i = 1, 2 \). Then \( P_k^{1+k}(\sup C_{1+k}) \cap P_k^{1+k}(\sup C_{2+k}) \) is controllable with respect to \( L(G_1) \) and \( \Sigma_{1+k} \).

**Proof:** The assumption \( L(G_k) \subseteq P_k(L(G_1)) \cap P_k(L(G_2)) \) implies that \( P_k(L(G_1)) \cap P_k(L(G_2)) \subseteq L(G_1) \).

Finally, as a consequence of Lemma 9 and Theorem 4, we obtain the following result.

**Theorem 11:** Consider the setting of Problem 1 with \( L(G_k) \subseteq P_k(L(G_1)) \cap P_k(L(G_2)) \) and the languages defined in (1). Assume that \( \sup C_{1+k} \) and \( \sup C_{2+k} \) are noncontrolling. Let \( P_k^{1+k} : (\Sigma_1 \cup \Sigma_2)^* \to \Sigma_k^* \) be an \( (P_k^{1+k})^{-1}L(G_1) \)-observer and OCC (resp. LCC) for \( (P_k^{1+k})^{-1}L(G_1) \), for \( i = 1, 2 \). Then

\[
\sup C_{1+k} \cap \sup C_{2+k} = \sup C(K, L, (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)),
\]

where \( L = L(G_1) \cap L(G_2) \). □

V. COORDINATION CONTROL WITH PARTIAL OBSERVATIONS

In this section, we study coordination control of modular DES, where both the coordinator supervisor and the local supervisors have incomplete (partial) information about occurrences of their events and, hence, they do not know the exact state of the coordinator and the local plants.

The contribution of this section is twofold. First, basic concepts of conditional observability and conditional normality are simplified in a similar way as it has been done in [7]. Then, we propose new sufficient conditions for a distributed computation of the supremal conditionally normal and conditionally controllable sublanguage. In particular, a weaker condition is presented that combines the weaker condition for distributed computation of the supremal conditionally normal and conditionally controllable sublanguage. In particular, a weaker condition is presented that is easy to check and that works also for non-prefix-closed specifications.

**A. Conditional Observability**

For coordination control with partial observations, the notion of conditional observability is of the same importance as observability for monolithic supervisory control theory with partial observations.

**Definition 12:** Let \( G_1 \) and \( G_2 \) be generators over \( \Sigma_1 \) and \( \Sigma_2 \), respectively, and let \( G_k \) be a coordinator over \( \Sigma_k \). A
language \( K \subseteq L_m(G_1 \parallel G_2 \parallel G_3) \) is conditionally observable with respect to generators \( G_1, G_2, G_3 \), controllable sets \( \Sigma_{1,c}, \Sigma_{2,c}, \Sigma_{3,c} \), and projections \( Q_{1+k}, Q_{2+k}, Q_k \), where \( Q_i : \Sigma_i \rightarrow \Sigma_{i,c} \), for \( i = 1 + k, 2 + k, k \), if

1. \( P_k(K) \) is observable with respect to \( L(G_k) \), \( \Sigma_{k,c}, Q_k \),
2. \( P_{1+k}(K) \) is observable with respect to \( L(G_1) \parallel P_k(K) \), \( \Sigma_{1,c+k}, Q_{1+k} \),
3. \( P_{2+k}(K) \) is observable with respect to \( L(G_2) \parallel P_k(K) \), \( \Sigma_{2,c+k}, Q_{2+k} \),

where \( \Sigma_{i,c+k} \subseteq \Sigma_i \cap (\Sigma_i \cup \Sigma_k) \), for \( i = 1, 2 \).

Analogously to the notion of \( L_m(G) \)-closed languages, we recall the notion of conditionally-closed languages defined in [3]. A nonempty language \( K \) over \( \Sigma \) is conditionally closed with respect to generators \( G_1, G_2, G_3 \) if

1. \( P_k(K) \) is \( L_m(G_k) \)-closed,
2. \( P_{1+k}(K) \) is \( L_m(G_1) \parallel P_k(K) \)-closed,
3. \( P_{2+k}(K) \) is \( L_m(G_2) \parallel P_k(K) \)-closed.

We can now formulate the main result for coordination control with partial observation. This is a generalization of a similar result for prefix-closed languages given in [4] stated moreover with the above defined simplified (but equivalent) form of conditional observability.

