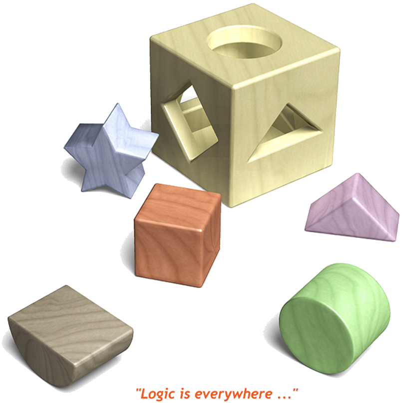
Deduction, Abduction and Induction

Steffen Hölldobler
International Center for Computational Logic
Technische Universität Dresden
Germany



- Introduction
- Deduction
- Sorts
- Abduction
- Induction

Introduction to Abduction

- Consider $\mathcal{K} \models F$
  where $\mathcal{K}$ is a set of formulas called knowledge base and F is a formula
- In the next example I will use the following propositional atoms:
  $grassIsWet$, $wheelsAreWet$, $sprinklerIsRunning$, $raining$
- Let $\mathcal{K} = \{g \rightarrow w, \ s \rightarrow w, \ r \rightarrow g\}$
  - Does $\mathcal{K} \models w$ hold?
- Idea Find an atom A such that $\mathcal{K} \cup \{A\} \models w$ and $\mathcal{K} \cup \{A\}$ is satisfiable
  - $A = w$
  - $A = g$
  - $A = s$ or $A = r$
- This process is called abduction

Introduction to Induction

- Let $\mathcal{K}_{plus} = \{ (\forall Y : number)\ plus(0, Y) \approx Y, (\forall X, Y : number)\ plus(s(X), Y) \approx s(plus(X, Y)) \}$
  - Does $\mathcal{K}_{plus} \models (\forall X, Y : number)\ plus(X, Y) \approx plus(Y, X)$ hold?
- Consider $\mathcal{D} = \mathbb{N} \cup \{\diamond\}$ and

I	0	s	$plus$
	0	f	$\oplus$

  where
  - $f(d) = \begin{cases} f(0) & \text{if } d = \diamond \\ d + f(0) & \text{if } d \in \mathbb{N} \end{cases}$
  - $d \oplus e = \begin{cases} 0 & \text{if } d = e = \diamond \\ \diamond & \text{if } d = 0 \text{ and } e = \diamond \\ d & \text{if } d \in \mathbb{N}^+ \text{ and } e = \diamond \\ e & \text{if } d = \diamond \text{ and } e \in \mathbb{N} \\ d + e & \text{if } d, e \in \mathbb{N} \end{cases}$
  - $+ : \mathbb{N} \to \mathbb{N}$ is the usual addition on $\mathbb{N}$ and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$
- Then $I \models \mathcal{K}_{plus}$ but $(\diamond \oplus 0) \neq (0 \oplus \diamond)$ $\rightsquigarrow$ Exercise

The Example Continued

- $\mathcal{K}_{plus} = \{ (\forall Y : number)\ plus(0, Y) \approx Y,\ (\forall X, Y : number)\ plus(s(X), Y) \approx s(plus(X, Y)) \}$
  - Does $\mathcal{K}_{plus} \models (\forall X, Y : number)\ plus(X, Y) \approx plus(Y, X)$ hold?
- In order to prove the commutativity of *plus* add Peano's induction principle

$$(P(0) \land (\forall M : number)\ (P(M) \rightarrow P(s(M)))) \rightarrow (\forall M : number)\ P(M)$$

  to $\mathcal{K}_{plus}$ (where P is a relational variable)
- For the induction base ($X = 0$) we replace $P(Y)$ by $plus(Y, 0) \approx Y$
- Let $\mathcal{K}_I$ be an appropriate set of induction axioms then

$$\mathcal{K}_{plus} \cup \mathcal{K}_I \models (\forall X, Y : number)\ plus(X, Y) \approx plus(Y, X)$$

