DEDUCTION SYSTEMS

Tableau Procedures I

Sebastian Rudolph
Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of \( ALC \) Concepts
- Correctness and Termination
- Summary
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- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  ~ idea: constructive decision procedure that tries to build models
- analog: truth tables in propositional logic
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\[(p \lor q) \rightarrow (\neg p \lor \neg q)\]
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\[\neg (p \lor q) \lor (\neg p \lor \neg q)\]
\[(\neg p \land \neg q) \lor (\neg p \lor \neg q)\]
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Negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:

\[\neg(p \lor q) \lor (\neg p \lor \neg q)\]
\[\neg(q \land \neg q) \lor (\neg p \lor \neg q)\]
\[\neg(p \land \neg q) \lor \neg p \lor \neg q\]
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Simple Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]
Simple Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches
Simple Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

\[\neg p \land \neg q\]
\[\neg p\]
\[\neg q\]

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches
Simple Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- \(-p\)
- \(-q\)

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches
- compare: truth table

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<th>(I(q))</th>
<th>(I(\neg p))</th>
<th>(I(\neg q))</th>
<th>(I(p \lor q))</th>
<th>(I(\neg p \lor \neg q))</th>
<th>(I((p \lor q) \rightarrow (\neg p \lor \neg q)))</th>
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</tbody>
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TU Dresden Deduction Systems
Simple Tableau with Contradiction

\((\neg p \lor q) \land p \land \neg q\)

- if a branch contains an atomic contradiction (clash), we call this branch **closed**
- a tableau is closed, if all its branches are
- a complete tableau without open branches shows the formula's unsatisfiability
Simple Tableau with Contradiction

\[ (\neg p \lor q) \land p \land \neg q \]
\[ \neg p \lor q \]
\[ p \]
\[ \neg q \]
Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

\[\neg p \lor q\]

\[p\]

\[\neg q\]

\[\neg p\]

\[q\]
Simple Tableau with Contradiction

\((\neg p \lor q) \land p \land \neg q\)

- \(\neg p \lor q\)
  - \(p\)
    - \(\neg q\)
  - \(\neg p\)
    - \(q\)
Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

\[
\neg p \lor q
\]

\[
p
\]

\[
\neg q
\]

\[
\neg p
\]

\[
q
\]

\[\bot\]

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Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

- \(\neg p \lor q\)
- \(p\)
- \(\neg q\)

\(\neg p\)
\(q\)

\(\bot\)
\(\bot\)

- if a branch contains an atomic contradiction (clash), we call this branch closed
- a tableau is closed, if all its branches are
Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

\[-p \lor q\]

\[p\]

\[-q\]

\[-p \quad q\]

\[\bot \quad \bot\]

- if a branch contains an atomic contradiction (clash), we call this branch closed
- a tableau is closed, if all its branches are
- a complete tableau without open branches shows the formula’s unsatisfiability
Constructing a Model from the Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

\[-p \land \neg q
\]

\[-p
\]

\[-q
\]

• given an open branch, we can construct a model
Constructing a Model from the Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

\[-\neg p \land \neg q\]
\[-\neg p\]
\[-\neg q\]

- given an open branch, we can construct a model
- let \( I(p) = \text{false} \) and let \( I(q) = \text{false} \)
Constructing a Model from the Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

\[\neg p \land \neg q\]

\[\neg p\]

\[\neg q\]

- given an open branch, we can construct a model
- let \(I(p)=\text{false}\) and let \(I(q)=\text{false}\)
- let \(I(p)=\text{false}\) (\(I(q)\) is irrelevant since not in the branch, default assignment false)
Constructing a Model from the Tableau

\[ (\neg p \land \neg q) \lor \neg p \lor \neg q \]

- given an open branch, we can construct a model
- let \( I(p) = \text{false} \) and let \( I(q) = \text{false} \)
- let \( I(p) = \text{false} \) (\( I(q) \) is irrelevant since not in the branch, default assignment \text{false})
- let \( I(q) = \text{false} \) (\( I(p) \) is irrelevant since not in the branch, default assignment \text{false})
Propositional Tableau

- not always exponentially many combinations have to be checked (as opposed to truth table method)
- branches can be built one after the other $\Rightarrow$ only polynomial space needed
- if we care about satisfiability we can stop after constructing the first complete open branch
Construction with only one Branch in Memory

\[(\neg p \lor q) \land p \land q\]
Construction with only one Branch in Memory

\[(\neg p \lor q) \land p \land q\]

\[\neg p^{1a} \lor q^{1b}\]

\[p\]

\[q\]

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
Construction with only one Branch in Memory

\((\neg p \lor q) \land p \land q\)

\(\neg p^{1a} \lor q^{1b}\)

\(p\)

\(q\)

\(\neg p^{1a}\)

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
Construction with only one Branch in Memory

\[ (\neg p \lor q) \land p \land q \]

\[ \neg p^{1a} \lor q^{1b} \]

\[ p \]

\[ q \]

\[ \neg p^{1a} \]

\[ \bot^{1a} \]

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
- when encountering a contradiction caused by a choice, remove marked formulae and try next choice

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Construction with only one Branch in Memory

\[(\neg p \lor q) \land p \land q\]

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- when encountering a disjunction we assign so-called choice points
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From Propositional Tableau to Tableau for DLs

How can the tableaux be extended for checking satisfiability of $\mathcal{ALC}$ concepts?