**Theorem 13:** Consider the setting of Problem [1]. There exist nonblocking supervisors \( S_1, S_2, S_k \) such that

\[
L_m(S_1/|G_1||S_k/G_k)) \parallel L_m(S_2/|G_2||S_k/G_k)) = K
\]

(1)

if and only if \( K \) is (i) conditionally observable with respect to \( G_1, G_2, G_3 \), \( \Sigma_{1,c}, \Sigma_{2,c}, \Sigma_{3,c} \), \( \Sigma_{k,c} \), and projections \( Q_{1+k}, Q_{2+k}, Q_k \) from \( \Sigma_i \) to \( \Sigma_{i,c+k} \), for \( i = 1 + k, 2 + k, k \).

**Proof:** (If) Since \( K \subseteq L_m(G_1||G_2||G_3) \), we have \( P_k(K) \subseteq L_m(G_k) \) is controllable with respect to \( L(G_k) \), \( \Sigma_{k,c} \), \( Q_k \). Then, see [1], that there exists a nonblocking supervisor \( S_k \) such that \( L_m(S_k/G_k) = P_k(K) \). Similarly, we have \( P_{1+k}(K) \subseteq L_m(G_1) \parallel L_m(G_1) \) and \( P_{1+k}(K) \subseteq (P_{k+1}(K) \parallel L_m(G_1) \parallel L_m(G_1)) \parallel P_{1+k}(K) \), \( \Sigma_1 \parallel L_m(G_1) \parallel P_{1+k}(K) \) is conditionally closed with respect to \( G_1, G_2, G_3 \), \( \Sigma_{1,c}, \Sigma_{2,c}, \Sigma_{3,c} \), and projections \( Q_{1+k}, Q_{2+k}, Q_k \) from \( \Sigma_i \) to \( \Sigma_{i,c+k} \), for \( i = 1 + k, 2 + k, k \).

(Only if) To prove this direction, projections \( P_k, P_{1+k}, P_{2+k} \) are applied to [1]. The closed-loop languages can be written as synchronized products, thus [1] can be written as

\[
K = L_m(S_1) \parallel L_m(G_1) \parallel L_m(S_2) \parallel L_m(G_2) \parallel L_m(S_k) \parallel L_m(G_k) \parallel L_m(S_k) \parallel L_m(G_k) \parallel L_m(S_k) \parallel L_m(G_k)
\]

which gives \( P_k(K) \subseteq L_m(S_k) \parallel L_m(G_k) = L_m(S_k) \parallel L_m(G_k) \). On the other hand, \( L_m(S_k) \subseteq P_k(K) \), see Problem [1], hence \( L_m(S_k/G_k) = P_k(K) \), which means, according to the basic theorem of supervisory control [1], that \( P_k(K) \) is controllable with respect to \( L(G_k) \) and \( \Sigma_{k,c} \), \( L_m(G_k) \)-closed, and observable with respect to \( L(G_k) \), \( \Sigma_{k,c} \), and \( Q_k \). Now, the application of \( P_{1+k} \) to [1] gives \( P_{1+k}(K) \subseteq L_m(S_1/G_1||S_k/G_k) \subseteq P_{1+k}(K) \). According to the basic theorem of supervisory control, \( P_{1+k}(K) \) is controllable with respect to \( L(G_1||S_k/G_k) \) and \( \Sigma_{1+k,c} \parallel L_m(G_1||S_k/G_k) \)-closed, and observable with respect to \( L(G_1||S_k/G_k) \), \( \Sigma_{1+k,c} \), and \( Q_{1+k} \). Similarly, \( P_{2+k}(K) \) is controllable with respect to \( L(G_2||S_k/G_k) \) and \( \Sigma_{2+k,c} \parallel L_m(G_2||S_k/G_k) \)-closed, and observable with respect to \( L(G_2||S_k/G_k) \), \( \Sigma_{2+k,c} \), and \( Q_{2+k} \), which was to be shown.

**B. Conditional normality**

It is well known that supremal observable sublanguages do not exist in general and it is also the case of conditionally observable sublanguages. Therefore, a stronger concept of language normality has been introduced.

Let \( G \) be a generator over \( \Sigma \), and let \( P : \Sigma^* \rightarrow \Sigma^o \) be a projection. A language \( K \subseteq L_m(G) \) is *normal* with respect to \( P(G) \) and \( P \) if \( K = P^{-1}(P(\bar{K})) \). It is known that normality implies observability [1].

