EXERCISE 5

Science of Computational Logic

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Problem 5.1

Show that first-order logic is monotonic.

Solution

Monotonic means: For all sets of sentences $\mathcal{F}$, $\mathcal{F}'$ and all sentences G :
$\mathcal{F} \models G$ implies $\mathcal{F} \cup \mathcal{F}' \models G$

- We assume $\mathcal{F} \models G$.
- To show: $\mathcal{F} \cup \mathcal{F}' \models G$, that is, every model for $\mathcal{F} \cup \mathcal{F}'$ is also a model for G .
- For every model I of $\mathcal{F} \cup \mathcal{F}'$:

  $I \models \mathcal{F} \cup \mathcal{F}'$ implies $I \models \mathcal{F}$ because $\mathcal{F} \subseteq \mathcal{F} \cup \mathcal{F}'$.
  $I \models \mathcal{F}$ implies $I \models G$ because $\mathcal{F} \models G$.

  Hence every model I of $\mathcal{F} \cup \mathcal{F}'$ is also a model for G .

Problem 5.2

Show that reasoning with CWA is non-monotonic.

Solution

Example: $\mathcal{L}(\{p|1, q|1\}, \{a|0, b|0\}, \mathcal{V})$

Consider $\mathcal{F} = \{p(a), q(b)\}$

Then $\mathcal{F}_{CWA} = \{\neg p(b), \neg q(a)\}$. and consequently $\mathcal{F} \models_{CWA} \neg p(b)$ since $C_{CWA}(\mathcal{F}) \models \neg p(b)$

However, with $\mathcal{F}' = \mathcal{F} \cup \{p(b)\}$ we get $\mathcal{F}'_{CWA} = \{\neg q(a)\}$

and consequently, $\mathcal{F} \not\models_{CWA} \neg p(b)$ since $C_{CWA}(\mathcal{F}) \not\models \neg p(b)$

Problem 5.3

Consider the language $\mathcal{L}(\mathcal{R}, \mathcal{F}, \mathcal{V})$, with $\mathcal{R} = \{p/0, q/0\}$.
Given the set of formulas $\mathcal{S} = \{p \leftarrow \neg q, q \leftarrow \neg p\}$

Compute $C_{CWA}(\mathcal{S})$.

Solution

$\mathcal{S} = \{p \leftarrow \neg q, q \leftarrow \neg p\} \equiv \{p \vee q\}$
$\mathcal{S}_{CWA} = \{\neg p, \neg q\}$, because $\{p \vee q\} \not\models p$ and $\{p \vee q\} \not\models q$
$\mathcal{S} \cup \mathcal{S}_{CWA} \equiv \{p \vee q, \neg p, \neg q\}$ is unsatisfiable.
Hence $C_{\mathrm{CWA}}(\mathcal{S}) = C(\mathcal{S} \cup \mathcal{S}_{CWA}) = \mathcal{L}(R, \mathcal{F}, \mathcal{V})$

Problem 5.4

Prove that the closed world assumption eliminated non-least Herbrand models:

If F is a formula and I is a non-least Herbrand model of F, then $I \not\models C_{CWA}(F)$.

Solution

If I is a non-least model of F, then there is a model J of F such that $J \subsetneq J$.

Then $I \neq J$ and $J \subseteq J$.

Then $D^I = D^J$ and $\cdot^I, \cdot^J$ are identical except that for all $q \in \mathcal{R}$ we find $q^J \subseteq q^I$.

Then J is also a Herbrand interpretation.

Since $I \neq J$, and I and J can only differ in the interpretation of relation symbols, there must exist $q \in \mathcal{R}$ and $d_1, \ldots, d_n \in D$ such that $(d_1, \ldots, d_n) \in q^I$, but $(d_1, \ldots, d_n) \notin q^J$.

Then $q(d_1, \ldots, d_n)$ is a ground term, and since J is a model of F, we conclude that $F \not\models q(d_1, \ldots, d_n)$.

Since $q(d_1, \ldots, d_n)$ is ground, we conclude that $\neg q(d_1, \ldots, d_n) \in F_{CWA}$.

Problem 5.5

Proof the following proposition:
Let $\mathcal{F}$ be a satisfiable set of Skolem formulas. Then it holds:

$C_{CWA}(\mathcal{F})$ is satisfiable $\Leftrightarrow$ $\mathcal{F}$ admits a least Herbrand model.

