Finite and Algorithmic Model Theory

Lecture 1 (Dresden 12.10.22, Revised version)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

TECHNISCHE UNIVERSITÄT DRESDEN

Uniwersytet Wrocławski

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Today's agenda

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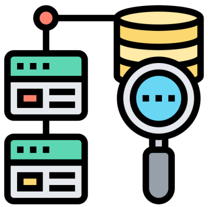

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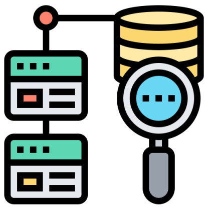

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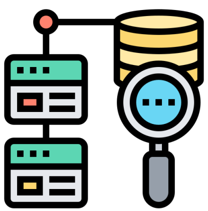

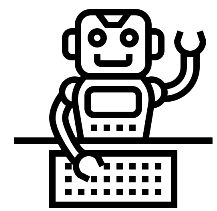

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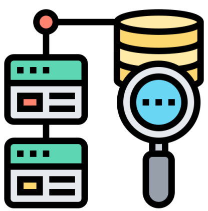

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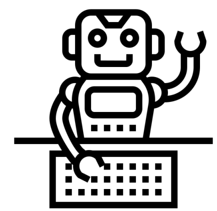

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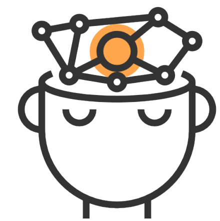

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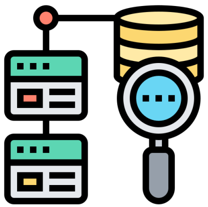

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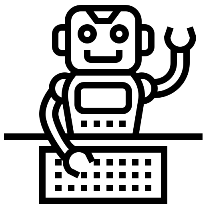

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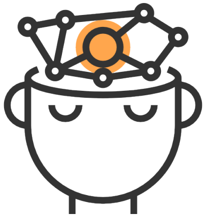

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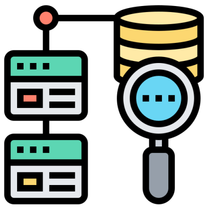

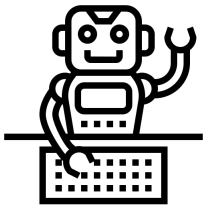

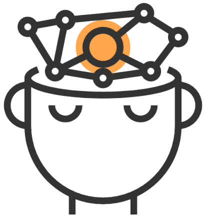

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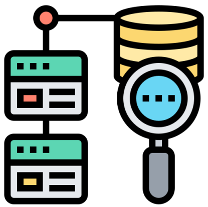

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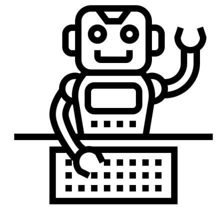

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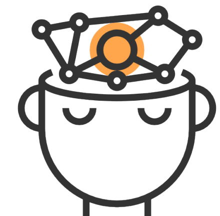

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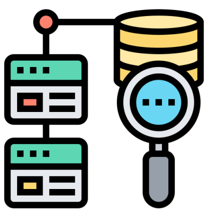

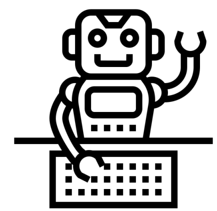


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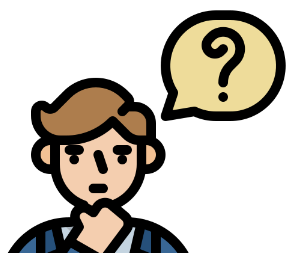

Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

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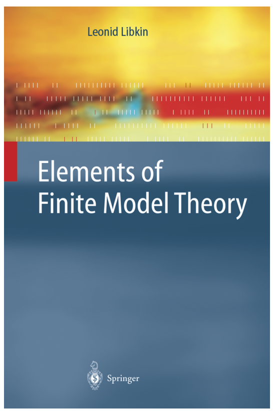

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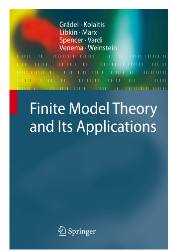

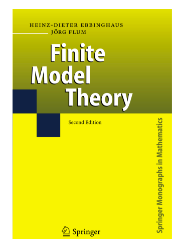

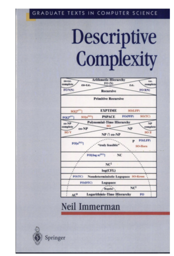

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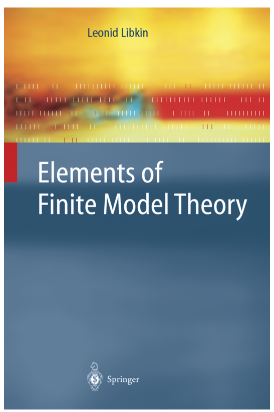

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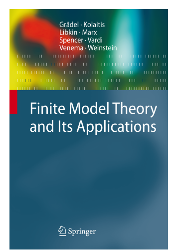

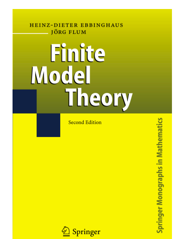

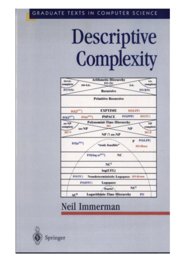

Last but Not Least: I offer MSc/PHD research projects for motivated students!

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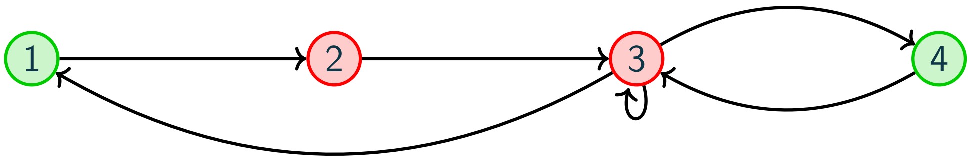
$\mathfrak{A} :=$ 

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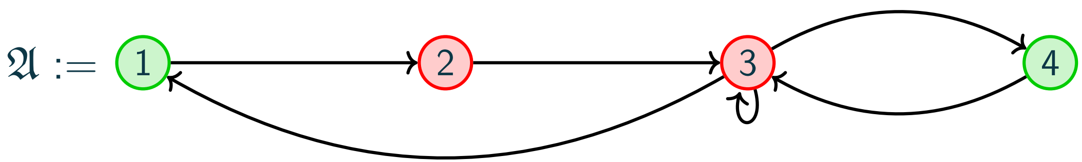

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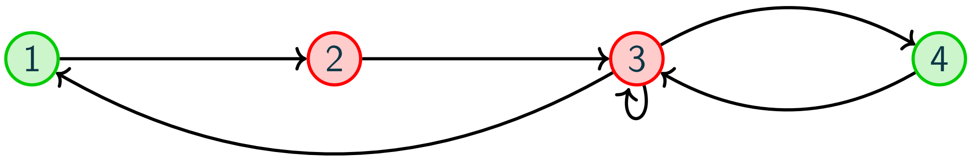
$\mathfrak{A} :=$ 

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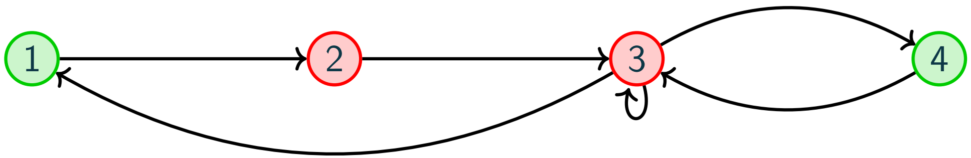
$\mathfrak{A} :=$ 

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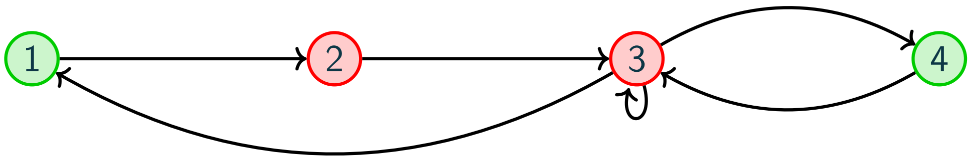
$\mathfrak{A} :=$ 

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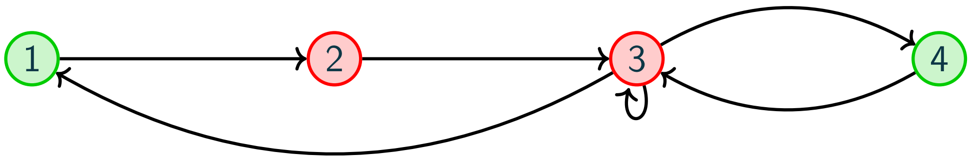
$\mathfrak{A} :=$ 

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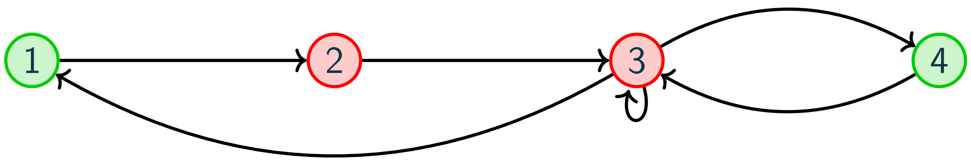
$\mathfrak{A} :=$ 

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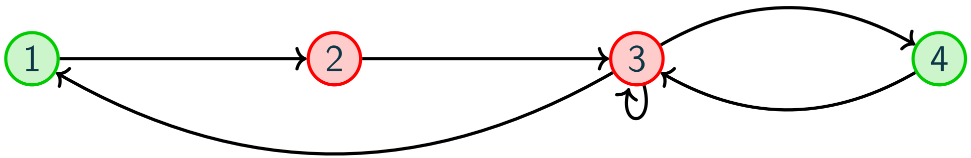
$\mathfrak{A} :=$ 

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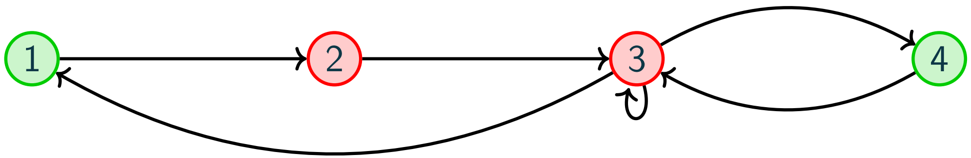
$\mathfrak{A} :=$ 

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Constants $\approx$ elements, unary relations $\approx$ colours, binary (resp. higher-arity) relations $\approx$ (hyper)edges

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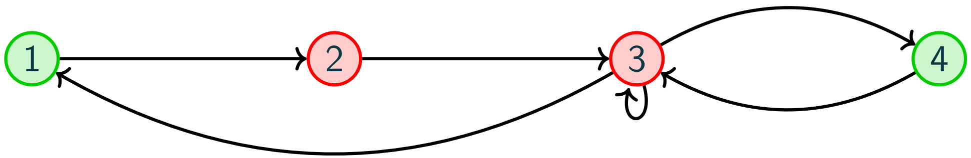
$\mathfrak{A} :=$ 

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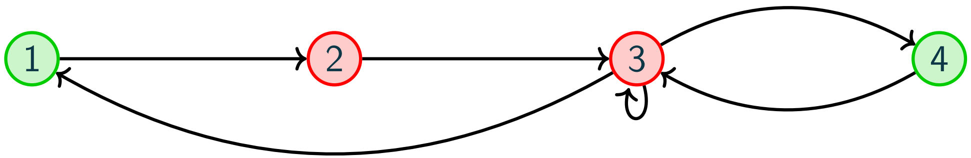
$\mathfrak{A} :=$ 

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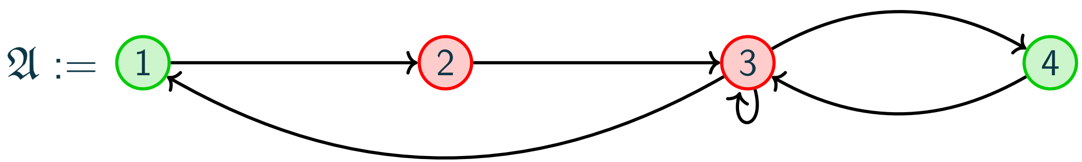

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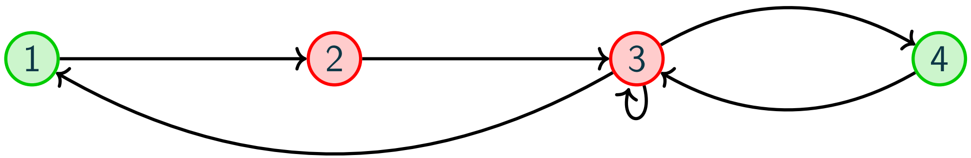