**Definition 14:** Let \( G_1 \) and \( G_2 \) be generators over \( \Sigma_1 \) and \( \Sigma_2 \), respectively, and let \( G_k \) be a coordinator over \( \Sigma_k \). A language \( K \subseteq L_m(G_1||G_2||G_k) \) is conditionally normal with respect to generators \( G_1, G_2, G_k \) and projections \( Q_{1+k}, Q_{2+k}, Q_k \), where \( Q_i : \Sigma_i \rightarrow \Sigma_i^o \) for \( i = 1 + k, 2 + k, k \), if

1. \( P_k(K) \) is normal with respect to \( L(G_k) \) and \( Q_k \),
2. \( P_{1+k}(K) \) is normal with respect to \( L(G_1) \parallel P_k(K) \) and \( Q_{1+k} \),
3. \( P_{2+k}(K) \) is normal with respect to \( L(G_2) \parallel P_k(K) \) and \( Q_{2+k} \).

The following result is an immediate application of conditional normality in coordination control.

**Theorem 15:** Consider the setting of Problem [1]. If the specification \( K \) is conditionally observable with respect to \( G_1, G_2, G_k \) and \( \Sigma_{1,c}, \Sigma_{2,c}, \Sigma_{k,c} \), conditionally closed with respect to \( G_1, G_2, G_k \), and conditionally normal with respect to \( G_1, G_2, G_k \) and projections \( Q_{1+k}, Q_{2+k}, Q_k \) from \( \Sigma_1 \rightarrow \Sigma_1^o \), \( i = 1 + k, 2 + k, k \), then there exist nonblocking supervisors \( S_1, S_2, S_k \) such that

\[
L_m(S_1/|G_1||S_k/G_k)) \parallel L_m(S_2/|G_2||S_k/G_k)) = K
\]

(2)

**Proof:** As normality implies observability, the proof follows immediately from Theorem 13.

The following result was proved for prefix-closed languages in [4]. Here we generalize it for not necessarily prefix-closed languages.

**Theorem 16:** The supremal conditionally normal sublanguage always exists and equals to the union of all conditionally normal sublanguages.

**Proof:** We show that conditional normality is preserved under union. Let \( I \) be an index set, and let \( K_i \) be conditionally normal sublanguages of \( K \subseteq L_m(G_1||G_2||G_k) \) with respect to generators \( G_1, G_2, G_k \) and projections \( Q_{1+k}, Q_{2+k}, Q_k \) to local observable event sets, for \( i \in I \). We prove that \( \bigcup_{i \in I} K_i \) is conditionally normal with respect to those generators and natural projections.
i) \( P_k(\bigcup_{i \in I} K_i) \) is normal with respect to \( L(G_k) \) and \( Q_k \) because \( Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \cap L(G_k) = \bigcup_{i \in \mathcal{I}} (Q_k^{-1}P_kP_k(K_i) \cap L(G_k)) = \bigcup_{i \in \mathcal{I}} P_k(K_i) = P_k(\bigcup_{i \in \mathcal{I}} K_i) \), where the second equality is by normality of \( P_k(K_i) \) with respect to \( L(G_k) \) and \( Q_k \), for \( i \in I \).

ii) Note that \( Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \cap L(G_1) \cap P_k(\bigcup_{i \in \mathcal{I}} K_i) = \bigcup_{i \in \mathcal{I}} Q_k^{-1}P_kP_k(K_i) \cap L(G_1) \cap P_k(K_i) \) and \( P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \cap L(G_1) \cap P_k(K_i) \subseteq Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \cap L(G_1) \cap P_k(\bigcup_{i \in \mathcal{I}} K_i) \). For the sake of contradiction, assume that there exist indices \( i \neq j \) in \( I \) such that \( Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \cap L(G_1) \cap P_k(\bigcup_{i \in \mathcal{I}} K_i) \neq P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i). \) Then the left-hand side must be nonempty, which implies that there exists \( x \in \bigcup_{i \in \mathcal{I}} Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \cap L(G_1) \cap P_k(\bigcup_{i \in \mathcal{I}} K_i), \) and \( x \notin \bigcup_{i \in \mathcal{I}} P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i). \) As \( x \in \bigcup_{i \in \mathcal{I}} Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i), \) there exists \( w \in K_i \) such that \( Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \cap L(G_1) \cap P_k(\bigcup_{i \in \mathcal{I}} K_i) \neq P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i). \) Applying the projection \( P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \), we get that \( P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \neq P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i). \) Since \( P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \neq P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \), we have \( P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \neq P_k^{-1}Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \). Hence, \( x \in \bigcup_{i \in \mathcal{I}} Q_k^{-1}P_kP_k(\bigcup_{i \in \mathcal{I}} K_i) \), which is a contradiction.

iii) As the last item of the definition is proven in the same way, the theorem holds.