Solution

"$\Leftarrow$": We assume $\mathcal{F}$ admits a least Herbrand model. Let I be this model.
We show that $I \models \mathcal{F} \cup \mathcal{F}_{CWA}$ as follows:

For every formula $G \in \mathcal{F} \cup \mathcal{F}_{CWA}$ one of the following must be the case:

case 1 $G \in \mathcal{F}$: Then $I \models G$ (because $I \models \mathcal{F}$)

case 2 $G \in \mathcal{F}_{CWA}$, which implies $G = \neg A$ for some ground atom A, and with $\mathcal{F} \not\models A$.
$\mathcal{F} \not\models A$ means that there is a model I' of $\mathcal{F}$ which is not a model of A.
Since $\mathcal{F}$ and A are formulae in Skolem form there exists also a Herbrand model I'' of $\mathcal{F}$, which is not a model of A. (E.g., take as I'' the Herbrand interpretation corresponding to I'.)
With the usual notation for Herbrand models we have: $A \notin I''$.
Because I is the least Herbrand model, $I \subseteq I''$ holds.
Hence $A \notin I$ and $I \models \neg A = G$

Thus $I \models \mathcal{F} \cup \mathcal{F}_{CWA}$ holds. $I \models C(\mathcal{F} \cup \mathcal{F}_{CWA}) = C_{\mathrm{CWA}}(\mathcal{F})$ follows immediately,
and consequently, $C_{\mathrm{CWA}}(\mathcal{F})$ is satisfiable.

"$\Rightarrow$": Assume $C_{\mathrm{CWA}}(\mathcal{F}) = C(\mathcal{F} \cup \mathcal{F}_{CWA})$ is satisfiable by a model I.

Then I is a model of $\mathcal{F} \cup \mathcal{F}_{CWA}$ in particular.

Since $\mathcal{F} \cup \mathcal{F}_{CWA}$ is a set of Skolem formulae, the Herbrand interpretation H corresponding to I is also a model of $\mathcal{F} \cup \mathcal{F}_{CWA}$.
Then $H \models \mathcal{F}$ because $\mathcal{F} \subseteq C(\mathcal{F} \cup \mathcal{F}_{CWA})$. Obviously H is a Herbrand model for $\mathcal{F}$.

Still to show: H is the least Herbrand model of $\mathcal{F}$.

For this we have to show that for all ground atoms A and all Herbrand models $J \models \mathcal{F}$ it holds: $A \in H \implies A \in J$.

Let $A \in H$ and $J \models \mathcal{F}$. Since $H \not\models \neg A$ and $H \models \mathcal{F}_{CWA}$, we can conclude that $\neg A \notin \mathcal{F}_{CWA}$. So by the definition of $\mathcal{F}_{CWA}$, $\mathcal{F}_{CWA} \models A$. Since $J \models \mathcal{F}_{CWA}$, this means that $J \models A$, i.e. that $A \in J$, as required.
$\Rightarrow$ For all Herbrand models $J \models \mathcal{F}$: $H \subseteq J$. In other words, H is the least Herbrand model of $\mathcal{F}$.

Problem 5.6

Reconsider the theorem from the lectures proved in the preceding problem.

1. Show that the condition that $\mathcal{F}$ a set of formulas in Skolem normal form is necessary for $\Rightarrow$-direction.
2. Show for the $\Leftarrow$-direction that without the condition that $\mathcal{F}$ a set of formulas in Skolem normal form the existence of a least Herbrand model of $\mathcal{F}$ does not entail the existence of a Herbrand model of $C_{CWA}(\mathcal{F})$.

Solution

- Consider $\mathcal{L}(\{p/1\}, \{a/0\}, \mathcal{V})$ and $\mathcal{F} = \{(\exists X)p(X), (\exists X)\neg p(X)\}$
  - Then $\mathcal{F}$ is satisfiable by $I = (D, \cdot^I)$: $D = \{1, 2\}$, $a^I = 2$, $p^I = \{1\}$

    Because $I \models \mathcal{F}$ and $I \not\models p(a)$ and $p(a)$ is the only ground atom we get: $\mathcal{F}_{CWA} = \{\neg p(a)\}$.
    $C_{\text{CWA}}(F)$ is satisfiable because $I \models \mathcal{F} \cup \mathcal{F}_{CWA}$.
  - Yet there is no Herbrand model (and hence no least one) for $\mathcal{F}$.
    ($I_1 = \emptyset$ and $I_2 = \{p(a)\}$ are the only possible Herbrand Interpretations.)
- Consider $\mathcal{L}(\{p/1\}, \{a/0\}, \mathcal{V})$ and $\mathcal{F} = \{(\exists X)p(X)\}$.
  - Then $\mathcal{F}_{CWA} = \{\neg p(a)\}$, and $\{p(a)\}$ is a least Herbrand model for $\mathcal{F}$.
  - But there is no Herbrand model for $\mathcal{F} \cup \mathcal{F}_{CWA} = \{(\exists X)p(X), \neg p(a)\}$
  - However, $\mathcal{F} \cup \mathcal{F}_{CWA}$ is satisfiable by $I = (D, \cdot^I)$: $D = \{1, 2\}$, $a^I = 2$, $p^I = \{1\}$