- How does $\mathcal{K}_I$ look like? $\leadsto$ Exercise

Deduction, Abduction and Induction

- Peirce 1931 $\mathcal{K}_{facts} \cup \mathcal{K}_{rules} \models G_{result}$
  - **Deduction**
    is an analytic process based on the application of general rules to particular facts, with the inference as a result
  - **Abduction**
    is synthetic reasoning which infers a fact from the rules and the result
  - **Induction**
    is synthetic reasoning which infers a rule from the facts and the result

Deduction

- **All reasoning processes considered in the module Foundations so far are deductions**
- **The logics (first-order, equational) are unsorted**
- **They can be easily extended to sorted logics**
- **We will use a sorted logic in the subsection on Induction**

Sorts

- $(\forall X, Y)\,(\text{number}(X) \wedge \text{number}(Y) \rightarrow plus(X, Y) \approx plus(Y, X))$
  - $(\forall X, Y : number)\; plus(X, Y) \approx plus(Y, X)$
- A first order language with sorts consists of
  - a first order language $\mathcal{L}(\mathcal{R}, \mathcal{F}, \mathcal{V})$ and
  - a function $sort : \mathcal{V} \rightarrow 2^{\mathcal{R}_S}$

  where $\mathcal{R}_S \subseteq \mathcal{R}$ is a finite set of unary predicate symbols called base sorts
- Elements of $2^{\mathcal{R}_S}$ are called sorts; $\emptyset \in 2^{\mathcal{R}_S}$ is called top sort
- We write $X : s$ if $sort(X) = s$
- We assume that for every sort s there are countably many variables $X : s \in \mathcal{V}$

Sorts – Semantics

- Let I be an interpretation with domain $\mathcal{D}$

$$I: \;\; s = \{p_1, \ldots, p_n\} \mapsto s^I = \mathcal{D} \cap p_1^I \cap \ldots \cap p_n^I$$

  - $I: \;\; \emptyset \mapsto \mathcal{D}$

- A variable assignment $\mathcal{Z}$ is sorted iff for all $X : s \in \mathcal{V}$ we find $X^{\mathcal{Z}} \in s^I$
- We assume that all sorts are non-empty
- $F^{I,\mathcal{Z}}$ is defined as usual except for

$[(\exists X : s)\, F]^{I,\mathcal{Z}} = \top$ iff there exists $d \in s^I$ such that $F^{I,\{X \mapsto d\}\mathcal{Z}} = \top$

$[(\forall X : s)\, F]^{I,\mathcal{Z}} = \top$ iff for all $d \in s^I$ we find $F^{I,\{X \mapsto d\}\mathcal{Z}} = \top$

Relativization

- Sorted formulas can be mapped onto unsorted ones by means of a **relativization function** *rel*

$$
\begin{array}{lcl}
rel(p(t_1, \ldots, t_n)) & = & p(t_1, \ldots, t_n) \\
rel(\neg F) & = & \neg rel(F) \\
rel(F_1 \wedge F_2) & = & rel(F_1) \wedge rel(F_2) \\
rel(F_1 \vee F_2) & = & rel(F_1) \vee rel(F_2) \\
rel(F_1 \rightarrow F_2) & = & rel(F_1) \rightarrow rel(F_2) \\
rel(F_1 \leftrightarrow F_2) & = & rel(F_1) \leftrightarrow rel(F_2) \\
rel((\forall X : s)\, F) & = & (\forall Y)\,(p_1(Y) \wedge \ldots \wedge p_n(Y) \rightarrow rel(F\{X \mapsto Y\})) \\
 & & \text{if } sort(X) = s = \{p_1, \ldots, p_n\} \text{ and } Y \text{ is a new variable} \\
rel((\exists X : s)\, F) & = & (\exists Y)\,(p_1(Y) \wedge \ldots \wedge p_n(Y) \wedge rel(F\{X \mapsto Y\})) \\
 & & \text{if } sort(X) = s = \{p_1, \ldots, p_n\} \text{ and } Y \text{ is a new variable}
\end{array}
$$

