Complexity Theory
Exercise 6: Diagonalization
2 Dec 2025

Exercise 6.1. Find the fault in the following proof of $P \neq NP$.

1. Assume that $P = NP$.

2. Then $SAT \in P$ and thus there exists a $k \in \mathbb{N}$ such that $SAT \in DTime(n^k)$.

3. Because every language in NP is poly-time reducible to SAT, we have $NP \subseteq DTime(n^k)$.

4. It follows that $P \subseteq DTime(n^k)$.

5. By the Time Hierarchy Theorem there exist languages in $DTime(n^{k+1})$ that are not in $DTime(n^k)$, contradicting $P \subseteq DTime(n^k)$.

6. Therefore, $P \neq NP$.

Exercise 6.2. Show the following.

1. $\text{TIME}(2^n) = \text{TIME}(2^{n+1})$

2. $\text{TIME}_*(2^n) \subset \text{TIME}_*(2^{2n})$

3. $\text{NTIME}(n) \subset \text{PSPACE}$

Exercise 6.3. Define a function that is computable but not time-constructible.

Exercise 6.4. Consider the function $\mathsf{pad}\colon \Sigma^* \times \mathbb{N} \to \Sigma^*\#^*$ defined as $\mathsf{pad}(s, \ell) = s\#^j$, where $j = \max(0, \ell - |s|)$. For some language $\mathbf{A} \subseteq \Sigma^*$ and $f\colon \mathbb{N} \to \mathbb{N}$ define $\mathsf{pad}(\mathbf{A}, f) = \{\, \mathsf{pad}(s, f(|s|)) \mid s \in \mathbf{A} \,\}$.
   Show all of the following satements.

1. Show that, if $\mathbf{A} \in \text{DTIME}(n^6)$, then $\mathsf{pad}(\mathbf{A}, n^2) \in \text{DTIME}(n^3)$.

2. Show that, if $\text{NExpTime} \neq \text{ExpTime}$, then $P \neq NP$.

3. Show for every $\mathbf{A} \subseteq \Sigma^*$ and $k \in \mathbb{N}$ that $\mathbf{A} \in P$ if and only if $\mathsf{pad}(\mathbf{A}, n^k) \in P$.

4. Show $P \neq \text{DSPACE}(n)$.

5. Show $NP \neq \text{DSPACE}(n)$.

Exercise 6.5. You are given two oracles and one of them is the set **TRUE QBF**, but you do not know which one. Design a polynomial algorithm that decides **TRUE QBF** using these oracles.