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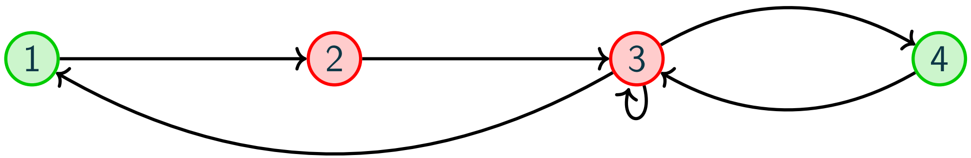
$\mathfrak{A} :=$ 

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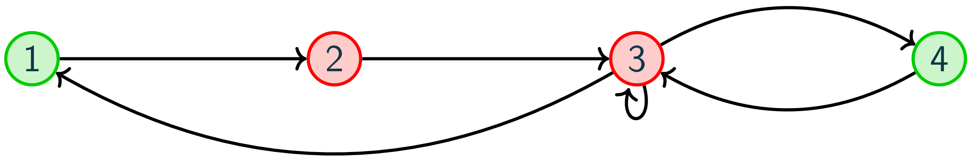
$\mathfrak{A} :=$ 

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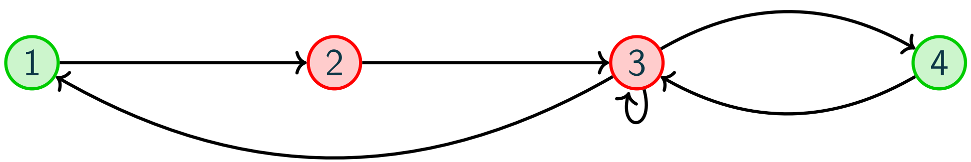
$\mathfrak{A} :=$ 

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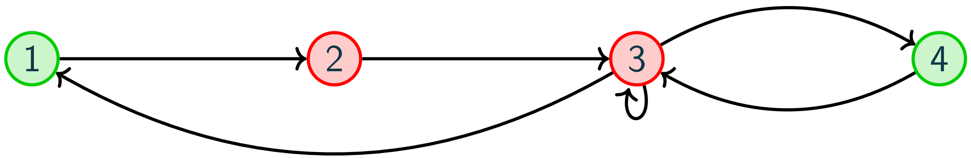
$\mathfrak{A} :=$ 

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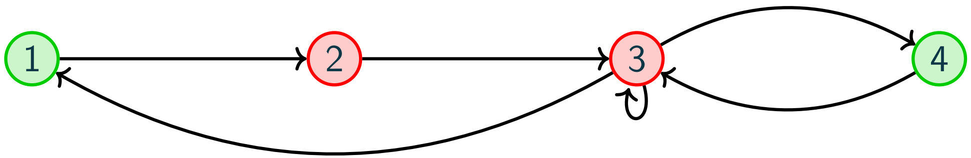
$\mathfrak{A} :=$ 

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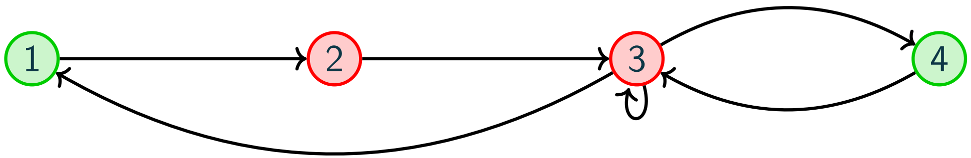

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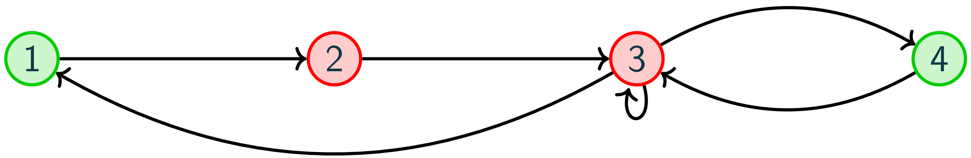
$\mathfrak{A} :=$ 

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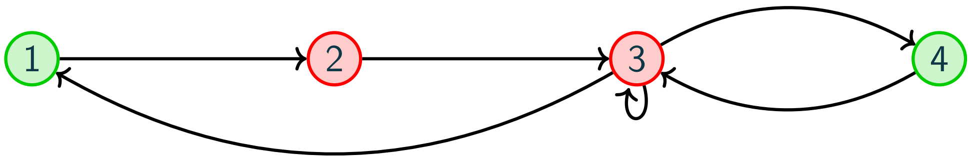
$\mathfrak{A} :=$ 

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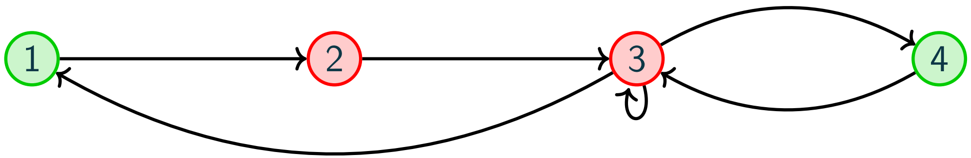

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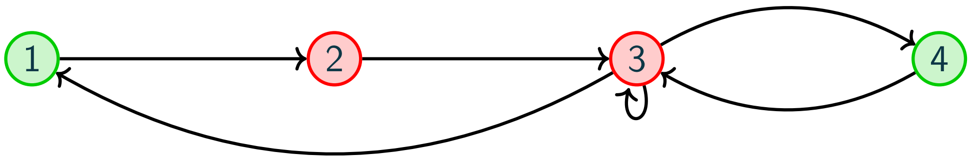

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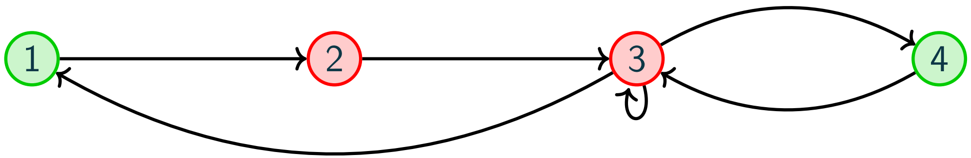
$\mathfrak{A} :=$ 

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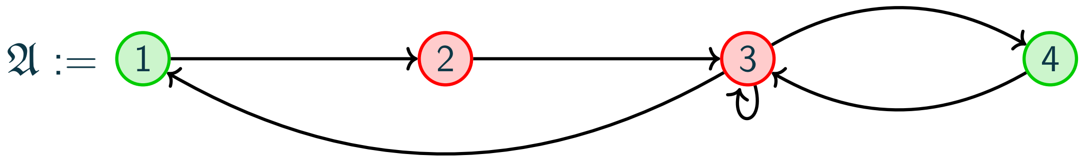

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Example (of a First-Order Logic (FO) Formula)

(in a coloured graph:) Any node is either green or red.

$\varphi := \forall x \; (G(x) \vee R(x)) \wedge (G(x) \leftrightarrow \neg R(x))$

We write $\mathfrak{A} \models \varphi$ to indicate that $\mathfrak{A}$ satisfies φ or $\mathfrak{A}$ is a model of φ.

Formulae often employ: Variables: $x, y, z, X, Y, \ldots$ Boolean connectives: $\wedge, \vee, \neg, \leftrightarrow, \bigvee_{i=0}^{\infty}, \ldots$

Quantifiers: $\forall, \exists, \exists^{even}, \exists^{=42}, \exists^{35\%}, \exists Set, \Diamond$, Predicates (relational symbols): $P, \in, =, \sim$,

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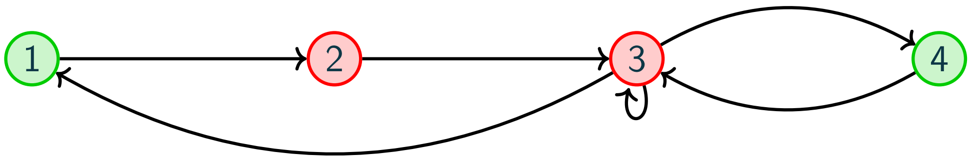
$\mathfrak{A} :=$ 

over a signature $\tau := \{\mathrm{G}^{(1)}, \mathrm{R}^{(1)}, \mathrm{E}^{(2)}\}$

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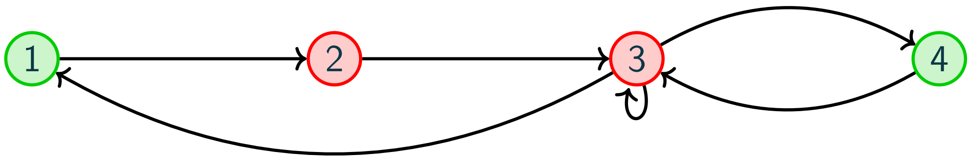
$\mathfrak{A} :=$ 

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More examples I.

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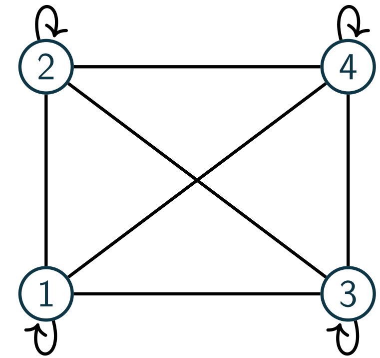

$\models \varphi_{\text{4clique}}$

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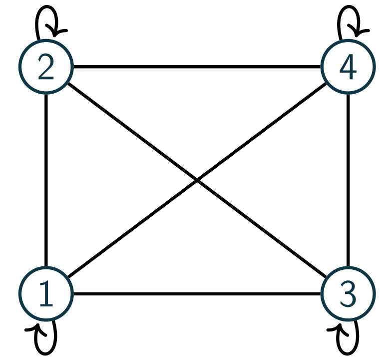

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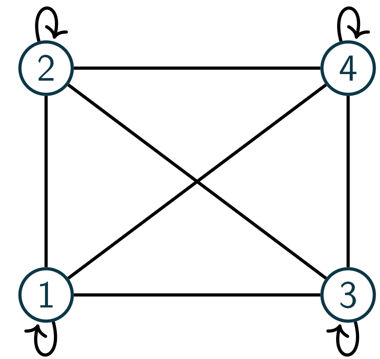

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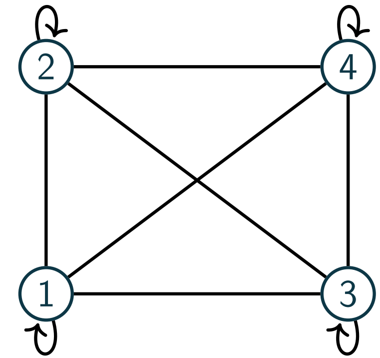

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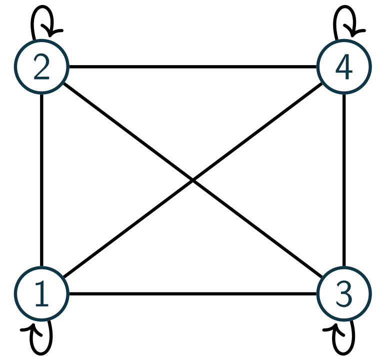

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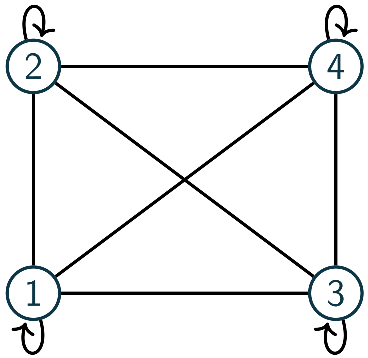

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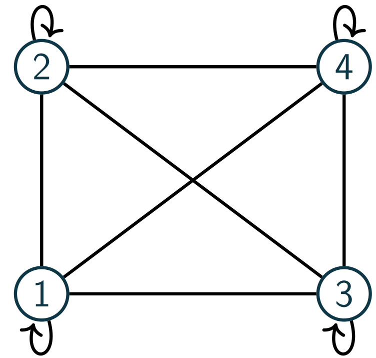

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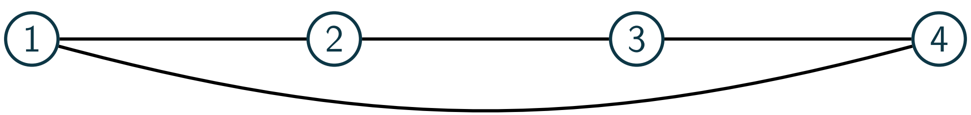
$\mathfrak{G} :=$ 

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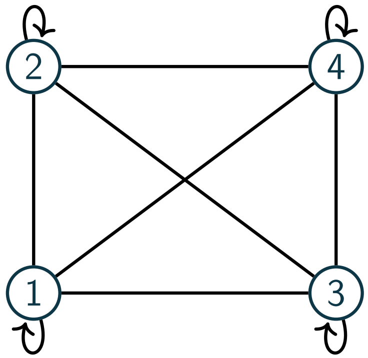

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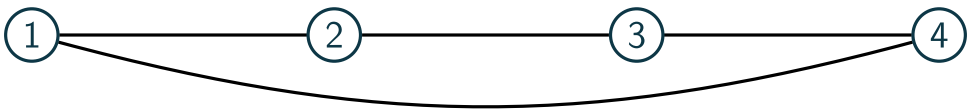

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More examples I.