Sorting Function and Relation Symbols

- Each atom of the form $p(t_1, \ldots, t_n)$ can be equivalently replaced by

$$(\forall X_1 \ldots X_n)\,(p(X_1, \ldots, X_n) \leftarrow X_1 \approx t_1 \wedge \ldots \wedge X_n \approx t_n)$$

- Each atom $A\lceil f(t_1, \ldots, t_n)\rceil$ can be equivalently replaced by

$$(\forall X_1 \ldots X_n)\, A\lceil f(t_1, \ldots, t_n)/f(X_1, \ldots, X_n)\rceil \leftarrow X_1 \approx t_1 \wedge \ldots \wedge X_n \approx t_n$$

- Each formula F can be transformed into an equivalent formula F', in which
  - all arguments of function and relation symbols different from $\approx$ are variables and
  - all equations are of the form $t_1 \approx t_2$ or $f(X_1, \ldots, X_n) \approx t$, where $X_1, \ldots, X_n$ are variables and t, t_1, and t_2 are variables or constants
- Sorting the variables occurring in F' effectively sorts the function and relation symbols

Sort Declaration

- F' is usually quite lengthy and cumbersome to read
- If $sort(X) = s$ then the sort declaration for the variable X is

$$X : s$$

- Let s_i, $1 \leq i \leq n$, and s be sorts, f a function and p a relation symbol, both with arity n. Then

$$f : s_1 \times \ldots \times s_n \rightarrow s$$

and

$$p : s_1 \times \ldots \times s_n$$

are sort declarations for f and p, respectively

Abduction

- **Example** Starting a car
- **Applications**
  - fault diagnosis
  - medical diagnosis
  - high level vision
  - natural language understanding
  - reasoning about states, actions, and causality
  - knowledge assimilation

A First Characterization of Abduction

- Given $\mathcal{K}$ and G; find explanation $\mathcal{K}'$ such that
  - $\mathcal{K} \cup \mathcal{K}' \models G$ and
  - $\mathcal{K} \cup \mathcal{K}'$ is satisfiable

  The elements of $\mathcal{K}'$ are said to be abduced
- Abducing atoms is no real restriction
- Weakness of this first characterization
  We want to abduce causes of effects but no other effects

Restrictions

- **Abducible formulas**
  - set of pre-specified and domain-dependent formulas
  - abduction is restricted to this set
  - default in logic programming: set of undefined predicates
- Typical criteria for choosing a set of abducible formulas
  - an explanation should be **basic**, i.e., it cannot be explained by another explanation
  - an explanation should be **minimal**, i.e., it cannot be subsumed by another explanation
  - additional information
  - domain-dependent preference criteria
  - integrity constraints

Abductive Framework

- **Abductive framework** $\langle \mathcal{K}, \mathcal{K}_A, \mathcal{K}_{IC} \rangle$ where
  - $\mathcal{K}$ is a set of formulas
  - $\mathcal{K}_A$ is a set of ground atoms called **abducibles**
  - $\mathcal{K}_{IC}$ is a set of integrity constraints
- Observation G is **explained** by $\mathcal{K}'$ **iff**
  - $\mathcal{K}' \subseteq \mathcal{K}_A$
  - $\mathcal{K} \cup \mathcal{K}' \models G$ and
  - $\mathcal{K} \cup \mathcal{K}'$ satisfies $\mathcal{K}_{IC}$
- $\mathcal{K} \cup \mathcal{K}'$ **satisfies** $\mathcal{K}_{IC}$ **iff**
  - $\mathcal{K} \cup \mathcal{K}' \cup \mathcal{K}_{IC}$ are satisfiable (**satisfiability view**) or
  - $\mathcal{K} \cup \mathcal{K}' \models \mathcal{K}_{IC}$ (**theoremhood view**)

