TECHNISCHE UNIVERSITÄT DRESDEN

VOTING FOR BINS

INTEGRATING IMPRECISE PROBABILISTIC BELIEFS INTO THE CONDORCET JURY THEOREM

Jonas Karge
Computational Logic Group, TU Dresden

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Introduction

Scenario: Multiple experts assess the likelihood of an event such as:

Example: Global sea level will rise at least 1,5 meters until the year 2100 above the level of 2000.

Two fundamental questions:

(1) How can we appropriately represent the probabilistic beliefs of experts?

(2) What constitutes a reasonable method for aggregation?

Outline

(i) Aggregation Method: Voting in a jury theorem setting
(ii) Representation: Imprecise probabilistic beliefs
(iii) Voting with this Representation: Supervaluationism
(iv) Embedding

The Condorcet Jury Theorem

The Condorcet Jury Theorem (CJT)

Marie Jean Antoine Nicolas Caritat Marquis de Condorcet

Theorem: For odd-numbered **homogenous** groups of **independent** and **reliable** agents in a **dichotomic** voting setting, the probability that majority voting identifies the correct alternative

- increases monotonically with the number of agents and (non-asymptotic part)
- converges to 1 as the number of agents goes to infinity. (asymptotic part)

Voting

Define **approval voting** and obtain simpler voting mechanisms as special cases.

Given: finite set of n agents $\mathcal{A} = \{a_1, \dots, a_n\}$
finite set of m choices $\mathcal{W} = \{\omega_1, \dots, \omega_m\}$

- **approval voting (instance)**: relation $V \subseteq \mathcal{A} \times \mathcal{W}$
  $(a_i, \omega_j) \in V$ means agent a_i approves choice ω_j
- given $\omega \in \mathcal{W}$, obtain **score** $\#_V\, \omega$ as overall number of votes that ω receives, i.e.,

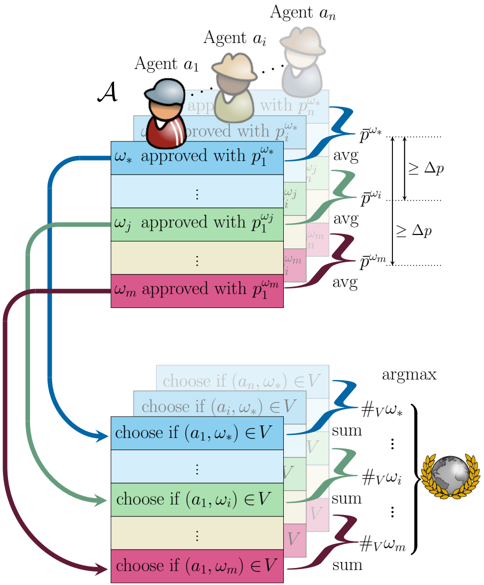
$$\#_V\, \omega = |\{a_i \in \mathcal{A}_n \mid (a_i, \omega) \in V\}|$$

- ω **wins approval vote** V if it receives strictly more votes than any other choice:

$$\#_V\, \omega > \max_{\omega' \in \mathcal{W} \setminus \{\omega\}} \#_V\, \omega'$$

The Voting Scenario

$\mathcal{W}$



Success:

CJT under approval voting

Asymptotic result:

Theorem: For odd-numbered **heterogenous** groups of **independent** and **Δ-reliable** agents in a voting setting with a finite number of alternatives, the probability that approval voting identifies the correct alternative

- converges to 1 as the number of agents goes to infinity.

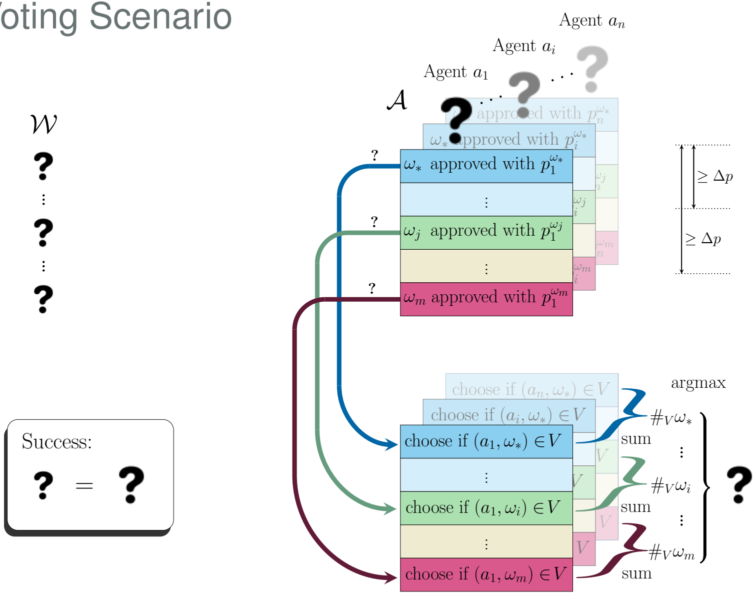
Beyond the convergence behavior in the infinite:

Theorem: In a Δp-group reliable setting with m choices, the worst case approval vote success probability is at least P_{min} whenever the number of agents is equal or higher than

$$\min\left(\tfrac{2}{\Delta p^2}\ln Q, 1+(\tfrac{1}{\Delta p^2}-1)Q\right), \qquad (1)$$

where $Q = 2\frac{m-1}{1-P_{\text{min}}}$ is the twofold ratio between the number of incorrect alternatives and the admissible error probability.

The Voting Scenario



Probabilistic Beliefs

Precise Probabilities

Starting point:

Definition: A probability function $\mathbb{P}$ is a function $\mathbb{P} : 2^{\Omega} \rightarrow \mathbb{R}$, satisfying the probability axioms.

$\Rightarrow$ Output of function reflects the agent's **degree of belief** in that proposition.

Reconsider:

Example: Global sea level will rise at least 1,5 meters until the year 2100 above the level of 2000.

Problem:

What probability is the expert supposed to assign to A?

Imprecise Probabilities

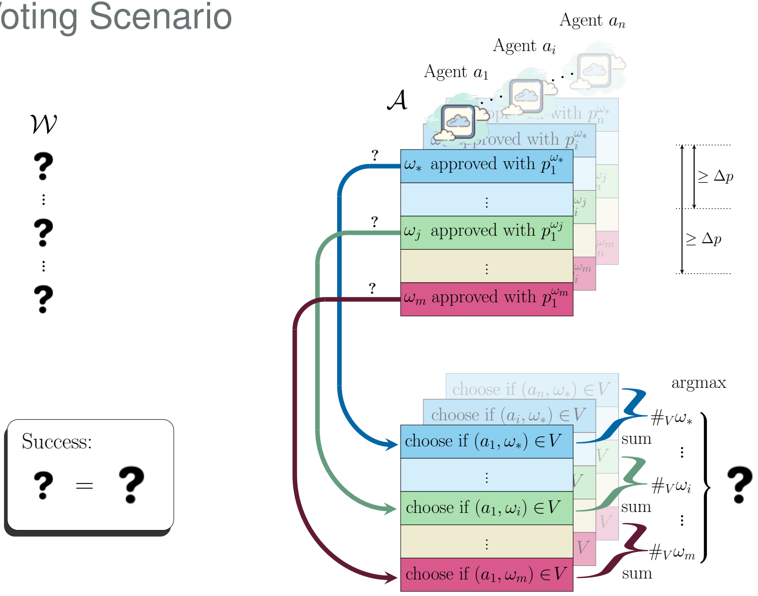
Definition:
Imprecise probabilities are sets of probability functions.

We refer to a specific set of probability functions as the agent's representor, denoted by $\mathcal{P}$.

Definition: An agent's imprecise degree of belief in a proposition H is represented by a function, $\mathcal{P}(H)$, with $\mathcal{P}(H) = \{\mathbb{P}(H) : \mathbb{P} \in \mathcal{P}\}$.

Example: Assume, the agent's representor consists of three probability functions that assign event A values from the set $\{0.4, 0.6, 0.8\}$. Assuming **convexity**, we may represent the agent's imprecise degree of belief with $\mathcal{P}(A) = [0.4, 0.8]$. Thus, our agent is $40 - 80\%$ confident that event A will occur, i.e., that proposition A is true.

The Voting Scenario



Supervaluationism and Voting

Standard Supervaluationism

Consider a **vague predicate** such as **tall**
⇒ can be made more precise by introducing cutoff points (i.e. 300cm, 180cm, 20cm).

Each cutoff point represents a **precisification** of that predicate.

Truth value of vague predicates:

- **Determinate truth** (true according to all admissible precisifications, person who is 400cm tall);
- **Determinate falsehood** (false according to all admissible precisifications, 10cm);
- **Indeterminate truth** (true and false according to some admissible precisifications, 190cm).

Modified Supervaluationism

Definition: A proposition is predominantly true if it is true according to a relative majority of admissible precisifications.

Definition: Given two propositions, A and B, an agent is considered to be predominantly more confident in proposition A than in proposition B if a greater proportion of elements within the agent's imprecise degree of belief satisfy the condition $Pr(A) > Pr(B)$.

Problem: we need to measure the proportion of possibly infinitely many elements.

⇒ For any closed, $[a, b]$, open, (a, b), or half open, $(a, b]$ or $[a, b)$, interval it holds that its **Lebesque measure** is of length $l = b - a$;

⇒ determine the proportion of elements in favor of a proposition by measuring the length of the corresponding interval.