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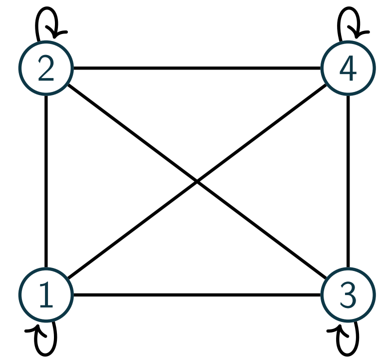

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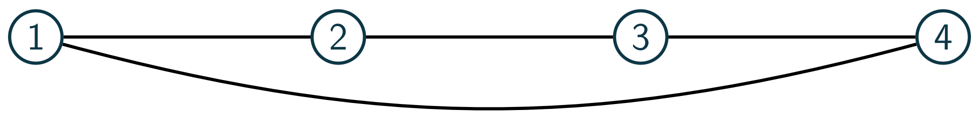

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More examples I.

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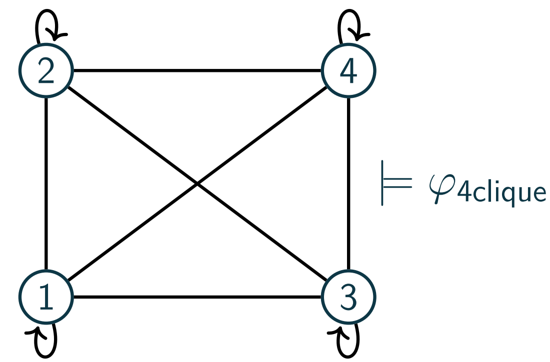

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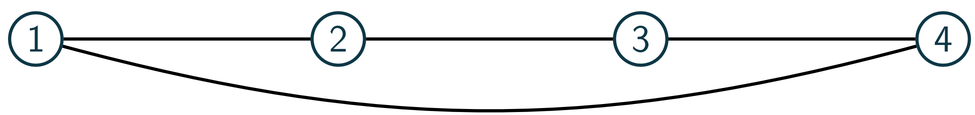
$\mathfrak{G} :=$ 

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Quantification over sets:

More examples I.

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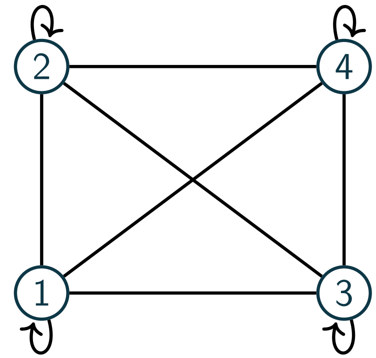

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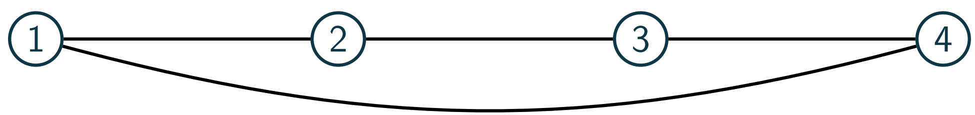

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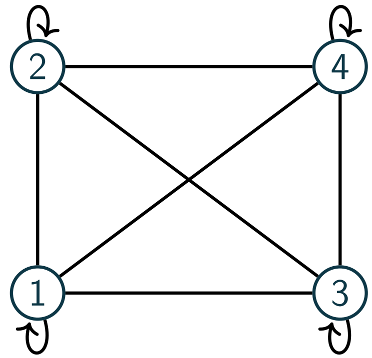

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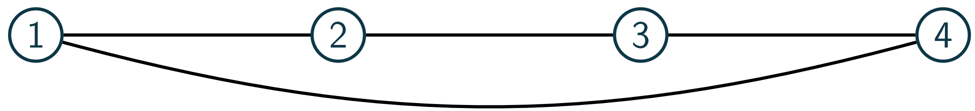

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There exists a colouring with G and R and it is correct

More examples I.

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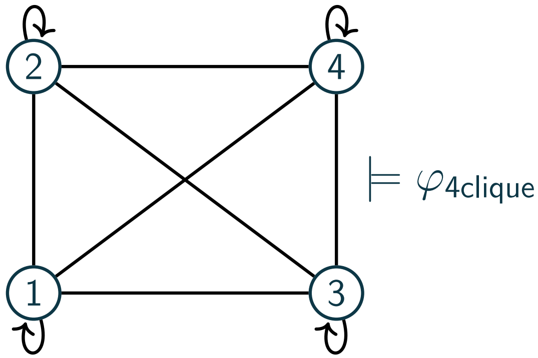

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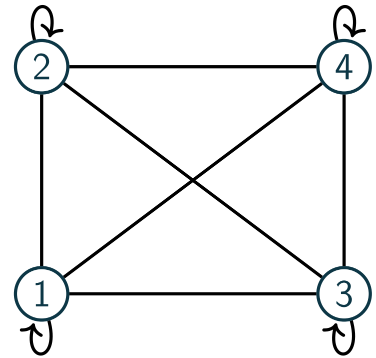

$\models \varphi_{\text{4clique}}$

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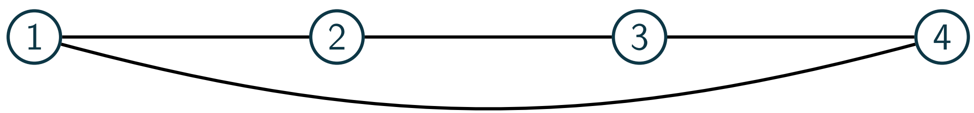

$\models \varphi_{2COL}$

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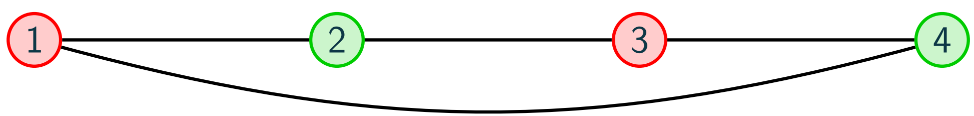
$\mathfrak{G}' :=$



$\models \varphi_{ok}$

More examples II.

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Exercise (Write an FO$[\{\mathrm{E}^{(2)}, \mathtt{a}, \mathtt{b}\}]$ formula $\varphi_k^{\mathsf{reach}(\mathtt{a},\mathtt{b})}$ testing if there is a path from $\mathtt{a}$ to $\mathtt{b}$ of length k.)

More examples II.

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More examples II.

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No. And we will show it today!

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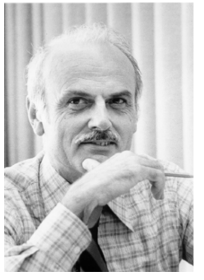

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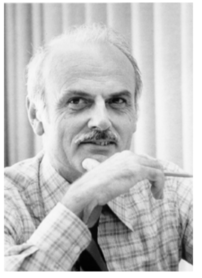

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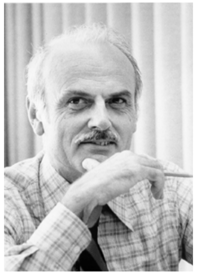

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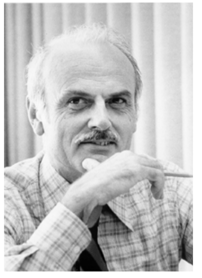

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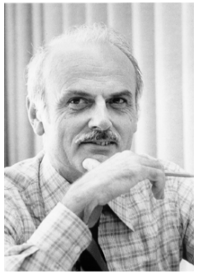

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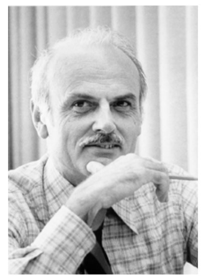

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Description logics: a family of logics for knowledge representation.

W3C® SNOMED CT The global language of healthcare protégé Dublin Core Metadata Initiative Making it easier to find information

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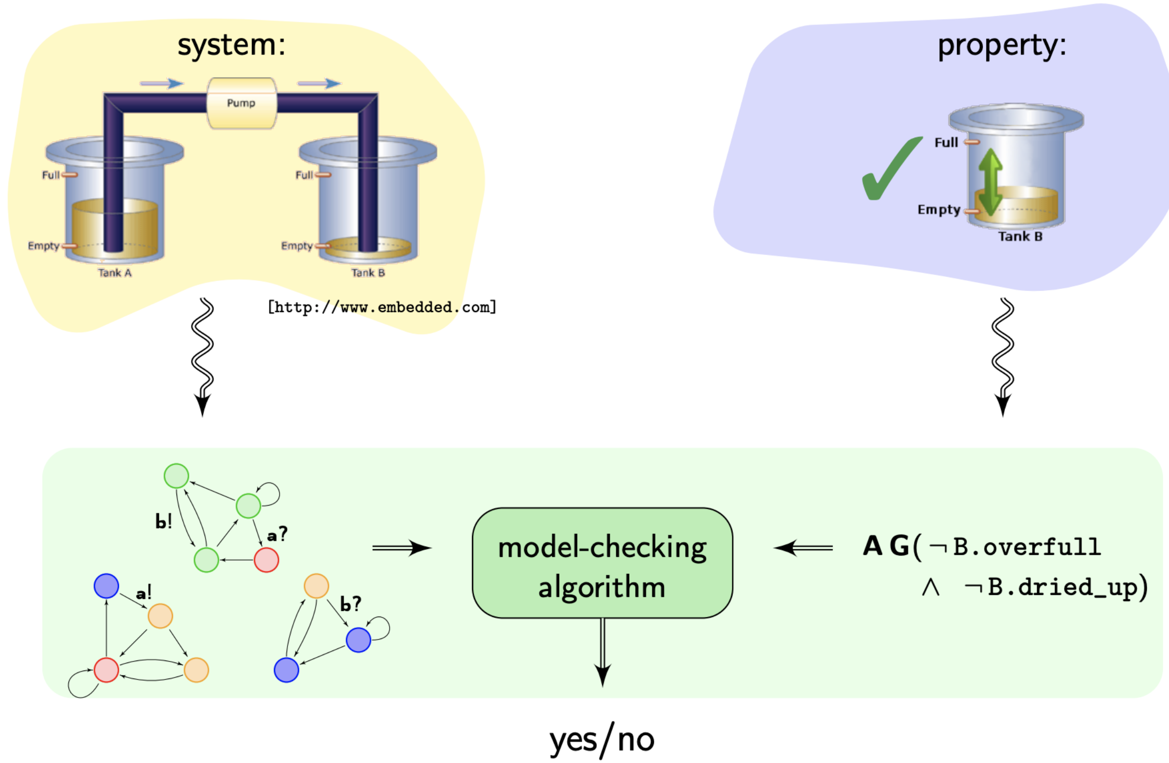
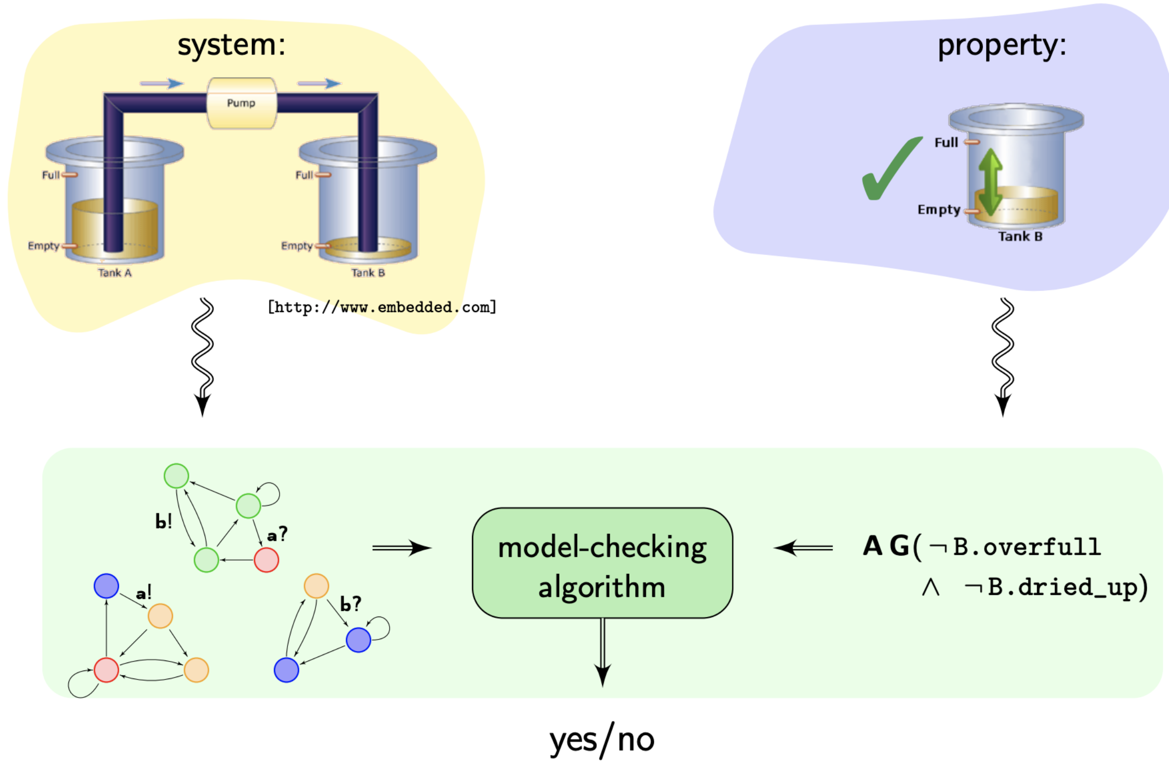


Motivations II: why do we care about logic?