Knowledge Assimilation

- **Task** assimilate new knowledge into a given knowledge base
- **Example**
  - $\mathcal{K} = \{\ sibling(X, Y) \leftarrow parents(Z, X) \wedge parents(Z, Y),$
    $parents(X, Y) \leftarrow father(X, Y),$
    $parents(X, Y) \leftarrow mother(X, Y),$
    $father(john, \text{mary}),$
    $mother(jane, \text{mary})\ \}$
  - $\mathcal{K}_{IC} = \{\ X \approx Y \leftarrow father(X, Z) \wedge father(Y, Z),$
    $X \approx Y \leftarrow mother(X, Z) \wedge mother(Y, Z)\ \}$
  - $\mathcal{K}_A = \{A \mid A$ is a ground instance of $father(john, Y)$ or $mother(jane, Y)\}$
  - $\approx$ is a 'built–in' predicate such that
    - $X \approx X$ holds and
    - $s \not\approx t$ holds for all distinct ground terms s and t
  - **Task** assimilate $sibling(\text{mary}, \text{bob})$

The Example Continued

- **Two minimal explanations**
  - $\{father(john, \text{bob})\}$
  - $\{mother(jane, \text{bob})\}$
- **What happens if we additionally observe that $mother(joan, \text{bob})$?**
  - **belief revision**

Theory Revision

- **Default reasoning and jumping to a conclusion**
- **Example**
  - $\mathcal{K} = \{$ $penguin(X) \rightarrow bird(X)$, $birdsFly(X) \rightarrow (bird(X) \rightarrow fly(X))$, $penguin(X) \rightarrow \neg fly(X)$, $penguin(tweedy)$, $bird(john)$ $\}$
  - $\mathcal{K}_{IC} = \emptyset$
  - $\mathcal{K}_A = \{A \mid A$ is a ground instance of $birdsFly(X)\}$
- **Task 1** Explain $fly(john)$
- **Task 2** Explain $fly(tweedy)$
- **What happens if we additionally observe** $penguin(john)$**?**

Abduction and Model Generation

- **Example**
  - $\mathcal{K} = \{ \quad \mathit{wobblyWheel} \leftrightarrow \mathit{brokenSpokes} \vee \mathit{flatTyre}, \; \mathit{flatTyre} \leftrightarrow \mathit{puncturedTube} \vee \mathit{leakyValve} \quad \}$
  - $\mathcal{K}_{IC} = \emptyset$
  - $\mathcal{K}_A = \{\mathit{brokenSpokes}, \; \mathit{puncturedTube}, \; \mathit{leakyValve}\}$

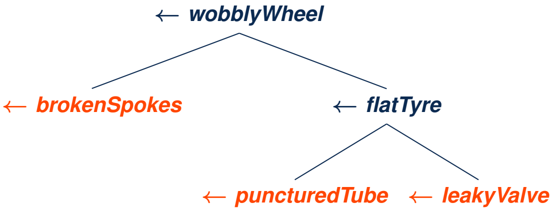
- $\mathcal{K} = \mathcal{K}_{\leftarrow} \cup \mathcal{K}_{\rightarrow}$ **where**
  - $\mathcal{K}_{\leftarrow} = \{ \quad \mathit{wobblyWheel} \leftarrow \mathit{brokenSpokes}, \; \mathit{wobblyWheel} \leftarrow \mathit{flatTyre}, \; \mathit{flatTyre} \leftarrow \mathit{puncturedTube}, \; \mathit{flatTyre} \leftarrow \mathit{leakyValve} \quad \}$
  - $\mathcal{K}_{\rightarrow} = \{ \quad \mathit{wobblyWheel} \rightarrow \mathit{brokenSpokes} \vee \mathit{flatTyre}, \; \mathit{flatTyre} \rightarrow \mathit{puncturedTube} \vee \mathit{leakyValve} \quad \}$

The Wobbly–Wheel Example

- **Observation** ***wobblyWheel***
- **What are the minimal and basic explanations?**
- **How can these explanation be computed?**
  - **SLD–resolution**
  - **Model generation**

Abduction and SLD–Resolution

- Consider the SLD-derivation tree for $\leftarrow$ *wobblyWheel* wrt $\mathcal{K}_{\leftarrow}$



Abduction and Model Generation

- Remember $\mathcal{K}_{\rightarrow} = \{$ *wobblyWheel* $\rightarrow$ *brokenSpokes* $\vee$ *flatTyre*, *flatTyre* $\rightarrow$ *puncturedTube* $\vee$ *leakyValve* $\}$
- Add *wobblyWheel* to $\mathcal{K}_{\rightarrow}$
- What are the minimal models of the extended knowledge base?