NB: initially, we assume no underlying knowledge base, thus unsatisfiability means that the concept is contradictory “by itself”.

- tableau represents an element of the domain (plus its “environment”)

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From Propositional Tableau to Tableau for DLs

How can the tableaux be extended for checking satisfiability of $\mathcal{ALC}$ concepts? NB: initially, we assume no underlying knowledge base, thus unsatisfiability means that the concept is contradictory “by itself”.

- tableau represents an element of the domain (plus its “environment”)
- tableau branch: finite set of propositions of the form $C(a)$, $r(a, b)$
- for existential quantifiers, new domain elements are introduced
- universal quantifiers propagate formulae (=concept expressions) to neighboring elements
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Propositional Logic – Some Logical Equivalences

- We aim at negations being present only in front of atomic concepts

\[
\begin{align*}
\varphi \land \psi & \equiv \psi \land \varphi \\
\varphi \lor \psi & \equiv \psi \lor \varphi \\
\varphi \land (\psi \land \omega) & \equiv (\varphi \land \psi) \land \omega \\
\varphi \lor (\psi \lor \omega) & \equiv (\varphi \lor \psi) \lor \omega \\
\varphi \land \varphi & \equiv \varphi \\
\varphi \lor \varphi & \equiv \varphi \\
\varphi \land (\psi \lor \varphi) & \equiv \varphi \\
\varphi \lor (\psi \land \varphi) & \equiv \varphi \\
\varphi \land (\psi \land \omega) & \equiv (\varphi \land \psi) \land (\varphi \lor \omega) \\
\varphi \lor (\psi \lor \omega) & \equiv (\varphi \lor \psi) \lor (\varphi \land \omega)
\end{align*}
\]
Further Logical Equivalences

\[
\neg (C \land D) \iff \neg C \lor \neg D
\]

\[
\neg (D \lor D) \iff \neg C \land \neg D
\]

\[
\neg \neg C \iff C
\]

\[
\neg (\forall r. C) \iff \exists r. (\neg C)
\]

\[
\neg (\exists r. C) \iff \forall r. (\neg C)
\]

\[
\neg (\leq n \cdot s. C) \iff \geq n + 1 \cdot s. C
\]

\[
\neg (\geq n \cdot s. C) \iff \leq n - 1 \cdot s. C, \quad n \geq 1
\]

\[
\neg (\geq 0 \cdot s. C) \iff \bot
\]

- apply these rules iteratively until none can be applied any more
- \iff equivalent concept in negation normal form
**NNF Transformation**

recursive definition of an NNF transformation:

if $C$ atomic:

$\text{NNF}(C) := C$

$\text{NNF}(\neg C) := \neg C$

otherwise:

$\text{NNF}(\neg\neg C) := \text{NNF}(C)$

$\text{NNF}(C \cap D) := \text{NNF}(C) \cap \text{NNF}(D)$

$\text{NNF}(\neg(C \cap D)) := \text{NNF}(\neg C) \cup \text{NNF}(\neg D)$

$\text{NNF}(C \cup D) := \text{NNF}(C) \cup \text{NNF}(D)$

$\text{NNF}(\neg(C \cup D)) := \text{NNF}(\neg C) \cap \text{NNF}(\neg D)$

$\text{NNF}(\forall r. C) := \forall r. (\text{NNF}(C))$

$\text{NNF}(\neg(\forall r. C)) := \exists r. (\text{NNF}(\neg C))$

$\text{NNF}(\exists r. C) := \exists r. (\text{NNF}(C))$

$\text{NNF}(\neg(\exists r. C)) := \forall r. (\text{NNF}(\neg C))$

$\text{NNF}(\leq n \ s. C) := \leq n \ s. (\text{NNF}(C))$

$\text{NNF}(\neg(\leq n \ s. C)) := \geq n + 1 \ s. (\text{NNF}(C))$

$\text{NNF}(\geq n \ s. C) := \geq n \ s. (\text{NNF}(C))$

$\text{NNF}(\neg(\geq n \ s. C)) := \leq n - 1 \ s. (\text{NNF}(C))$

if $n \geq 1$

$\text{NNF}(\geq 0 \ s. C) := \top$

$\text{NNF}(\neg(\geq 0 \ s. C)) := \bot$

otherwise
NNF Transformation – Example

\[
\begin{align*}
\text{NNF}(\neg(\neg C \sqcap (\neg D \sqcup E))) & = \text{NNF}(\neg\neg C) \sqcup \text{NNF}(\neg(\neg D \sqcup E)) \\
& = \text{NNF}(C) \sqcup \text{NNF}(\neg(\neg D \sqcup E)) \\
& = C \sqcup \text{NNF}(\neg(\neg D \sqcup E)) \\
& = C \sqcup (\text{NNF}(\neg D) \sqcap \text{NNF}(\neg E)) \\
& = C \sqcup (\text{NNF}(D) \sqcap \text{NNF}(\neg E)) \\
& = C \sqcup (D \sqcap \text{NNF}(\neg E)) \\
& = C \sqcup (D \sqcap \neg E)
\end{align*}
\]
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Tableau for $\mathcal{ALC}$ Concepts

- tableau for a propositional formulal $\alpha$: one element, labeled with subformulae of $\alpha$
- tableau for an $\mathcal{ALC}$ concept $C$: graph (more precisely: tree) where the nodes are labeled with subformulae of $C$
- root labeled with $C$
- represents model for $C$ (if complete and clash-free)
- non-root nodes are enforced by existential quantifiers

**Definition**

Let $C$ be an $\mathcal{ALC}$ concept, $\text{SF}(C)$ the set of all subformulae of $C$ and $\text{Rol}(C)$ the set of all roles occurring in $C$. A tableau for $C$ is a tree $G = \langle V, E, L \rangle$, with nodes $V$, edges $E \subseteq V \times V$ and a labeling function $L$ with $L: V \rightarrow 2^{\text{SF}(C)}$ and $L: V \times V \rightarrow 2^{\text{Rol}(C)}$. 
Properties of the $\mathcal{ALC}$ Tableau Algorithm

- the algorithm is specified as a set of rules
- every rule breaks down a complex concept into its parts
- rules applicable in any order
- the algorithm is non-deterministic (due to disjunction)
- check for atomic contradictions

Tableau algorithm for checking satisfiability of $\mathcal{ALC}$ concepts

**Input:** an $\mathcal{ALC}$ concept in NNF

**Output:**
- true if there is a clash-free tableau where no more rules can be applied
- false otherwise (tableau closed)
Tableau Rules for $\mathcal{ALC}$ Concepts

$\sqcap$-rule: For an arbitrary $v \in V$ mit $C \sqcap D \in L(v)$ and 
\{C, D\} \not\subseteq L(v), let \(L(v) := L(v) \cup \{C, D\}\).

$\sqcup$-rule: For an arbitrary $v \in V$ with $C \sqcup D \in L(v)$ and 
\{C, D\} \cap L(v) = \emptyset, choose $X \in \{C, D\}$ and let \(L(v) := L(v) \cup \{X\}\).

$\exists$-rule: For an arbitrary $v \in V$ with $\exists r.C \in L(v)$ such that 
there is no $r$-successor $v'$ of $v$ with $C \in L(v')$,
let $V = V \cup \{v'\}$, $E = E \cup \{(v, v')\}$, \(L(v) := \{C\}\) and \(L(v, v') := \{r\}\) for $v'$ a new node.

$\forall$-rule: For arbitrary $v, v' \in V$, $v'$ $r$-neighbor of $v$,
\(\forall r.C \in L(v)\) and $C \notin L(v')$, let \(L(v') := L(v') \cup \{C\}\).

- a node $v'$ is an $r$-neighbor of a node $v$ if $(v, v') \in E$ and $r \in L(v, v')$
# Tableau Rules for $\mathcal{ALC}$ Concepts

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- a node $v'$ is an $r$-neighbor of a node $v$ if $\langle v, v'\rangle \in E$ and $r \in L(v, v')$
- rule application order: “don’t care” non-determinism
Tableau Rules for $\mathcal{ALC}$ Concepts

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\forall\)-rule: For arbitrary \(v, v' \in V\), \(v'\) \(r\)-neighbor of \(v\), \(\forall r. C \in L(v)\) and \(C \not\in L(v')\), let \(L(v') := L(v') \cup \{C\}\).