1. Temporal logics as specification languages



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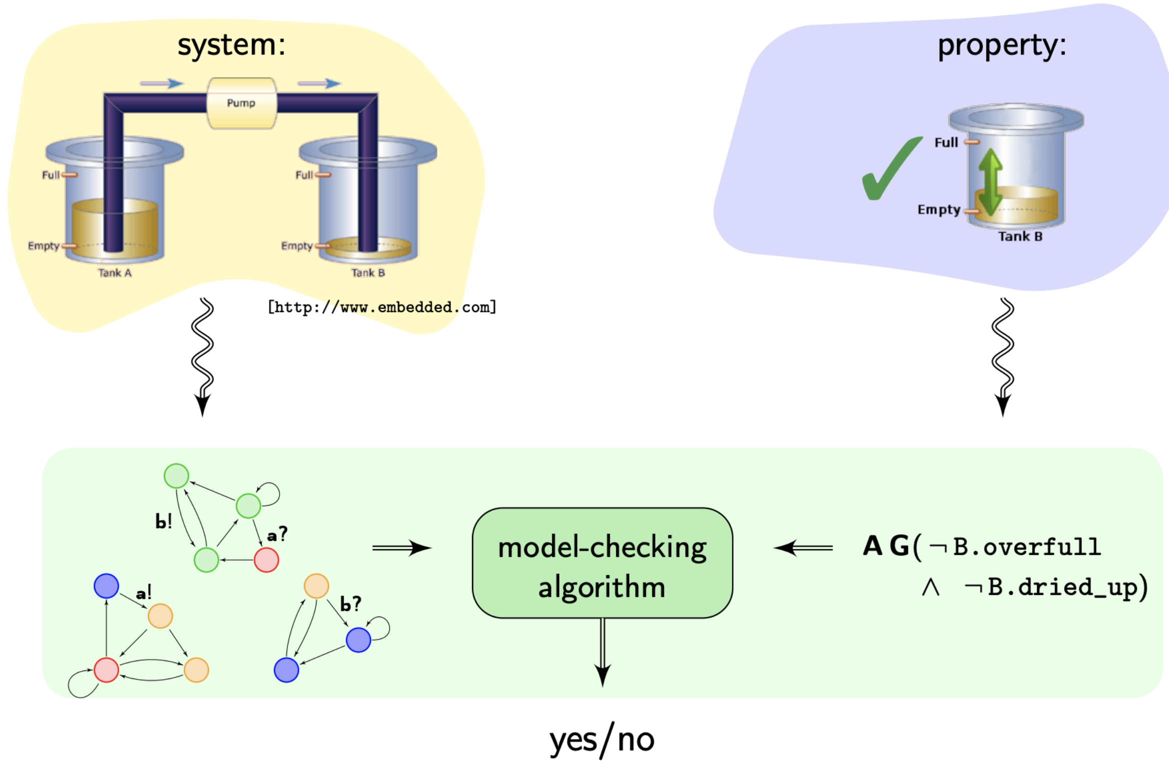

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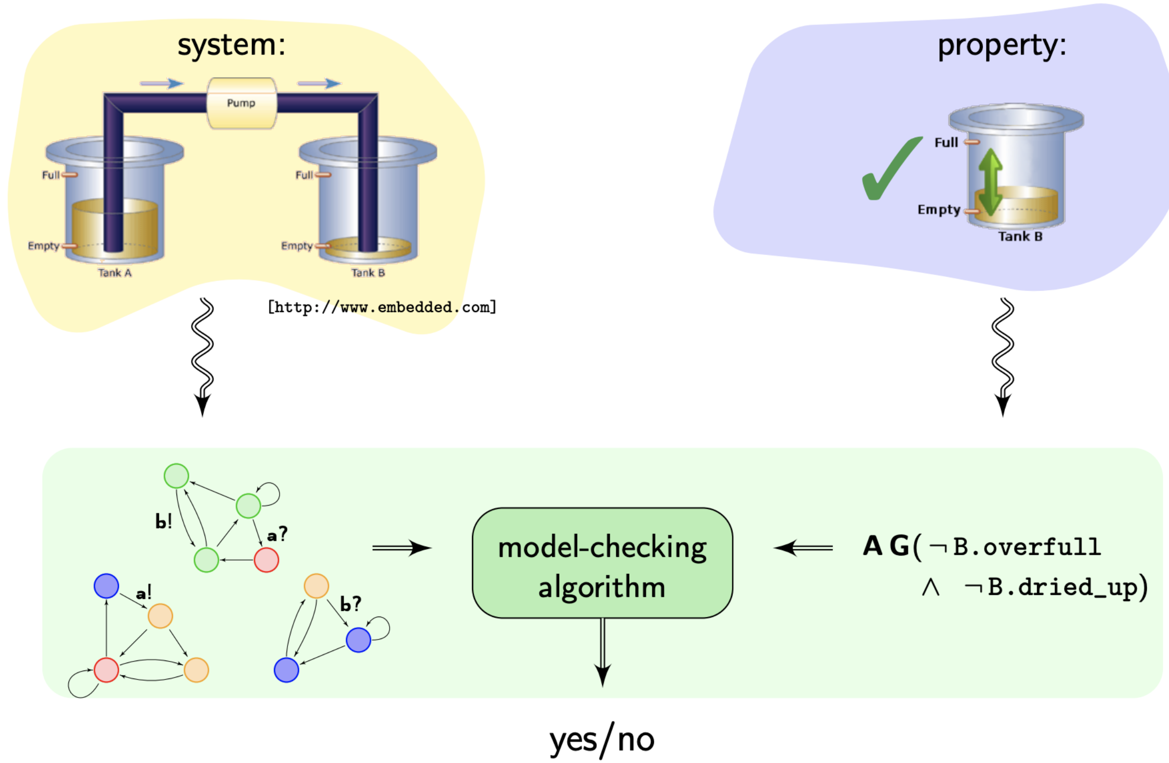

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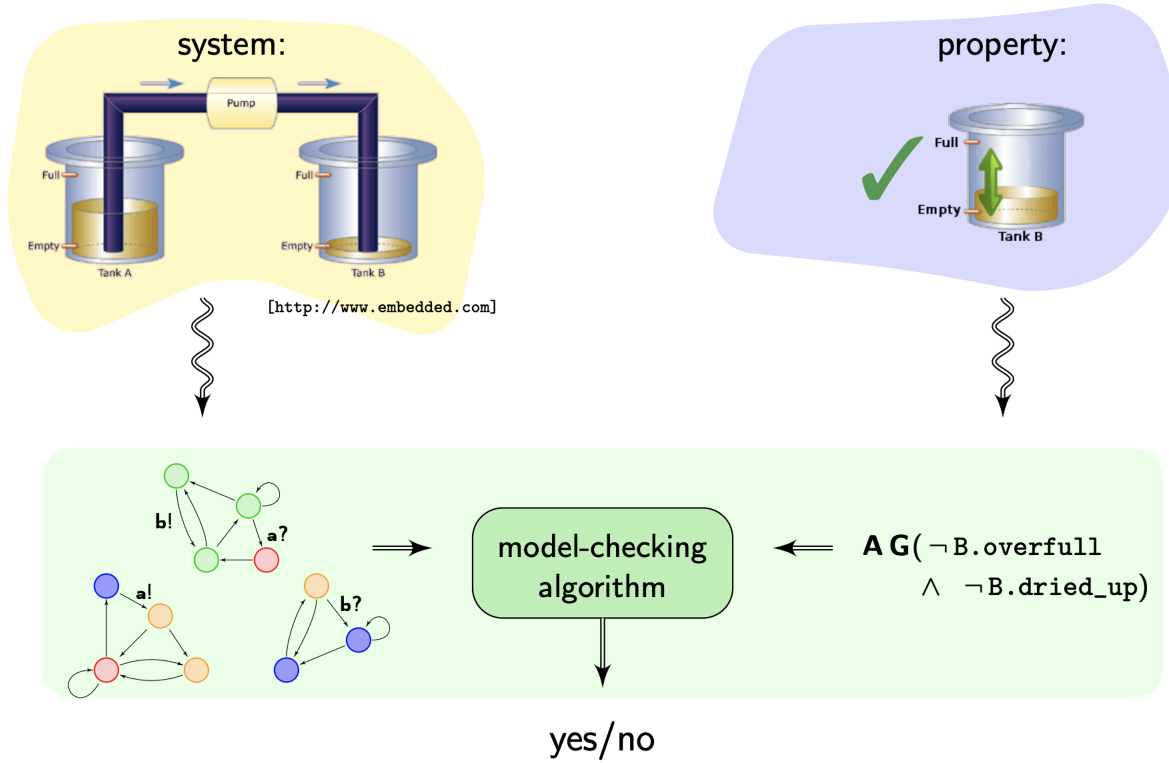

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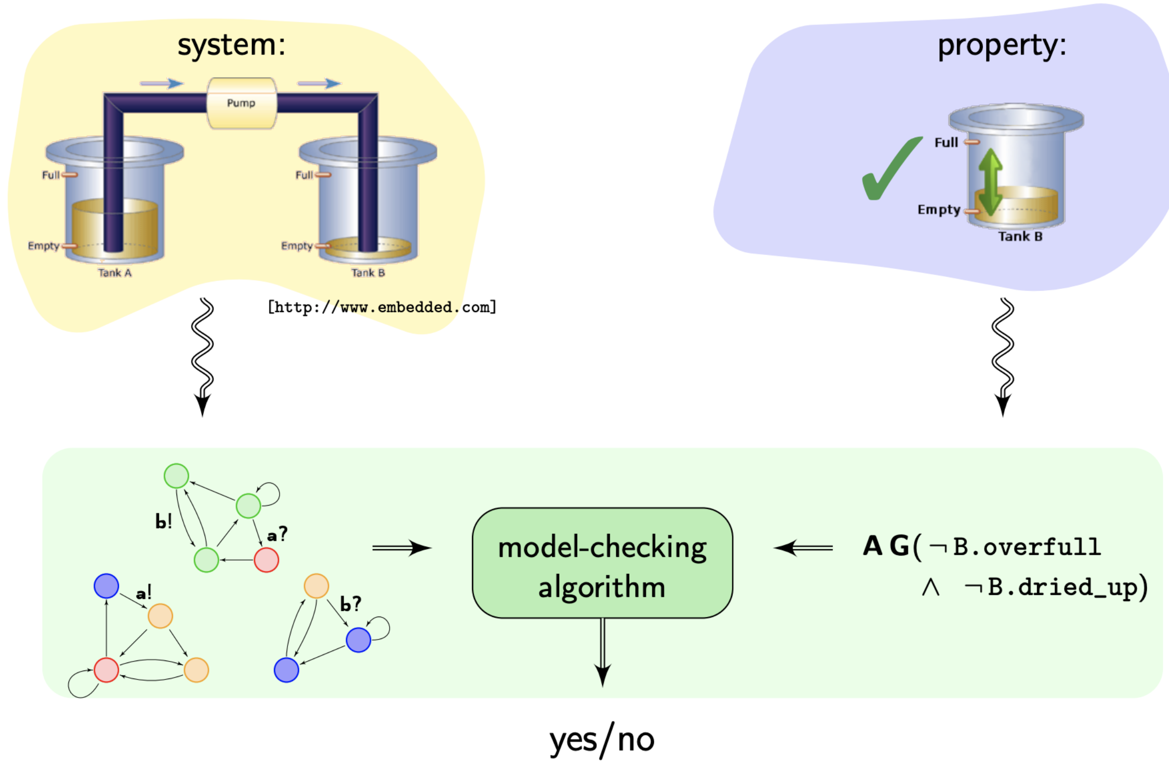