{*wobblyWheel*, *flatTyre*, *puncturedTube*}
{*wobblyWheel*, *flatTyre*, *leakyValve*}
{*wobblyWheel*, *brokenSpokes*}

- Restrict these models to the abducible predicates

Mathematical Induction

- **Essential proof technique used to verify properties about recursively defined objects like natural numbers, lists, trees, logic formulas, etc.**
- **Central role in the fields of mathematics, algebra, logic, computer science, formal language theory, etc.**

Some Typical Questions

- **Should induction be really used to prove a statement?**
- **Should the statement be generalized before an attempt is made to prove it by induction?**
- **Which variable should be the induction variable?**
- **What induction principle should used?**
- **What property should be used within the induction principle?**
- **Should nested induction be taken into account?**

Data Structures

- Function symbols are split into constructors and defined function symbols
- Let $\mathcal{F}$ be the set of function symbols
  - Constructors $\mathcal{C} \subseteq \mathcal{F}$
  - Defined function symbols $\mathcal{D} \subseteq \mathcal{F}$
  - $\mathcal{C} \cap \mathcal{D} = \emptyset$
  - $\mathcal{C} \cup \mathcal{D} = \mathcal{F}$
  - $\mathcal{T}(\mathcal{C})$ is called the set of constructor ground terms
- Data structures (or sorts) are sets of constructor ground terms

Data Structures – Examples

- $0 : number$
  $s : number \to number$
  - $\mathcal{T}(\{0,\ s\}) = \{0,\ s(0),\ s(s(0)),\ \ldots\}$ is called the sort *number*
- $\top : bool$
  $\bot : bool$
  - $\mathcal{T}(\{\top,\ \bot\}) = \{\top,\ \bot\}$ is called the sort *bool*
- $[\,] : list(number)$
  $: : number \times list(number) \to list(number)$
  - $\mathcal{T}([\,],\ :\}) = \{[\,],\ [0],\ [0,0],\ [s(0)],\ \ldots\}$ is called the sort *list*(*number*)

Well-Sortedness and Selectors

- **Well-Sortedness**
  - Constants and variables are well-sorted
  - If $f : sort_1 \times \ldots \times sort_n \to sort$
    and for all $1 \leq i \leq n$ we find that t_i is well-sorted and of sort $sort_i$
    then $f(t_1, \ldots, t_n)$ is well-sorted and of sort $sort$
- **Assumption** All terms are well-sorted!
- **Selectors**
  - For each n-ary constructor c we have n unary selectors s_i
    such that for all $1 \leq i \leq n$ we find $s_i(c(t_1, \ldots, t_n)) \approx t_i$

Data Structures – Requirements

- Different constructors denote different objects
- Constructors are injective
- Each object can be denoted as an application of some constructor to its selectors (if any exist)
- Each selector is 'inverse' to the constructor it belongs to
- Each selector returns a so-called witness term if applied to a constructor it does not belong to

Requirements for Numbers

- The requirements can be translated into first order formulas
- The requirements for *number* are

$$\mathcal{K}_{\text{number}} = \{ \begin{array}{l} (\forall N : number)\, 0 \not\approx s(N), \\ (\forall N, M : number)\, (s(N) \approx s(M) \rightarrow N \approx M), \\ (\forall N : number)\, (N \approx 0 \vee N \approx s(p(N))), \\ (\forall N : number)\, p(s(N)) \approx N, \\ p(0) \approx 0, \end{array} \}$$

  where
  - *p* is the selector for the only argument of the constructor *s* and
  - 0 is the witness term assigned to $p(0)$
- Note *p* is a defined function symbol!