Given generators \( G_1, G_2, \) and \( G_k, \)

\[
\text{supcCN}(K, \Sigma, (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4); (Q_{1+k}, Q_{2+k}, Q_k))
\]
denote the supcnal conditionally controllable and conditionally normal sublanguage of the specification language \( K \) with respect to the plant language \( L = L(G_1) \cap G_2, \) the sets of uncontrollable events \( \Sigma_1, \Sigma_2, \Sigma_3, \) and \( \Sigma_4, \) and projections \( Q_1, Q_2, Q_3, Q_4, \) where \( Q_1 : \Sigma_1 \rightarrow \Sigma_1, Q_2 : \Sigma_2 \rightarrow \Sigma_3, Q_3 : \Sigma_3 \rightarrow \Sigma_3, \) for \( i = 1, 2, +k, k. \)

In the sequel, the computation of the supcnal conditionally controllable and conditionally normal sublanguage is investigated. In the same way as in \([4]\), the following notation is adopted.

Consider the setting of Problem 1 with a prefix-closed specification \( K \) and the languages defined in \([2]\). Let \( P_k^{-1}+k \) be an \((P_i^{-1+k})^{-1}L(G_i)\)-observer and OCC (resp. LCC) for \((P_i^{-1+k})^{-1}L(G_i)\), for \( i = 1, 2 \). Assume that the language \( P_k^{-1+k}(\text{supcCN}_{1+k}) \cap P_k^{-2+k}(\text{supcCN}_{2+k}) \) is normal with respect to \( L(G_k) \) and \( Q_k \). Then

\[
\text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} = \text{supcCN}(K, L, (\Sigma_1, u, \Sigma_2, u, \Sigma_3), (Q_{1+k}, Q_{2+k}, Q_k)),
\]
where \( L = L(G_1) || G_2 || G_k \).

We can now further improve the above result as follows.

**Theorem 18:** Consider the setting of Problem 1 and the languages defined in \([2]\). Assume that \( \text{supcCN}_{1+k} \) and \( \text{supcCN}_{2+k} \) are nonconflicting and that \( P_k^{-1+k}(\text{supcCN}_{1+k}) \cap P_k^{-2+k}(\text{supcCN}_{2+k}) \) is controllable and normal with respect to \( L(G_k) \), \( \Sigma_3, u, \) and \( Q_k \). Then

\[
\text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} = \text{supcCN}(K, L, (\Sigma_1, u, \Sigma_2, u, \Sigma_3), (Q_{1+k}, Q_{2+k}, Q_k)),
\]
where \( L = L(G_1) || G_2 || G_k \).

**Proof:** Let \( M = \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \) and \( \text{supcCN} = \text{supcCN}(K, L, (E_{1+k}, u, E_{2+k}, u, E_{3+k}), (Q_{1+k}, Q_{2+k}, Q_k)). \)

To prove \( M \subseteq \text{supcCN} \), we show that \( M \subseteq \text{supcCN}(K, L, (E_{1+k}, u, E_{2+k}, u, E_{3+k}), (Q_{1+k}, Q_{2+k}, Q_k)). \) (which needs to be shown). However, \( P_k(M) = P_k^{-1+k}(\text{supcCN}_{1+k}) \cap P_k^{-2+k}(\text{supcCN}_{2+k}) \) is normal with respect to \( L(G_k) \) and \( Q_k \) by the assumption. Furthermore, \( P_k(x) = \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \). Since \( P_k(x) \subseteq \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \), \( P_k(x) \subseteq \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \). Therefore, \( x \in \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \) by the nonconflictingness of the supervisors. The case for \( P_k(M) \) is analogous, hence \( M \subseteq \text{supcCN} \).