- a node \(v'\) is an \(r\)-neighbor of a node \(v\) if \((v, v') \in E\) and \(r \in L(v, v')\)
- rule application order: “don’t care” non-determinism
- choice of disjunction: “don’t know” non-determinism
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \cup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \]

\[ L(u) = \{ C, \exists r. (A \cup \exists r. B), \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \} \]

\[ L(v) = \{ A \cup \exists r. B \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \sqcup A)) \]

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\[ L(v) = \{ A \sqcup \exists r. B \} \]

\[ L(w) = \{ \neg A \} \]
Tableau Algorithmus Example

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\[ L(u) = \{ C, \exists r. (A \cup \exists r. B), \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \} \]

\[ L(v) = \{ A \cup \exists r. B, \neg A, \forall r. (\neg B \cup A) \} \]

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L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \\
               \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} 
\]

\[
L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), A \} 
\]

\[
L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} 
\]
Tableau Algorithmus Example

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\[
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\[ L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \times, \exists r. B \} \]

\[ L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \]

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\[ L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} \]

\[ L(x) = \{ B \} \]
Tableau Algorithmus Example

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\[
\begin{align*}
L(u) &= \{ C, \exists r. (A \sqcup \exists r. B), \\
&\quad \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \\
L(v) &= \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \times, \exists r. B \} \\
L(w) &= \{ \neg A, \forall r. (\neg B \sqcup A) \} \\
L(x) &= \{ B, \neg B \sqcup A \}
\end{align*}
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \cup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \]

\[
L(u) = \{ C, \exists r. (A \cup \exists r. B), \\
\exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \}
\]

\[
L(v) = \{ A \cup \exists r. B, \neg A, \forall r. (\neg B \cup A), \overline{X}, \exists r. B \}
\]

\[
L(w) = \{ \neg A, \forall r. (\neg B \cup A) \}
\]

\[
L(x) = \{ B, \neg B \cup A, \neg B \}
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \uplus \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \uplus A)) \]

\[ L(u) = \{ C, \exists r. (A \uplus \exists r. B), \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \uplus A)) \} \]

\[ L(v) = \{ A \uplus \exists r. B, \neg A, \forall r. (\neg B \uplus A), \times, \exists r. B \} \]

\[ L(w) = \{ \neg A, \forall r. (\neg B \uplus A) \} \]

\[ L(x) = \{ B, \neg B \uplus A, \times \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \cup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \]

\[
\begin{align*}
L(u) &= \{ C, \exists r. (A \cup \exists r. B), \\
& \quad \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \}\?
\end{align*}
\]

\[
\begin{align*}
L(v) &= \{ A \cup \exists r. B, \neg A, \forall r. (\neg B \cup A), \times, \exists r. B \}\?
\end{align*}
\]

\[
\begin{align*}
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\end{align*}
\]

\[
\begin{align*}
L(x) &= \{ B, \neg B \cup A, \times, A \}\?
\end{align*}
\]
Tableau Algorithm Example

the model $\mathcal{I}$ constructed by the algorithm is the following:

\[
\begin{align*}
\Delta^\mathcal{I} &= \{u, v, w, x\} \\
A^\mathcal{I} &= \{x\} \\
B^\mathcal{I} &= \{x\} \\
r^\mathcal{I} &= \{\langle u, v \rangle, \langle u, w \rangle, \langle v, x \rangle\}
\end{align*}
\]

Check that indeed $C^\mathcal{I} = \{u\}$, given the defined semantics of $\mathcal{ALC}$
Tableau Algorithm Properties

1. the model is finite: only finitely many elements in the domain
2. the model is tree-shaped: the tableau is a labeled tree

The algorithm will always construct finite trees
- from a clash-free tableau, we can construct a finite model
- if there is no clash-free tableau, there is no model
Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary
Tableau Properties

- the depth (number of nested quantifiers) decreases in every node
- every node is labeled only with subformulae of $C$
- $C$ has only polynomially many subformulae
- if the output is true we can build a model out of the constructed tableau
- on the other hand, we can use a model of a satisfiable concept to construct a clash-free tableau for this concept
Tableau Algorithm for $\mathcal{ALC}$ Concepts

**Theorem**

1. The algorithm terminates for every input.
2. If the output is $true$, then the input concept is satisfiable.
3. If the input concept is satisfiable, then the output is $true$. 

**Corollary**

Every $\mathcal{ALC}$ concept $C$ has the following properties:

1. Finite model property: If $C$ has a model, then it has a finite one.
2. Tree model property: If $C$ has a model, then it has a tree-shaped one.
Tableau Algorithm for $\mathcal{ALC}$ Concepts

**Theorem**

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Summary

- we now have a constructive method for building model abstractions
- satisfiable $\mathcal{ALC}$ concepts always have a finite model that we can construct
- the algorithm is correct, complete and terminating
- serves as basis for practically implemented algorithms
- next: extension to knowledge bases