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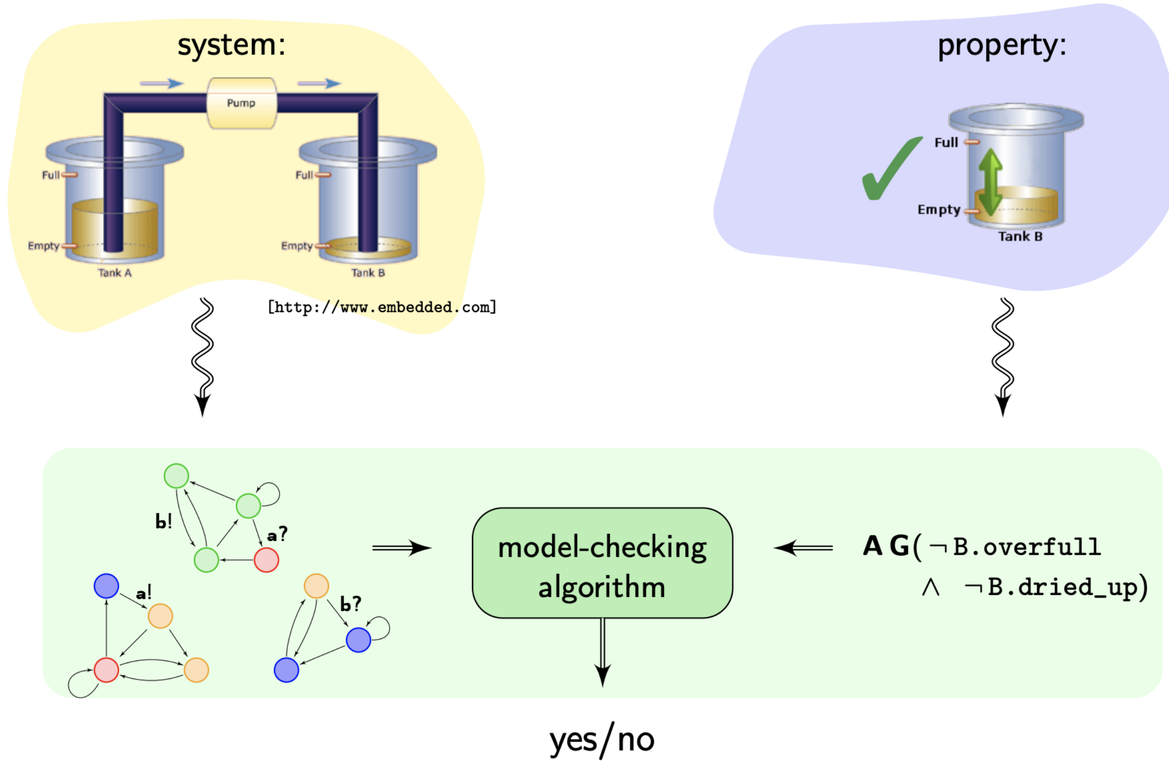

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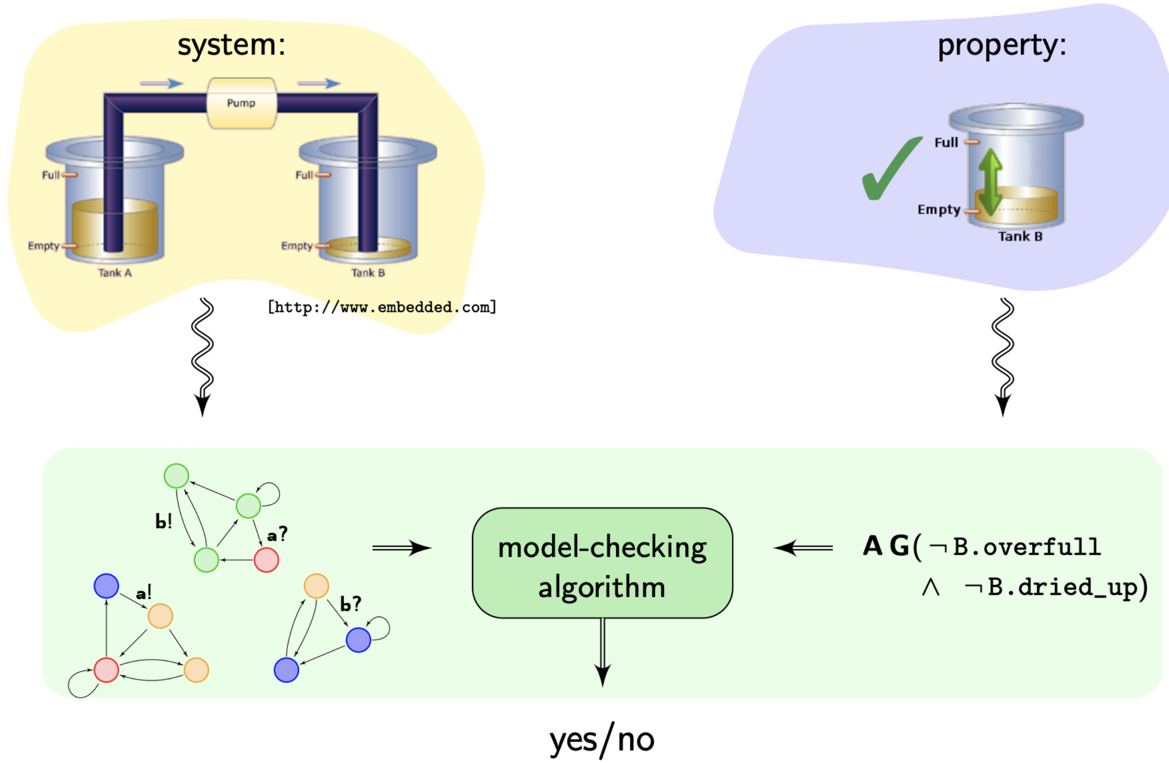

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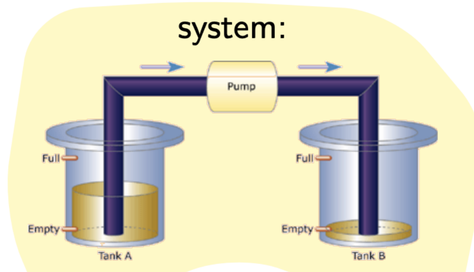

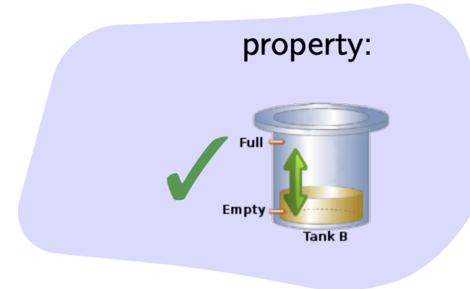


```
vim hello.c
// hello.c
#include <stdlib.h>

void test() {
  int *s = NULL;
  *s = 42;
}
```

```
bartoszbednarczyk@Minsky-Machine: ~/Downloads/Infer
$ infer run -- gcc -c hello.c

Capturing in make/cc mode...
Found 1 source file to analyze in /Users/bartoszbednarczyk/Downloads/Infer/infer-out


Analysis finished in 775mss

Found 1 issue

hello.c:6: error: NULL_DEREFERENCE
  pointer `s` last assigned on line 5 could be null and is dereferenced at line 6, column 3.
  4.   void test() {
  5.     int *s = NULL;
  6. >   *s = 42;
  7.   }


Summary of the reports

  NULL_DEREFERENCE: 1
```

Motivations III: why do we care about logic?

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In "standard" computational complexity we measure resources, e.g. space and time.

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$O(n)$ time

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$\Theta(n \log(n))$ memory?

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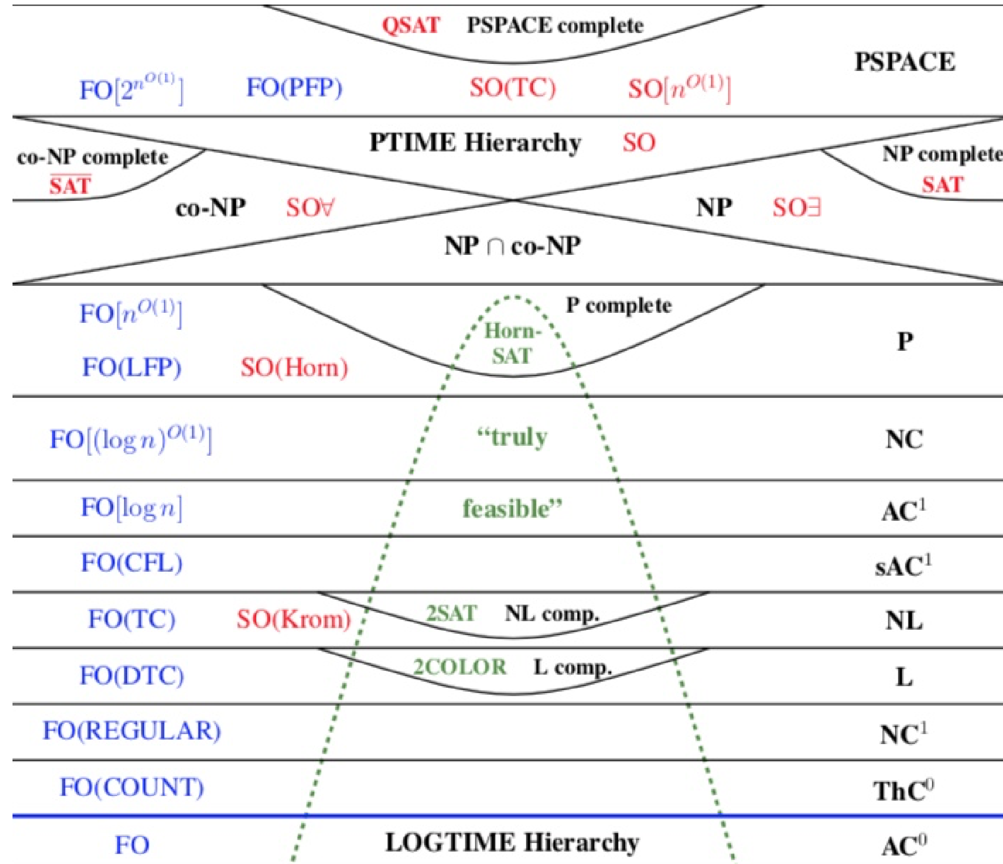

