

Pushing the Boundaries of Tractable Multiperspective Reasoning: A Deduction Calculus for Standpoint \mathcal{EL}^+

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Abstract

Standpoint \mathcal{EL} is a multi-modal extension of the popular description logic \mathcal{EL} that allows for the integrated representation of domain knowledge relative to diverse standpoints or perspectives. Advantageously, its satisfiability problem has recently been shown to be in PTIME, making it a promising framework for large-scale knowledge integration.

In this paper, we show that we can further push the expressivity of this formalism, arriving at an extended logic, called Standpoint \mathcal{EL}^+ , which allows for axiom negation, role chain axioms, self-loops, and other features, while maintaining tractability. This is achieved by designing a satisfiability-checking deduction calculus, which at the same time addresses the need for practical algorithms. We demonstrate the feasibility of our calculus by presenting a prototypical Data-log implementation of its deduction rules.

1 Introduction

The Semantic Web enables the exploitation of artefacts of knowledge representation (e.g., ontologies, knowledge bases, etc.) to support increasingly sophisticated automated reasoning tasks over linked data from various sources. *Description logics* (DLs) (Baader et al. 2017; Rudolph 2011) are a prominent class of logic-based KR formalisms in this context since they provide the theoretical underpinning of the Web Ontology Language (OWL 2), the main KR standard by the W3C (OWL Working Group 2009).

In particular, the lightweight description logic \mathcal{EL} (Baader, Brandt, and Lutz 2005) serves as the core of a popular family of DLs which is the formal basis of OWL 2 EL (Motik et al. 2009), a widely used tractable profile of OWL 2. One major appeal of the \mathcal{EL} family is that basic reasoning tasks can be performed in polynomial time with respect to the size of the ontology, enabling reasoning-supported creation and maintenance of very large ontologies. An example is SNOMED CT (Donnelly 2006), which is the largest healthcare ontology and has a broad user base including clinicians, patients, and researchers.

Beyond the scalable reasoning, the Semantic Web must provide mechanisms for the combination and integrated exploitation of the many knowledge sources available. Yet, its decentralised nature has led to the proliferation of ontologies of overlapping knowledge, which inevitably reflect different points of view and follow diverging modelling principles.

For instance, in the medical domain, some sources may use the concept `Tumour` to denote a process and others to denote a piece of tissue. Similarly, `Allergy` may denote an allergic reaction or just an allergic disposition. These issues pose well-known challenges in the area of knowledge integration.

Common ontology management approaches fully merge knowledge perspectives, which often requires logical weakening in order to maintain consistency. For instance, an initiative proposed the integration of SNOMED CT with the FMA1140 (Foundational Model of Anatomy) and the NCI (National Cancer Institute Thesaurus), into a unified combination called LargeBio (Osman, Ben Yahia, and Diallo 2021), and reported ensuing challenges. Beyond the risk of causing inconsistencies or unintended consequences, the unifying approach promotes weakly axiomatised ontologies designed to avoid conflict in any context of application at the expense of richer theories that would allow for further inferencing. Hence, while frameworks supporting the integrated representation of multiple perspectives seem preferable to recording the distinct views in a detached way, entirely merging them comes with significant downsides.

This need of handling multiple perspectives in the Semantic Web has led to several logic-based proposals. The closest regarding goals are multi-viewpoint ontologies (Hemam and Boufaïda 2011; Hemam 2018), which often model the intuition of viewpoints in a tailored extension of OWL for which no complexity bounds are given. Related problems are also addressed in work on contextuality, e.g. C-OWL, DDL, and CKR (Bouquet et al. 2003; Borgida and Serafini 2003; Serafini and Homola 2012)).

Modal logics are natural frameworks for modelling contexts and perspectives (Klarman and Gutiérrez-Basulto 2013; Gómez Álvarez and Rudolph 2021), and in contrast to tailored multi-perspective frameworks, they benefit from well-understood semantics. However, the interplay between DL constructs and modalities is often not well-behaved and can easily endanger the decidability of reasoning tasks or increase their complexity (Baader and Ohlbach 1995; Mosurović 1999; Wolter and Zakharyashev 1999). Notable examples are NEXPTIME-completeness of the multi-modal description logic $\mathbf{K}_{\mathcal{ALC}}$ (Lutz et al. 2002) and 2EXPTIME-completeness of $\mathcal{ALC}_{\mathcal{ALC}}$ (Klarman and Gutiérrez-Basulto 2013), a modal contextual logic framework in the style proposed by McCarthy and Buvac (1998).