To prove \( \text{supcCN} \subseteq M \), it is sufficient by Lemma \( 24 \) to show that \( P_k(x) \subseteq \text{supcCN}_{1+k} \), for \( i = 1, 2 \). To do this, note that \( P_k(x) \subseteq \text{supcCN}_{1+k} \subseteq \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \cap \text{supcCN}_{3+k} \), and \( Q_1 \) by definition. Since \( P_k(x) \subseteq \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \cap \text{supcCN}_{3+k} \), it is also controllable and normal with respect to \( \text{supcCN}_{1+k} \), \( \text{supcCN}_{2+k} \), and \( \text{supcCN}_{3+k} \) by Lemma \( 24 \), and \( \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \cap \text{supcCN}_{3+k} \), is also controllable and normal with respect to \( L(G_k) \) because \( \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \cap \text{supcCN}_{3+k} \), is also controllable and normal with respect to \( L(G_k) \) by Lemma \( 24 \), and \( \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \cap \text{supcCN}_{3+k} \), is also controllable and normal with respect to \( L(G_k) \) by Lemma \( 24 \). Therefore, \( P_k(x) \subseteq \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \cap \text{supcCN}_{3+k} \). Thus, \( x \in \text{supcCN}_{1+k} \cap \text{supcCN}_{2+k} \cap \text{supcCN}_{3+k} \) by the nonconflictingness of the supervisors. The case for \( P_k(M) \) is analogous, hence \( \text{supcCN} \subseteq M \) and the proof is complete.

Note that the sufficient condition in Theorem 18 is not practical for verification, although the intersection is only
over the coordinator alphabet that is hopefully small. Unlike controllability, normality is not preserved by natural projections under observer and OCC assumptions. This would require results on hierarchical control under partial observations that are not known so far. Therefore, we propose a condition that is (similarly as in the case of complete observations) stronger than the one of Theorem 18 but is easy to check and, moreover, is sufficient for a distributed computation of the supremal conditionally controllable and conditionally normal sublanguage even in the case of non-prefix-closed specifications. Namely, we observe that controllability and normality conditions of Theorem 18 are weaker than to require that $\text{supCN}_k \subseteq P_2(\text{supCN}_{i+k})$, for $i = 1, 2$. The intuition behind the condition $\text{supCN}_k \subseteq P_2(\text{supCN}_{i+k})$, for $i = 1, 2$, is that local supervisors (given by $\text{supCN}_{i+k}$) do not need to improve the action by the supervisor for the coordinator on the coordinator alphabet. In this case, the intuition is the same as if the three supervisors (the supervisor for the coordinator and local supervisors) would operate on disjoint alphabets (namely $\Sigma_k, \Sigma_1 \setminus \Sigma_k$ and $\Sigma_2 \setminus \Sigma_k$) and it is well known that there is no problem with blocking and maximal permissiveness in this case (nonconflictness and mutual controllability of modular controller) are trivially satisfied.

Proposition 19: Consider the setting of Problem 1 and the languages defined in (2). If $\text{supCN}_k \subseteq P_2(\text{supCN}_{i+k})$, for $i = 1, 2$, then $P_1(\text{supCN}_{i+k}) \cap P_2(\text{supCN}_{i+k})$ is controllable and normal with respect to $L(G_k)$, $\Sigma_{k,u}$, and $Q_k$.

Proof: First of all, we shown that the inclusion $\text{supCN}_k \supseteq P_2(\text{supCN}_{i+k})$, for $i = 1, 2$ always holds true. From its definition, $P_2(\text{supCN}_{i+k}) \subseteq P_2(L(G_i)) \cap \text{supCN}_k \subseteq \text{supCN}_k$ and, clearly, $P_1(\text{supCN}_{i+k}) \subseteq P_1(K)$ as well. In order to show that $P_1(\text{supCN}_{i+k}) \subseteq \text{supCN}_k$, it suffices to show that $\text{supCN}_k \cap P_1(K) \subseteq \text{supCN}_k$. This can be proven by showing that $\text{supCN}_k \cap P_1(K)$ is controllable and normal with respect to $L(G_k)$, $\Sigma_{k,u}$, and $Q_k$.