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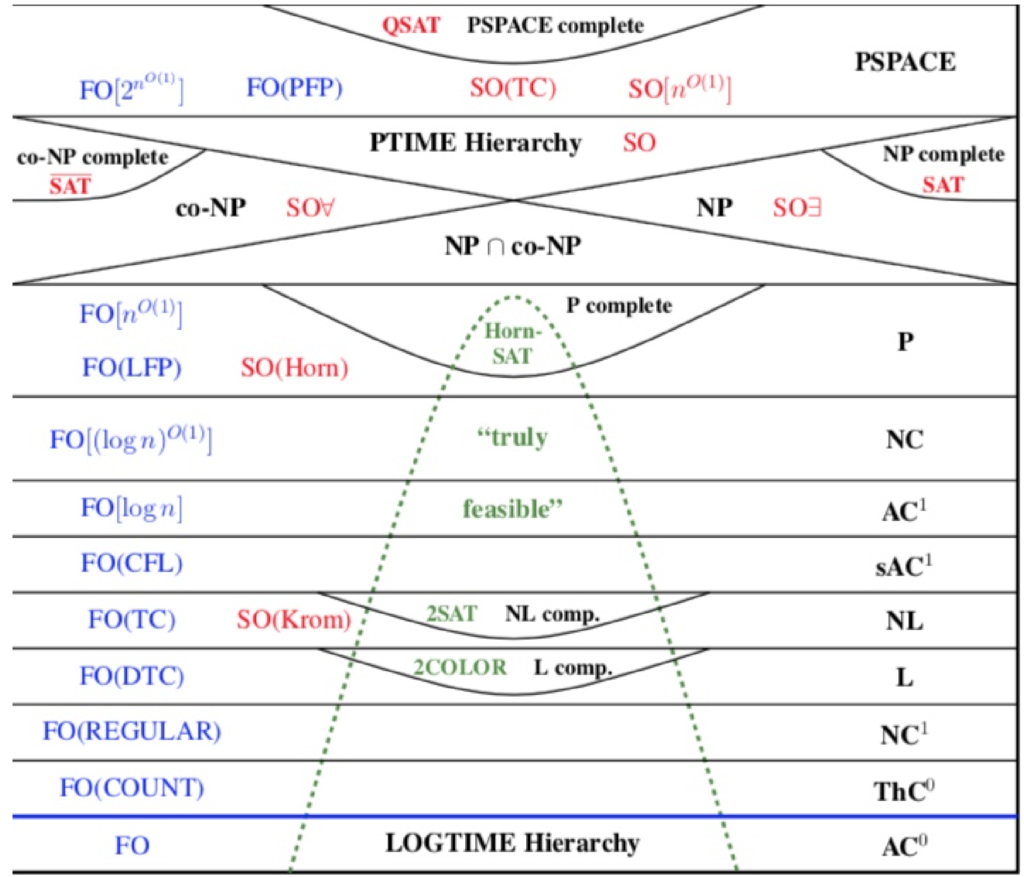

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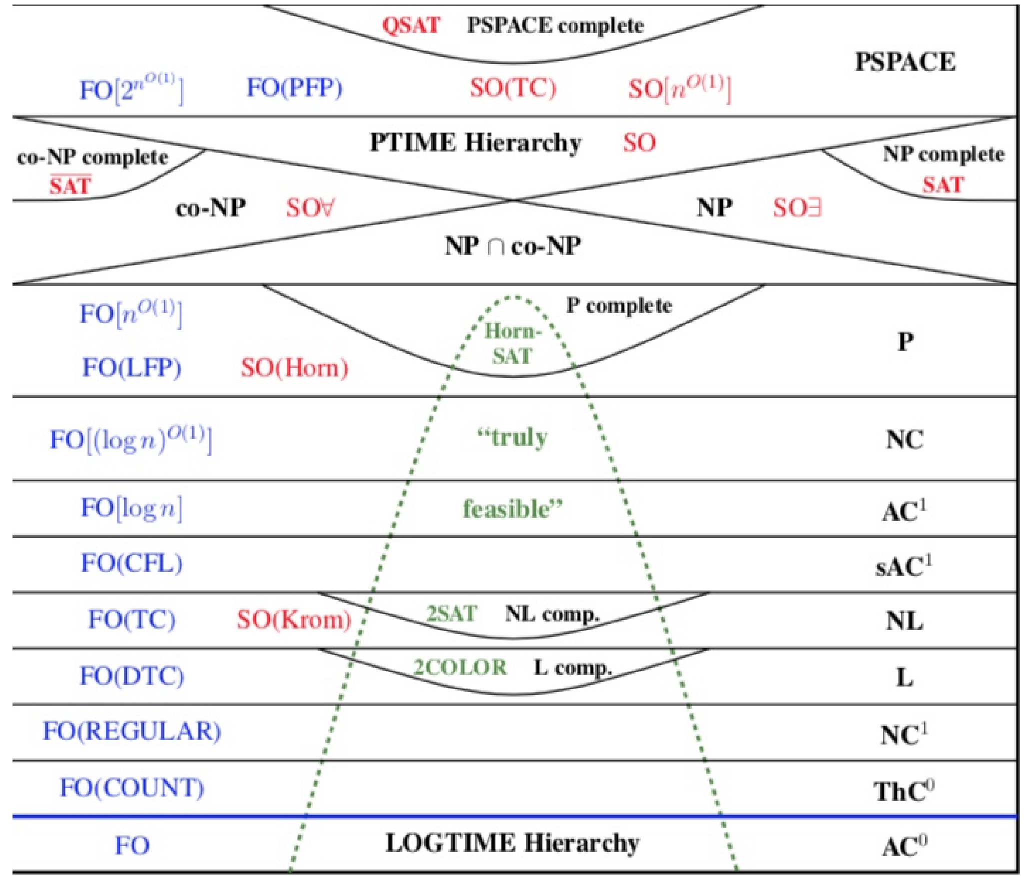

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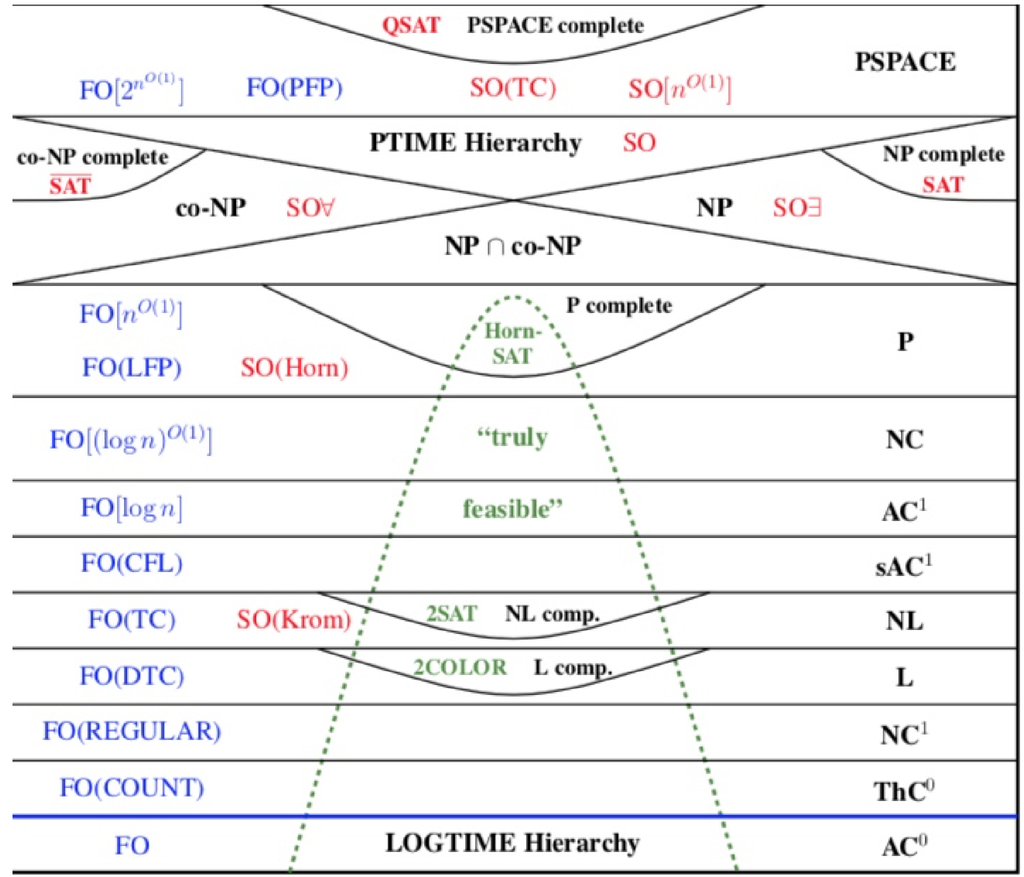

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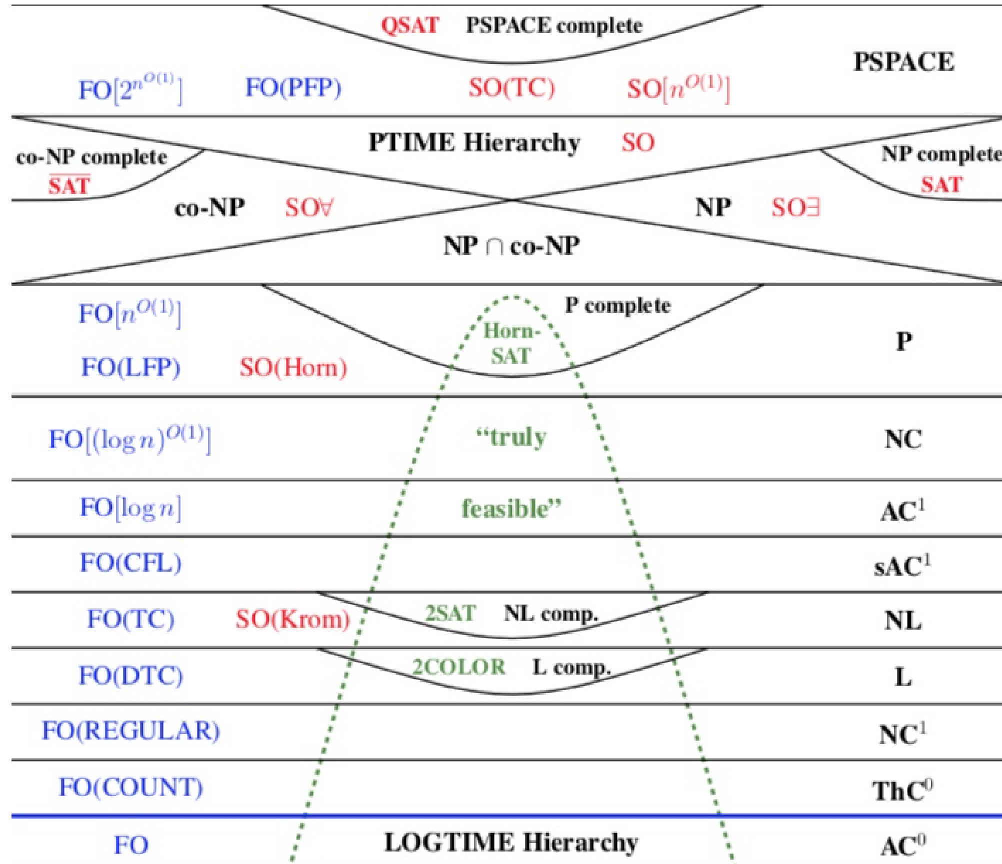

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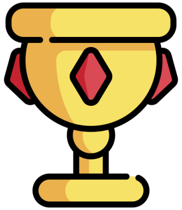
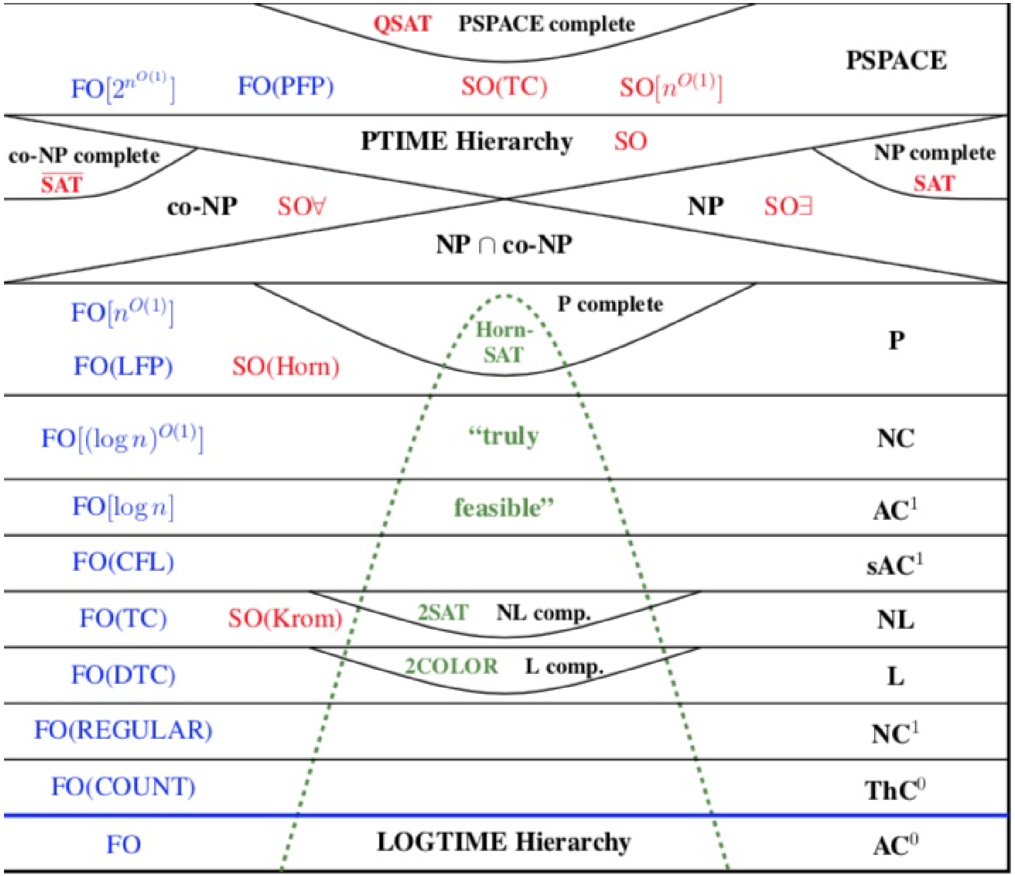


Is there a logic for PTime?