Standpoint logic (Gómez Álvarez and Rudolph 2021) is a recently proposed formalism that is rooted in modal logic and allows for the simultaneous representation of multiple, potentially contradictory viewpoints and the establishment of alignments between them. This is achieved by extending a given base logic (propositional logic in the case of Gómez Álvarez and Rudolph, description logic \mathcal{EL} herein) with labelled modal operators, where propositions $\Box_S\phi$ and $\Diamond_S\phi$ express information relative to the *standpoint* S and read, respectively: “according to S , it is *unequivocal/conceivable* that ϕ ”. Semantically, standpoints are represented by sets of *precisifications*,¹ such that $\Box_S\phi$ and $\Diamond_S\phi$ hold if ϕ is true in all/some of the precisifications associated with S .

The following example illustrates the use of standpoint logic for knowledge integration in the medical domain.

Example 1 (Tumour Disambiguation). A hospital H and a laboratory L have developed their own knowledge bases and aim to make them interoperable. Hospital H gathers clinical data about patients, which may be used to determine whether a person has priority at the emergency service. According to H , a *Tumour* is a process by which abnormal or damaged cells grow and multiply (Formula 1), and patients that conceivably have a *Tumour* have a *HighRisk* priority (Formula 2). The laboratory L annotates patients’ radiographs, and models *Tumour* as a lump of tissue (Formula 3).

$$\Box_H[\text{Tumour} \sqsubseteq \text{Process}] \quad (1)$$

$$\Box_H[\text{Patient} \sqcap \Diamond_H[\exists \text{HasProcess.Tumour}] \sqsubseteq \text{HighRisk}] \quad (2)$$

$$\Box_L[\text{Tumour} \sqsubseteq \text{Tissue}] \quad (3)$$

Both institutions inherit from SN , which contains the original SNOMED CT as well as patient data (Formula 4, with the operator \preceq encoding the inheritance). Among the background knowledge in SN , we have that *Tissue* and *Process* are disjoint classes (Formula 5) and that everything that has a part which has a process, has that process itself (Formula 6).

$$H \preceq SN \quad L \preceq SN \quad (4)$$

$$\Box_{SN}[\text{Tissue} \sqcap \text{Process} \sqsubseteq \perp] \quad (5)$$

$$\Box_{SN}[\text{HasPart} \circ \text{HasProcess} \sqsubseteq \text{HasProcess}] \quad (6)$$

While clearly incompatible due to Formula 5, the perspectives of H and L are semantically close and we may be aware of further complex relations between their perspectives. For instance, we might assert that whenever a clinician at L deems that a cancerous lump of tissue is large enough to conceivably be a *Tumour* (tissue), then it is unequivocally undergoing a *Tumour* (process) according to H (Formula 7). Or we might want to specify negative information such as non-subsumption between the classes of unequivocal instances of *Process* according to H and to L (Formula 8).

$$\Diamond_L[\text{Tumour}] \sqsubseteq \Box_H[\text{Tissue} \sqcap \exists \text{HasProcess.Tumour}] \quad (7)$$

$$\neg(\Box_H[\text{Process}] \sqsubseteq \Box_L[\text{Process}]) \quad (8)$$

Finally, these sources may also have assertional knowledge:

$$\Box_{SN}[\text{Patient}(p1) \wedge \text{HasPart}(p1, a) \wedge \text{Colon}(a)] \quad (9)$$

$$\Box_H[\text{HasPart}(a, b)] \quad (10)$$

$$\Diamond_L[\text{Tumour}(b)] \quad (11)$$

¹Precisifications are analogous to the *worlds* of modal-logic frameworks with possible-worlds semantics.