Modified Supervaluationism and Voting

Example: Consider proposition A and its complement B, i.e. global sea level will **not** rise at least 1,5 meters until the year 2100 above the level of 2000. Suppose we have $\mathcal{P}(A) = [0.4, 1]$ as our agent's imprecise degree of belief. For those elements represented by $(0.5, 1]$ it holds true that $Pr(A) > Pr(B)$. For those represented by $[0.4, 0.5)$ we have $Pr(B) > Pr(A)$. Taking their Lebesque measure, we obtain $l(A) = 0.5$ as well as $l(B) = 0.1$. Thus, the agent is predominantly more confident in proposition A.

Predominant confidence and voting:

Definition: Given a set of alternatives $\mathcal{W} = \{\omega_1, \ldots, \omega_m\}$ and set of agents $\mathcal{A} = \{a_1, \ldots, a_n\}$, agent a_i approves alternative ω_j if the agent is predominantly more confident in that alternative than in its competitors.

Embedding

What is an Alternative?

Recall: We are given a **finite** set of alternatives $\omega_1, ..., \omega_m$ and one $\omega_k \in \mathcal{W}$ represents the **correct probability** for an event to occur.

Suppose, proposition A has a probability of 40.1862345% to occur.

First idea: Each alternative represents a precise probability value.
$\Rightarrow$ Similar problems as on the belief level.

Second idea: Each alternative represents an **interval of probability values** of the form: $[X_{min}, X_{max}]$.

Simplest (theoretically excluded) case: A **single alternative** with $[X_{min}, X_{max}] = [0, 1]$.

More Alternatives?

To obtain more alternatives, $[X_{\text{min}}, X_{\text{max}}]$ is further divided into **subintervals**.

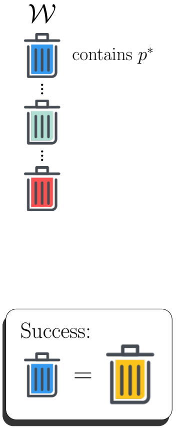
$\Rightarrow$ Partition the unit interval $[0, 1]$ into $2k$ subintervals of equal Lebesgue measure, denoted as $[X_{j-1}, X_j)$, where $j = 1, 2, ..., 2k$.

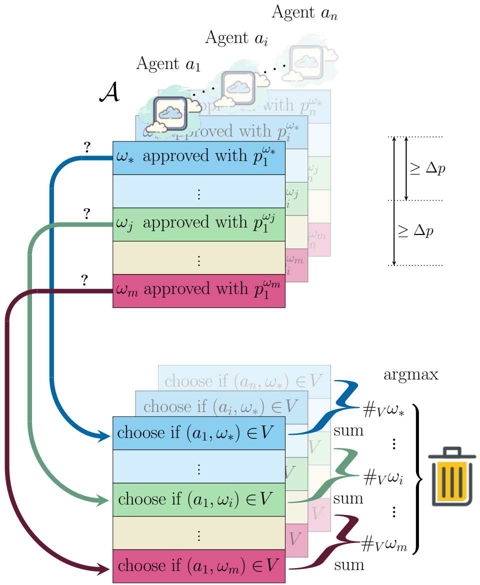
Note: The value of k depends on the desired precision.

Example: If a small group of agents is highly reliable in their estimates, a moderate precision of 10% may be adequate, achieved by setting $k = 5$ (yielding $2k = 10$) and providing 10 subintervals of equal Lebesgue measure.

Note: Each subinterval is referred to as a bin.

The Voting Scenario





Confidence in Bins

Problem: **predominant confidence** compares the beliefs in two propositions.

⇒ compare confidence in two probabilistic assessments for the same proposition.

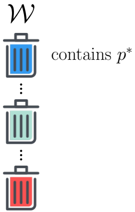
Definition: Let A be a proposition, $\mathcal{P}(A) = [a, b]$ be an agent's imprecise degree of belief in A, and let $[X_{j-1}, X_j)$, $j = 1, 2, ..., 2k$ be $2k$ bins defined on the unit interval reflecting probability values for A to occur. Given two bins B_1 and B_2, we say that an agent is predominantly more confident in B_1 if the intersection of $\mathcal{P}(A)$ and B_1 is of greater Lebesque measure than the one of $\mathcal{P}(A)$ and B_2. That is, $l(\mathcal{P}(A) \cap B_1) \geq l(\mathcal{P}(A) \cap B_2)$.