No idea since 1988.



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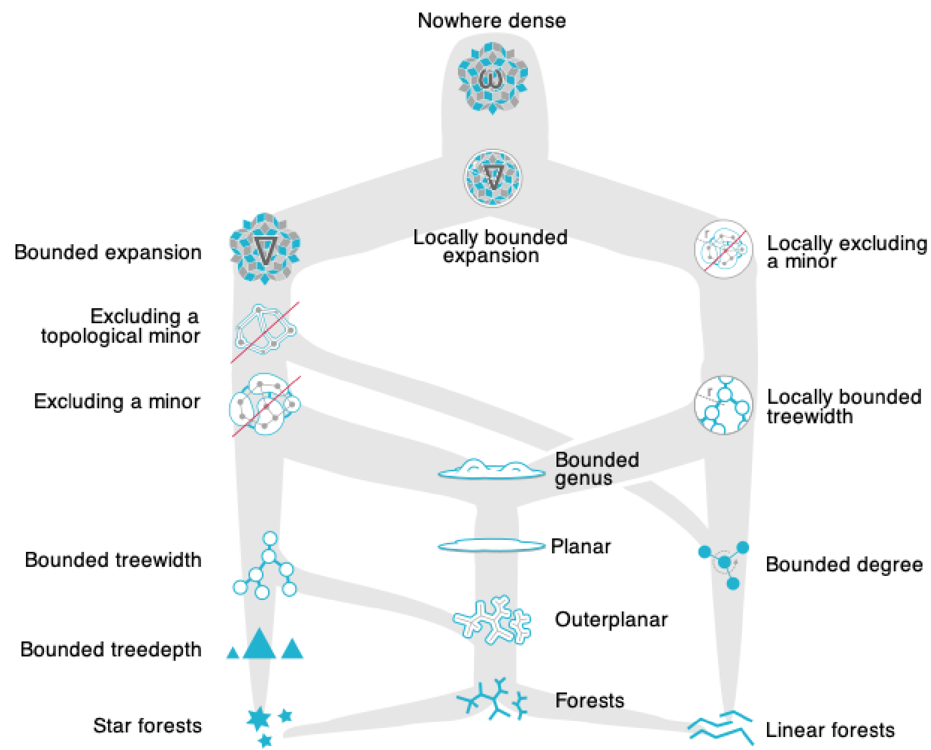

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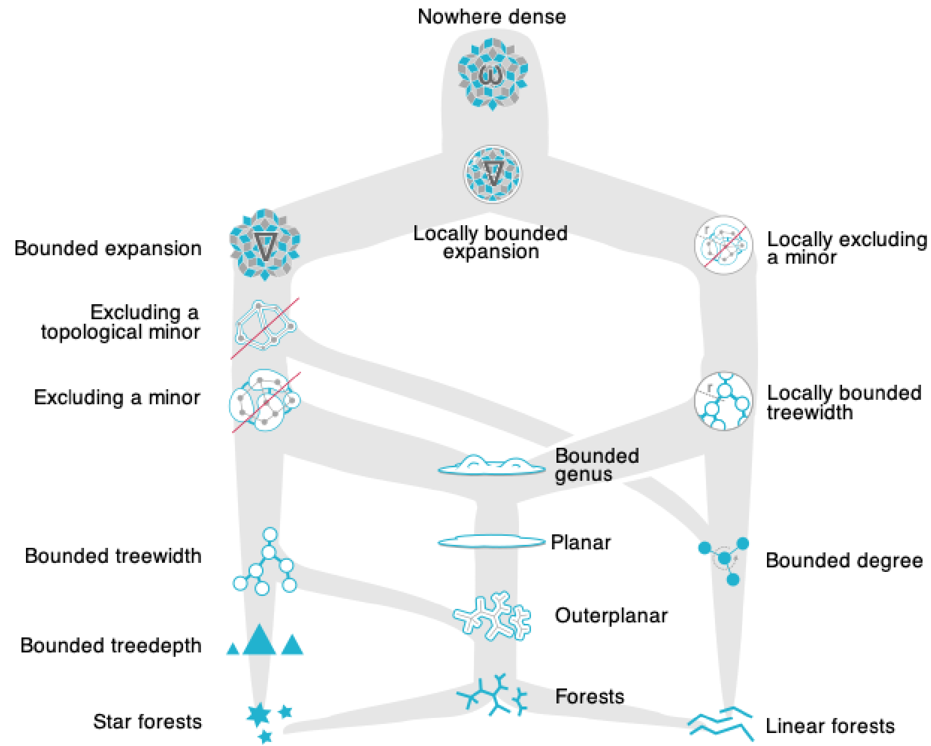

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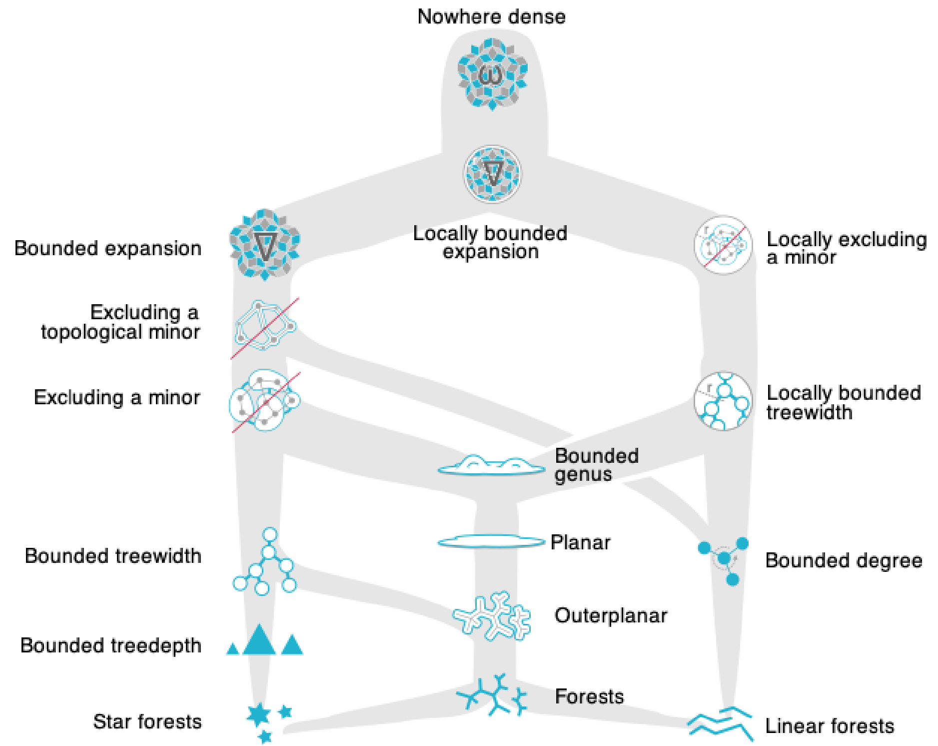

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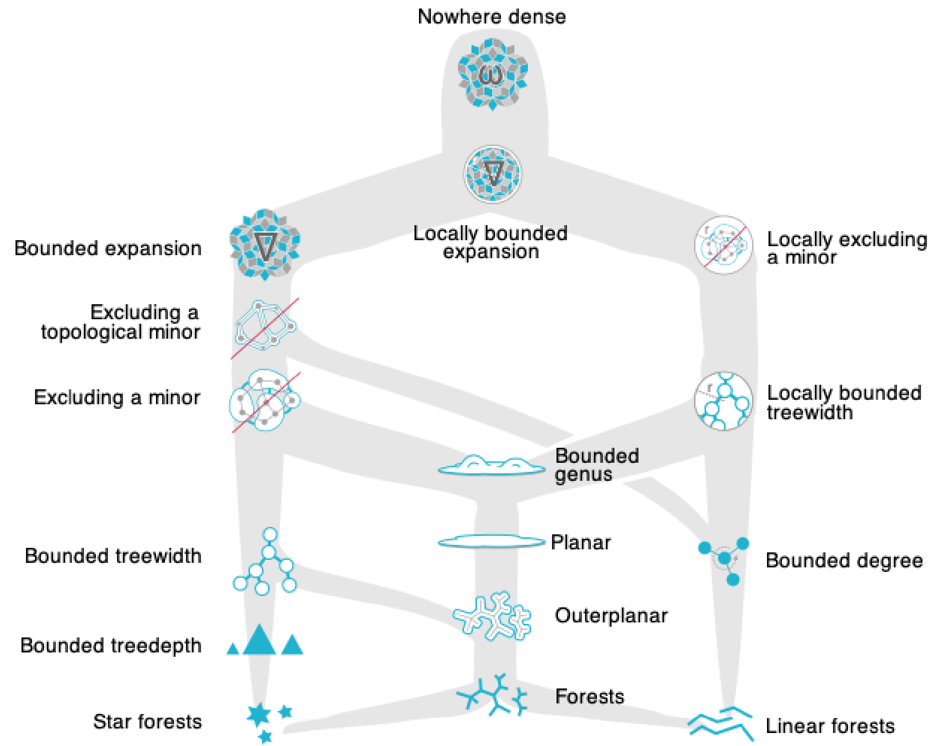

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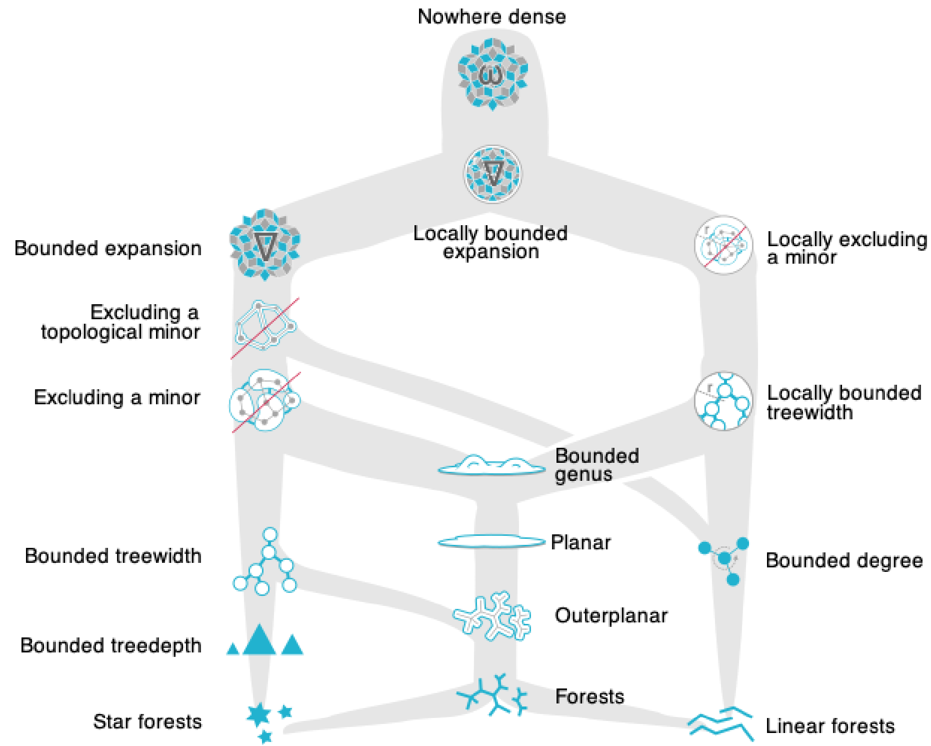

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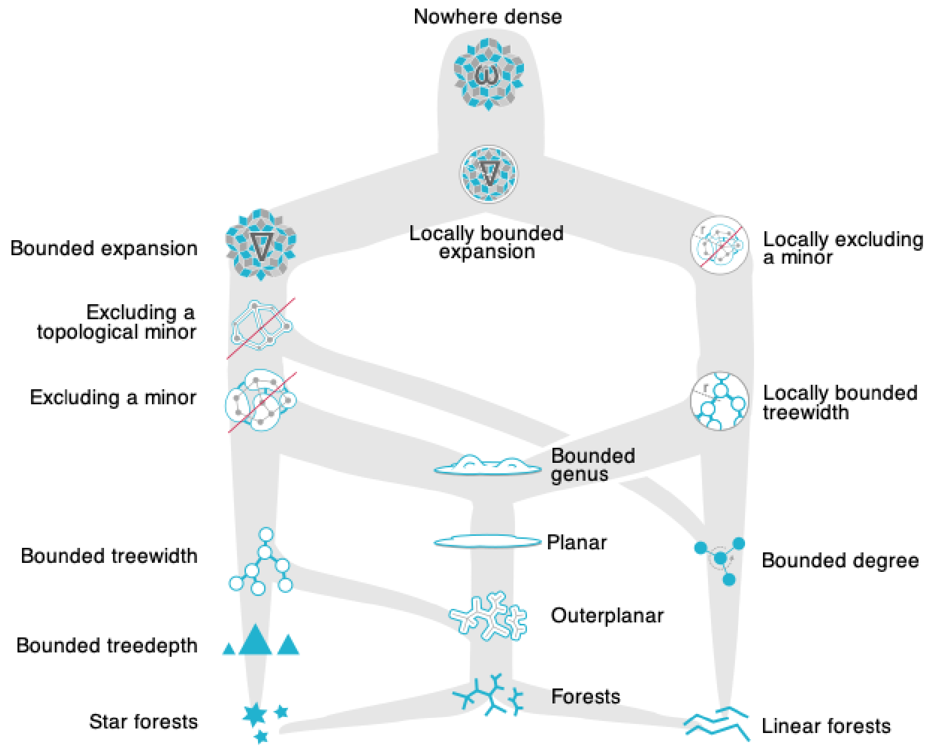
Theorem (Grohe, Kreutzer, Siebertz 2014)

$\mathcal{O}(|\varphi|^{1+\varepsilon})$ for $\mathcal{C} :=$ nowhere-dense graphs.