Defined Function Symbols

- Functions are defined on top of data structures
- We define functions with the help of a set of conditional equations, i.e., universally closed equations of the form

$$\forall\, l \approx r \leftarrow Body$$

  such that l is a non-variable term (i.e. of the form $g(s_1, \ldots, s_n)$),

$$var(l) \supseteq var(r) \cup var(Body)$$

  and *Body* denotes a conjunction of literals

- We sometimes omit the universal quantifiers in writing conditional equations
- g is a defined function symbol wrt a set $\mathcal{K}$ of conditional equations if $\mathcal{K}$ contains a conditional equation of the form

$$g(t_1, \ldots, t_n) \approx r \leftarrow Body$$

  The set of conditional equations of this form in $\mathcal{K}$ is called definition of g wrt $\mathcal{K}$

Defined Function Symbols – Examples

- **Predecessor** on *number* $\mathcal{K}_p$

$$(\forall N : number)\ p(s(N)) \approx N$$
$$p(0) \approx 0$$

- **Addition** on *number* $\mathcal{K}_{plus}$

$$(\forall X, Y : number)\ (plus(X, Y) \approx Y \quad \leftarrow \quad X \approx 0)$$
$$(\forall X, Y : number)\ (plus(X, Y) \approx s(plus(p(X), Y)) \quad \leftarrow \quad X \not\approx 0)$$

- **Less-than** on *number* $\mathcal{K}_{\mathrm{lt}}$

$$(\forall X, Y : number)\ (\mathrm{lt}(X, Y) \approx \bot \quad \leftarrow \quad Y \approx 0)$$
$$(\forall X, Y : number)\ (\mathrm{lt}(X, Y) \approx \top \quad \leftarrow \quad X \approx 0 \land Y \not\approx 0)$$
$$(\forall X, Y : number)\ (\mathrm{lt}(X, Y) \approx \mathrm{lt}(p(X), p(Y)) \quad \leftarrow \quad X \not\approx 0 \land Y \not\approx 0)$$

Rewriting Extended to Conditional Equations

- Let $\mathcal{K}$ be a finite set of conditional equations
- A term t can be rewritten wrt $\mathcal{K}$ iff
  1. t is well-sorted and ground
  2. t contains a subterm of the form $g(t_1, \ldots, t_n)$ where for all $1 \leq i \leq n$ we find that t_i is a constructor ground term
  3. $g(s_1, \ldots, s_n) \approx r \leftarrow Body \in \mathcal{K}$ and
  4. we find an mgu θ for $g(s_1, \ldots, s_n)$ and $g(t_1, \ldots, t_n)$ such that $\mathcal{K} \models Body\theta$
- In this case t is rewritten to the term obtained from t by replacing $g(t_1, \ldots, t_n)$ by $r\theta$
- Note θ is a matcher because t is ground

Cases

- Let $g(s_1, \ldots, s_n) \approx r \leftarrow Body$ be a rule and $X_1, \ldots, X_n$ new variables

$$g(X_1, \ldots, X_n) \approx r \leftarrow X_1 \approx s_1 \wedge \ldots \wedge X_n \approx s_n \wedge Body$$

  is called **homogeneous form** of this rule
- **Example**

$$(\forall X, N : number)\ (p(X) \approx N \leftarrow X \approx s(N))$$

  is the homogeneous form of

$$(\forall N : number)\ p(s(N)) \approx N$$

- **Obervation** A rule is semantically equivalent to its homogeneous form
- The **case** of a rule is the condition of its homogeneous form

Programs

- **A program is a set of clauses consisting of data structure declarations and function definitions**
- **Example** $\mathcal{K}_{\mathbf{number}} \cup \mathcal{K}_{\boldsymbol{plus}}$ **is a program**

Properties of Programs

- A program $\mathcal{K}$ is
  - **well-formed** iff it can be ordered such that each function symbol occurring in the definition of a function g in $\mathcal{K}$ either is introduced before by a data structure declaration or another function definition or, otherwise, is g in which case the function is **recursive**
  - **well-sorted** iff each term occurring in $\mathcal{K}$ is well-sorted
  - **deterministic** iff
    for each function definition occurring in $\mathcal{K}$ the cases are mutually exclusive
  - **case-complete** iff for each function definition of an n-ary function g occurring in $\mathcal{K}$ and each well-sorted n-tuple of constructor ground terms given as input to g there is at least one of the cases which is satisfied
  - **terminating** iff
    there is no infinite rewriting sequence for any well-sorted ground term
  - **admissible** iff
    it is well-formed, well-sorted, deterministic, case-complete and terminating
- The rewrite relation wrt an admissible program is confluent $\rightsquigarrow$ **Exercise**