That is, through SN , both H and L know of a patient $p1$ and their body parts (Formula 9) and, in view of some radiograph requested by H on part b of this patient’s colon (Formula 10), L suspects there may be tumour tissue (Formula 11). \diamond

In the first place, one should notice that a naive, standpoint-free integration of the knowledge bases without the standpoint infrastructure would trigger an inconsistency. Specifically, from a *Tumour*(b) instance we could infer both that *Tissue*(b) and *Process*(b) using the background knowledge of H and L , which in turn would lead to inconsistency with the SN axiom stating $\text{Tissue} \sqcap \text{Process} \sqsubseteq \perp$. Instead, with Standpoint $\mathcal{EL}+$, the logical statements (3)–(11) formalising Example 1 are not inconsistent, so all axioms can be jointly represented. On the one hand, we will be able to infer that H and L are indeed incompatible, denoted by $H \sqcap L \preceq \mathbf{0}$ and obtained from Formulas (3), (4), (5) and (10). On the other hand, beyond preserving consistency, the use of standpoint logic supports reasoning with and across individual perspectives.

Example 2 (Continued from Example 1). Assume that patient $p1$, of which laboratory L detected a tumour tissue (Formula 10), registers at emergencies in hospital H . From the knowledge expressed in Formulas (3)–(11), we can infer

$$\text{via (7) and (11)} \quad \Box_H[(\exists \text{HasProcess.Tumour})(b)] \quad (12)$$

$$\text{via (6),(10) and (12)} \quad \Box_H[(\exists \text{HasProcess.Tumour})(a)] \quad (13)$$

$$\text{via (6),(9) and (13)} \quad \Box_H[(\exists \text{HasProcess.Tumour})(p1)] \quad (14)$$

$$\text{via (4) and (9)} \quad \Box_H[\text{Patient}(p1)] \quad (15)$$

$$\text{via (2),(14) and (15)} \quad \Box_H[\text{HighRisk}(p1)] \quad (16)$$

meaning that, according to H , $p1$ has a tumour process and is classified as ‘high risk’. \diamond

Formally, *Standpoint logics* are multi-modal logics characterised by a simplified Kripke semantics, which brings about beneficial computational properties in different settings. For instance, it is known that adding sentential standpoints (where applying modal operators to formulas with free variables is disallowed) does not increase the complexity of numerous NP-hard FO-fragments (Gómez Álvarez, Rudolph, and Strass 2022), including the expressive DL \mathcal{SROIQb}_s , a logical basis of OWL 2 DL (Motik, Patel-Schneider, and Cuenca Grau 2009).

Yet, a fine-grained terminological alignment between different perspectives requires concepts preceded by modal operators, as in Axiom (7), which falls out of the sentential fragment. Recently, Gómez Álvarez, Rudolph, and Strass (2023) introduced a *standpoint version* of the description logic \mathcal{EL} , called Standpoint \mathcal{EL} , and established that it exhibits \mathcal{EL} ’s favourable PTIME standard reasoning, while showing that introducing additional features like empty standpoints, rigid roles, and nominals makes standard reasoning tasks intractable. In this paper, we show that we can push the expressivity of Standpoint \mathcal{EL} further while retaining tractability. We present an extended logic, called Standpoint $\mathcal{EL}+$, which allows for axiom negation, role chain axioms, self-loops, and other features. This result is achieved by designing the first satisfiability-checking deduction calculus for a standpoint-enhanced DL, thus at the same time addressing the need for practical algorithms.

Our paper is structured as follows. After introducing the syntax and semantics of Standpoint \mathcal{EL}^+ (denoted $\mathbb{S}_{\mathcal{EL}^+}$) and a suitable normal form (Section 2), we establish our main result: satisfiability checking and statement entailment in $\mathbb{S}_{\mathcal{EL}^+}$ is tractable. We show this by providing a particular Hilbert-style deduction calculus (Section 3) that operates on axioms of a fixed shape and bounded size, which immediately warrants that saturation can be performed in PTIME. For said calculus, we establish soundness and refutation-completeness. In Section 4, we briefly describe a proof-of-concept implementation of our approach, showing portions of the actual code to illustrate the key ideas. We conclude the paper in Section 5 with a discussion of future work.

An extended version of the paper with proofs of all results is available as a technical appendix.