Signatures (vocabularies)

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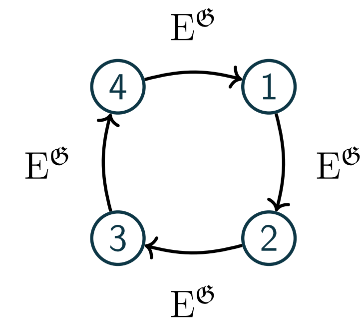

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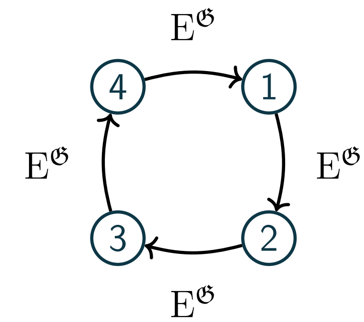

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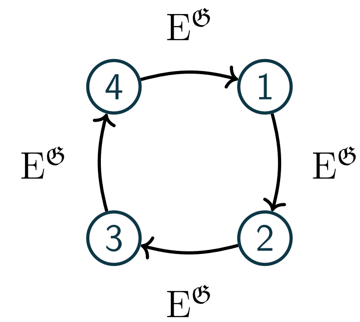

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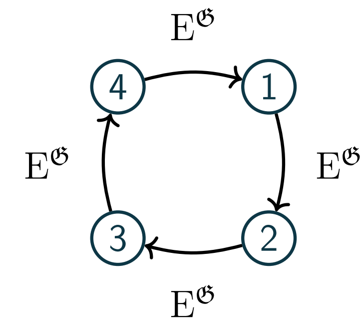

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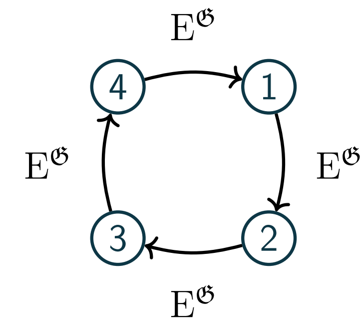

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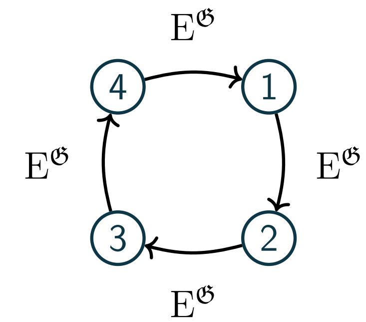

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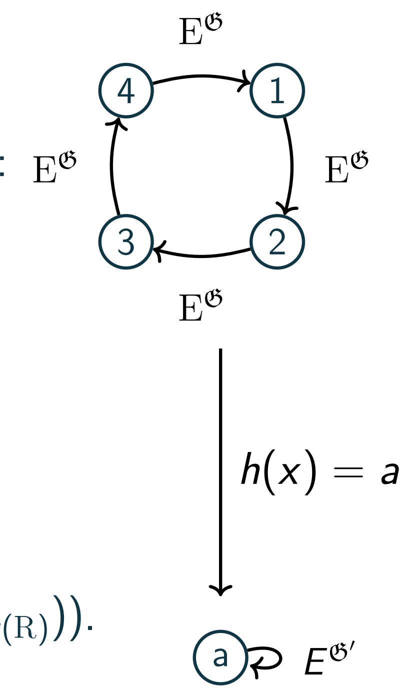

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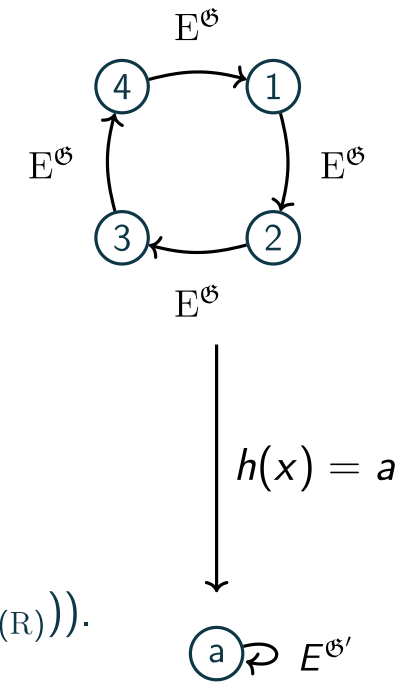

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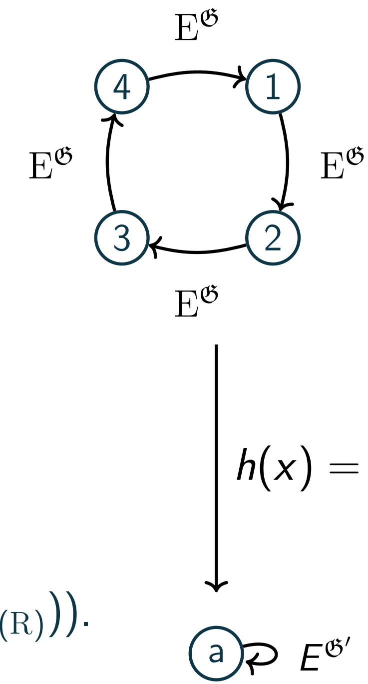

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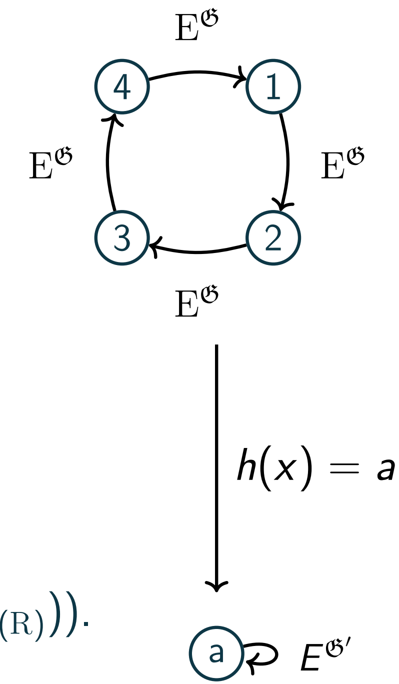

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We write $\mathcal{T} \models \varphi$ to say that every model of $\mathcal{T}$ is a model of φ. Note: $\mathcal{T} \models \bot$ iff $\mathcal{T}$ is unSAT.

Warning! Models can be of any size: finite, countably-infinite and larger!

Löwenheim–Skolem 1922: If a countable $\mathcal{T}$ has a model then $\mathcal{T}$ has a countable one.

FO has dedicated proof systems, e.g. Gentzen's sequents. Check Tim Lyon's lectures! [HERE]

$\mathcal{T} \vdash \varphi$ means φ is provable from $\mathcal{T}$ with sequents.

(we treat $\mathcal{T}$ as extra axioms, note that proofs are finite)

Gödel 1929: $\mathcal{T} \models \varphi$ iff $\mathcal{T} \vdash \varphi$

SAT for FO is Recursively Enumerable

$$\dfrac{\dfrac{Ax}{\forall x[P(x)], \forall x[P(x) \to Q(x)], Q(a) \vdash Q(a)} \qquad \dfrac{\dfrac{Ax}{\forall x[P(x)], \forall x[P(x) \to Q(x)] \vdash P(a), Q(a)}}{\forall x[P(x)], \forall x[P(x) \to Q(x)], \neg P(a) \vdash Q(a)}\,[\neg\vdash]}{\forall x[P(x)], \forall x[P(x) \to Q(x)], \neg P(a) \vee Q(a) \vdash Q(a)}\,[\vee\vdash]$$

$$\dfrac{\dfrac{\dfrac{\forall x[P(x)], \forall x[P(x) \to Q(x)], \neg P(a) \vee Q(a) \vdash Q(a)}{\forall x[P(x)], \forall x[P(x) \to Q(x)], P(a) \to Q(a) \vdash Q(a)}\,[\to\vdash\ r.w.]}{\forall x[P(x)], \forall x[P(x) \to Q(x)] \vdash Q(a)}\,[\forall\vdash]}{\forall x[P(x)], \forall x[P(x) \to Q(x)] \vdash \forall x[Q(x)}\,[\vdash\forall]$$

The Gödel's Compactness Theorem

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Showing

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Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \bot$. By (1) there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ s.t. $\mathcal{T}_0 \models \bot$.

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Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \bot$. By (1) there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ s.t. $\mathcal{T}_0 \models \bot$. A contradiction!

Employing compactness I: Reachability in $\{\mathrm{E}\}$-structures

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The general proof scheme to show that the property $\mathcal{P}$ is not FO-definable.

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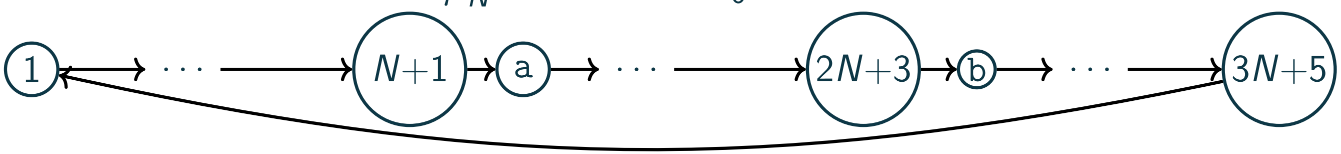

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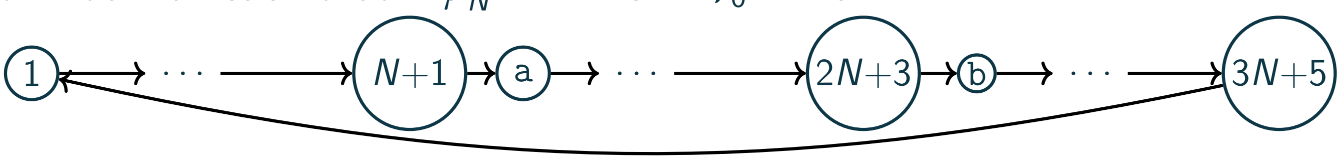

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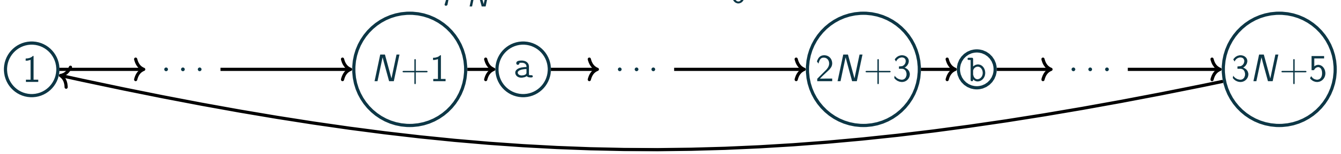

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By $\mathfrak{A} \models \mathcal{T}_1$ we get $\mathfrak{A} \models \varphi$. A contradiction (with the semantics of $\models$)!

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