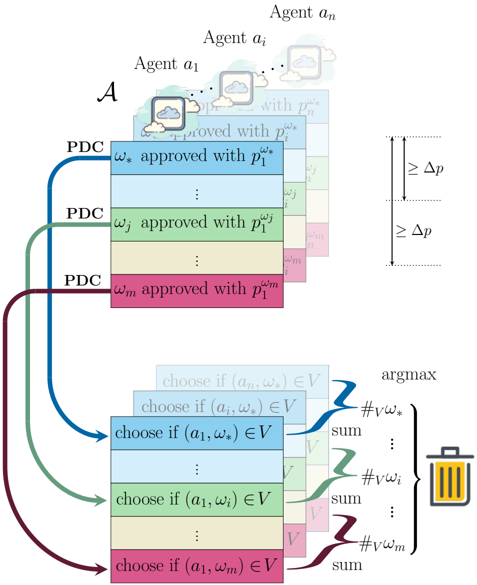
Voting for Bins

Example: Suppose there are only two bins for proposition A with $B_1 = [0, 0.5)$ and $B_2 = [0.5, 1]$ and let $\mathcal{P}(A) = [0.3, 0.9]$. We then have $\mathcal{P}(A) \cap B_1 = [0.3, 0.5)$ and $\mathcal{P}(A) \cap B_2 = [0.5, 0.9]$. This results in $l(\mathcal{P}(A) \cap B_1) = 0.2$ as well as $l(\mathcal{P}(A) \cap B_2) = 0.4$. Thus, the agent is predominantly more confident in the second bin.

Definition: Let $m = \omega_1, ..., \omega_m$ be a set of alternatives where each ω_i represents a bin of the form $[X_{j-1}, X_j)$. Moreover, let $a_1, ..., a_n$ be a set of agents and let V represent a single election. We say that an agent a_i votes for an alternative ω_j if she is predominantly confident in that alternative. That is, if $(l(\mathcal{P}(A) \cap \omega_j) \geq l(\mathcal{P}(A) \cap \omega_k))$ for all $j \neq k$ then $(a_i, \omega_j) \in V$.

The Voting Scenario





Estimate Permitted Precision

- In typical application scenarios, the number of voters on the expert board is known beforehand;
- i.e. studies on forecasting capabilities involving 42 climate scientists or 140 experts on COVID-19 outbreaks;

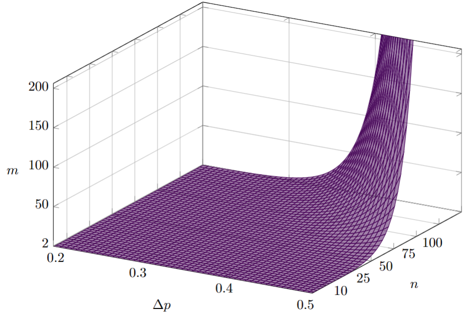
$\Rightarrow$ determine the maximal precision that can be allowed.

Theorem: In a Δp-group reliable setting where $\Delta p \in (0, 1)$ with n agents, the worst case approval vote success probability is at least $P_{\min}$ whenever the number of alternatives is equal or lower than

$$max(\frac{(1-p_{\min})}{(2e^{-\frac{1}{2}n\Delta p^2})} + 1, \frac{(1-p_{\min})(1+(n-1)\Delta p^2)}{2(1-\Delta p^2)} + 1). \tag{2}$$

$\Rightarrow$ Directly translates into the maximal allowed precision in percentage (C) with $C = \frac{100}{m}$.

Illustration



Selection of data points			
Number of Experts	Δp	Number of Bins	Precision
50	0.3	< 2	N.A.
50	0.4	4	25%
75	0.3	2	50%
75	0.4	21	4.8%
100	0.3	6	16%
100	0.4	150	$0.\overline{6}\%$
150	0.3	44	2.2%
150	0.4	8139	0.01%
200	0.3	406	0.25%
200	0.4	444307	0.0002%

Figure: Maximal number of permitted bins for $P_{min} = 0.9$ and varying Δp and n (left) as well as a selection of data points (right).

Summary and Future Work

Summary

We embedded imprecise probabilistic beliefs into a generalization of the **Condorcet Jury Theorem**:

- combined the epistemological account of **imprecise degrees of belief** as well as their interpretation in the **supervaluationinstic theory of vagueness** with a voting setting;
- each alternative represents an interval of **probability assessments (a bin)** for the same proposition as the imprecise degree of belief;
- established a direct correspondence between the number of **bins** in the voting process and the **maximal permitted precision** during aggregation;
- gave **estimate** for the allowed precision in the aggregation procedure.

Future work

- Compare the performance of **voting-based aggregation** of imprecise probabilistic beliefs with traditional methods of **probabilistic opinion pooling**;
- investigate applications at the intersection of ensemble learning in **ML and social choice**;
- consider **rough set theory** instead of **supervaluationism.**