Evaluation

- **Admissible programs $\mathcal{K}$ define a unique evaluator $eval_{\mathcal{K}}$ which maps terms to their normal form**
- $eval_{\mathcal{K}} : \mathcal{T}(\mathcal{F}) \to \mathcal{T}(\mathcal{C})$
- **$eval_{\mathcal{K}}(t)$ is called value of t**
- **$eval_{\mathcal{K}}$ is an interpretation with domain $\mathcal{T}(\mathcal{C})$**
- **$eval_{\mathcal{K}}$ is called standard interpretation of $\mathcal{K}$**
- **Example** **Consider $\mathcal{K}_{\text{number}} \cup \mathcal{K}_{plus}$**

$$
\begin{array}{ll}
 & plus(s(0), s(0)) \\
\to & s(plus(p(s(0)), s(0))) \\
\to & s(plus(0, s(0))) \\
\to & s(s(0))
\end{array}
$$

Evaluation – Example

- Consider $\mathcal{K} = \mathcal{K}_{\text{number}} \cup \mathcal{K}_{plus}$
  - $eval_{\mathcal{K}} \models \mathcal{K}$
    $eval_{\mathcal{K}} \models (\forall X, Y : number)\ plus(X, Y) \approx plus(Y, X)$,
    $eval_{\mathcal{K}} \models (\forall X : number)\ X \not\approx s(X)$
    - $\rightsquigarrow$ **Exercise**
  - $\mathcal{K} \not\models (\forall X, Y : number)\ plus(X, Y) \approx plus(Y, X)$
    $\mathcal{K} \not\models (\forall X : number)\ X \not\approx s(X)$
    - $\rightsquigarrow$ **Exercise**

Theory of Admissible Programs

- Let $\mathcal{K}$ be an admissible program
- We consider $\{G \mid eval_{\mathcal{K}} \models G\}$
- In other words, we restrict us to one specific interpretation
  This interpretation is sometimes called standard or intended interpretation
- Idea Add formulas to $\mathcal{K}$ such that non-standard interpretations are no longer models of $\mathcal{K}$
  - These formulas are called induction axioms
  - Let $\mathcal{K}_I$ be a decidable set of induction axioms such that $eval_{\mathcal{K}} \models \mathcal{K}_I$

Induction – Example

- Let $\mathcal{K} = \mathcal{K}_{\text{number}} \cup \mathcal{K}_{plus}$
- Let $\mathcal{K}_I$ be the set of all formulas of the form

$$(P(0) \wedge (\forall X : number)\,(P(X) \rightarrow P(s(X)))) \rightarrow (\forall X : number)\,P(X)$$

- This scheme can be instantiated by, e.g., replacing $P(X)$ by $X \not\approx s(X)$

$$(0 \not\approx s(0) \wedge (\forall X : number)\,(X \not\approx s(X) \rightarrow s(X) \not\approx s(s(X)))) \rightarrow (\forall X : number)\,X \not\approx s(X) \quad (1)$$

- $eval_{\mathcal{K}} \models (1) \quad \rightsquigarrow$ **Exercise**
- $\mathcal{K} \cup \{(1)\} \models (\forall X : number)\,X \not\approx s(X) \quad \rightsquigarrow$ **Exercise**
  - The proof is finite
    (in contrast to a proof of $eval_{\mathcal{K}} \models (\forall X : number)\,X \not\approx s(X)$)

Inductive Theorem Proving

- Theorem proving by induction is incomplete (Gödel's incompleteness theorem)
- Induction axioms may be computed from inductively defined data structures
- Heuristics may guide selection of
  - the induction variable
  - the induction schema and
  - the induction axiom
- Several theorem provers based on induction are available, e.g.,
  - NQTHM
  - OYSTER-CLAM
  - INKA