For controllability, let $s \in \text{supCN}_k \cap P_1(K)$, $u \in \Sigma_{k,u}$ with $su \in L(G_k)$. Since there exists $t \in \Sigma_k$ such that $st \in \text{supCN}_k \cap P_1(K) \subseteq \text{supCN}_k$, we have that $s \in \text{supCN}_k$ as well. Since $\text{supCN}_k$ is controllable with respect to $L(G_k)$ and $\Sigma_{k,u}$, $su \in \text{supCN}_k \cap P_1(K)$. Hence, there exists $v \in \Sigma_k$ such that $sv \in \text{supCN}_k \cap P_1(K)$. Altogether, $sv \in \text{supCN}_k \cap P_1(K)$, i.e., $sv \in \text{supCN}_k \cap P_2(K)$.

For normality, let $s \in \text{supCN}_k \cap P_2(K)$ and $s' \in L(G_k)$ with $Q_k(s) = Q_k(s')$. Recall that $s \in \text{supCN}_k$ as well. Again, normality of $\text{supCN}_k$ with respect to $L(G_k)$ and $Q_k$ implies that $s' \in \text{supCN}_k$. Thus, there exists $v \in \Sigma_k$ such that $sv' \in \text{supCN}_k \subseteq P_1(K)$. This implies that $sv' \in \text{supCN}_k \cap P_1(K)$, i.e., $s' \in \text{supCN}_k \cap P_1(K)$, which completes the proof of the inclusion $\text{supCN}_k \supseteq P_1(\text{supCN}_{i+k})$, for $i = 1, 2$.

According to the assumption that the other inclusions also hold, we have the equalities $\text{supCN}_k = P_i(\text{supCN}_{i+k})$, for $i = 1, 2$. Therefore, $P_1(\text{supCN}_{i+k}) \cap P_2(\text{supCN}_{i+k}) = \text{supCN}_k$, which is controllable and normal with respect to $L(G_k)$, $\Sigma_{k,u}$, and $Q_k$ by definition of $\text{supCN}_k$.

Now, combining Proposition 19 and Theorem 18 we obtain the corollary below.

Corollary 20: Consider the setting of Problem 1 and the languages defined in (2). If $\text{supCN}_k \subseteq P_2(\text{supCN}_{i+k})$, for $i = 1, 2$, then

$$\text{supCN}_{i+k} \parallel \text{supCN}_{i+2k} = \text{supCN}(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}), (Q_{1+k}, Q_{2+k}, Q_k)),$$

where $L = L(G_1) \parallel G_2).$

Proof: Let $\text{supCN} = \text{supCN}(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}), (Q_{1+k}, Q_{2+k}, Q_k))$ and $M = \text{supCN}_{i+k} \parallel \text{supCN}_{i+2k}$. To prove that $M$ is a subset of $\text{supCN}$, we show that (i) $M$ is controllably decomposable with respect to generators $G_1$, $G_2$, $G_k$ and uncontrollable event sets $\Sigma_{1,u}$, $\Sigma_{2,u}$, $\Sigma_{k,u}$, and (iii) $M$ is conditionally normal with respect to generators $G_1$, $G_2$, $G_k$ and projections $Q_{1+k}$, $Q_{2+k}$, $Q_k$. To this aim, notice that $M$ is a subset of $P_1(\text{supCN}) \parallel P_2(\text{supCN}) = K$, because $K$ is conditionally decomposable. Moreover, by Lemma 23 and the fact shown in the proof of Proposition 19 that $\text{supCN}_k \supseteq P_1(\text{supCN}_{i+k})$, for $i = 1, 2$, the language $P_1(M) = P_1(\text{supCN}_{i+k}) \cap P_2(\text{supCN}_{i+k}) = \text{supCN}_k$ is controllable and normal with respect to $L(G_k)$, $\Sigma_{k,u}$, and $Q_k$. Similarly, $P_2(\text{supCN}_{j+k}) = \text{supCN}_{i+k}$ for $j \neq i$, which is controllable and normal with respect to $L(G_i) \parallel P_j(M)$. Hence, $M$ is a subset of $\text{supCN}$.

The opposite inclusion is shown in Theorem 18 because nonconflictness is not needed in this direction of the proof.

VI. CONCLUSION

In this paper, we have further generalized several results of coordination control of concurrent automata with both complete and partial observations. We have presented weaker sufficient conditions for the computation of supremal conditionally controllable sublanguages and supremal conditionally controllable and conditionally normal sublanguages with simplified concepts of conditional observability and conditional normality. Since our results admit quite a straightforward extension to a multi-level coordination control framework, in a future work we would apply our framework to DES models of engineering systems.
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REFERENCES


APPENDIX

In this section, we list the auxiliary results.

**Lemma 21 (Proposition 4.6 in [2]):** Let \( L_i \subseteq \Sigma_i \), for \( i = 1, 2 \), be prefix-closed languages, and let \( K_f \subseteq L_i \) be controllable with respect to \( L_i \) and \( \Sigma_i \). Let \( \Sigma = \Sigma_1 \cup \Sigma_2 \). If \( K_1 \) and \( K_2 \) are synchronously nonconflicting, then \( K_1 \parallel K_2 \) is controllable with respect to \( L_1 \parallel L_2 \) and \( \Sigma \).

**Lemma 22 ([6]):** Let \( K \subseteq L \subseteq M \) be languages over \( \Sigma \) such that \( K \) is controllable with respect to \( L \) and \( \Sigma \), and \( L \) is controllable with respect to \( M \) and \( \Sigma \). Then \( K \) is controllable with respect to \( M \) and \( \Sigma \).

**Lemma 23 ([13]):** Let \( P_k : \Sigma^* \rightarrow \Sigma_k^* \) be a projection, and let \( L_i \subseteq \Sigma_i^* \), where \( \Sigma_i \subseteq \Sigma \), for \( i = 1, 2 \), and \( \Sigma_1 \cap \Sigma_2 \subseteq \Sigma \). Then \( P_k(L_1 \parallel L_2) = P_k(L_1) \parallel P_k(L_2) \).

**Lemma 24 ([6]):** Let \( L_i \subseteq \Sigma_i^* \), for \( i = 1, 2 \), and let \( P_i : (\Sigma_1 \cup \Sigma_2)^* \rightarrow \Sigma_i^* \) be a projection. Let \( A \subseteq (\Sigma_1 \cup \Sigma_2)^* \) such that \( P_1(A) \subseteq L_1 \) and \( P_2(A) \subseteq L_2 \). Then \( A \subseteq L_1 \parallel L_2 \).

**Lemma 25 ([9]):** Let \( L_i \subseteq \Sigma_i^* \), for \( i \in J \), be languages, and let \( \cup_{k \in J} \subseteq (\Sigma_i \cap \Sigma_i) \subseteq \Sigma_i \subseteq (\cup_{i \in J} \Sigma_i)^* \). If \( P_{i_0} : \Sigma_i \rightarrow (\Sigma_i \cap \Sigma_i)^* \) is an \( L_i \)-observer, for \( i \in J \), then \( P_{o_0} : (\cup_{i \in J} \Sigma_i)^* \rightarrow \Sigma_{o_0}^* \) is an \((\cup_{i \in J} L_i)\)-observer.

**Lemma 26:** Let \( K \subseteq L \subseteq M \) be languages such that \( K \) is normal with respect to \( L \), and \( L \) is normal with respect to \( M \) and \( Q \). Then, \( K \) is normal with respect to \( M \) and \( Q \).

**Proof:** \( Q^{-1}Q(K) \cap L = K \) and \( Q^{-1}Q(L) \cap M = L \), hence \( Q^{-1}Q(K) \cap M \subseteq Q^{-1}Q(L) \cap M = L \). It implies \( Q^{-1}Q(K) \cap M = Q^{-1}Q(K) \cap M = K \).

**Lemma 27:** Let \( K_1 \subseteq L_1 \) over \( \Sigma_1 \) and \( K_2 \subseteq L_2 \) over \( \Sigma_2 \) be nonconflicting languages such that \( K_1 \) is normal with respect to \( L_1 \) and \( Q_1 : \Sigma_1 \rightarrow \Sigma_{1,o}^* \), and \( K_2 \) is normal with respect to \( L_2 \) and \( Q_2 : \Sigma_2 \rightarrow \Sigma_{2,o}^* \). Then \( K_1 \parallel K_2 \) is normal with respect to \( L_1 \parallel L_2 \) and \( Q : (\Sigma_1 \cup \Sigma_2)^* \rightarrow (\Sigma_{1,o} \cup \Sigma_{2,o})^* \).

**Proof:** \( Q^{-1}Q(K_1 \parallel K_2) \cap L_1 \parallel L_2 \subseteq Q^{-1}Q(K_1) \cap Q_2^{-1}Q_2(K_2) \parallel L_1 \parallel L_2 = K_1 \parallel K_2 \). As the other inclusion always holds, the proof is complete.