2 Syntax, Semantics, and Normalisation

We now introduce syntax and semantics of Standpoint \mathcal{EL}^+ (referred to as $\mathbb{S}_{\mathcal{EL}^+}$) and propose a normal form that is useful for subsequent algorithmic considerations.

Syntax A *Standpoint DL vocabulary* is a traditional DL vocabulary consisting of sets N_C of *concept names*, N_R of *role names*, and N_I of *individual names*, extended by an additional set N_S of *standpoint names* with $*$ $\in N_S$. A *standpoint operator* is of the form \diamond_s (“diamond”) or \square_s (“box”) with $s \in N_S$; we use \odot_s to refer to either.²

- *Concept terms* are defined via (where $A \in N_C$, $R \in N_R$)
 $C ::= \top \mid \perp \mid A \mid C_1 \sqcap C_2 \mid \exists R.C \mid \odot_s C \mid \exists R.\text{Self}$
- A *general concept inclusion (GCI)* is of the form $C \sqsubseteq D$, where C and D are concept terms.
- A *role inclusion axiom (RIA)* is of the form $R_1 \circ \dots \circ R_n \sqsubseteq R$ where $n \geq 1$, $R_1, \dots, R_n, R \in N_R$.
- A *concept assertion* is of the form $C(a)$, where C is a concept term and $a \in N_I$.
- A *role assertion* is of the form $R(a, b)$, with $a, b \in N_I$ and $R \in N_R$.
- An *axiom* ξ is a GCI, RIA, or concept/role assertion.
- A *literal* λ is an axiom ξ or a negated axiom $\neg\xi$.
- A *monomial* μ is a conjunction $\lambda_1 \wedge \dots \wedge \lambda_m$ of literals.
- A *formula* φ is of the form $\odot_s \mu$ for a monomial μ and $s \in N_S$.
- A *sharpening statement* is of the form $s_1 \sqcap \dots \sqcap s_n \preceq s$ where $n \geq 1$, $s_1, \dots, s_n \in N_S$, and $s \in N_S \cup \{0\}$.³

Note that in particular, monomials subsume (finite) knowledge bases of the \mathcal{EL} family; monomials even go beyond that in allowing for the occurrence of *negated* axioms. Yet, monomials do not have the full expressive power of arbitrary Boolean combinations of axioms, which is a necessary restriction in order to maintain tractability.

A $\mathbb{S}_{\mathcal{EL}^+}$ *knowledge base (KB)* is a finite set of formulae and possibly negated sharpening statements. We refer to arbitrary elements of \mathcal{K} as *statements*. Note that all statements except sharpening statements are preceded by modal operators (“modalised” for short).

²We use brackets $[\dots]$ to delimit the scope of the operators.

³ 0 is used to express standpoint disjointness as in $s \sqcap s' \preceq 0$.

Semantics The semantics of $\mathbb{S}_{\mathcal{EL}^+}$ is defined via standpoint structures. Given a Standpoint DL vocabulary $\langle N_C, N_R, N_I, N_S \rangle$, a *description logic standpoint structure* is a tuple $\mathfrak{D} = \langle \Delta, \Pi, \sigma, \gamma \rangle$ where:

- Δ is a non-empty set, the *domain* of \mathfrak{D} ;
- Π is a set, called the *precisifications* of \mathfrak{D} ;
- σ is a function mapping each standpoint symbol to a non-empty⁴ subset of Π while we set $\sigma(0) = \emptyset$;
- γ is a function mapping each precisification from Π to an “ordinary” DL interpretation $\mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle$ over the domain Δ , where the interpretation function $\cdot^{\mathcal{I}}$ maps:
 - each concept name $A \in N_C$ to a set $A^{\mathcal{I}} \subseteq \Delta$,
 - each role name $R \in N_R$ to a binary relation $R^{\mathcal{I}} \subseteq \Delta \times \Delta$,
 - each individual name $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta$,
and we require $a^{\gamma(\pi)} = a^{\gamma(\pi')}$ for all $\pi, \pi' \in \Pi$ and $a \in N_I$.

Note that by this definition, individual names (also referred to as constants) are interpreted rigidly, i.e., each individual name a is assigned the same $a^{\gamma(\pi)} \in \Delta$ across all precisifications $\pi \in \Pi$. We will refer to this uniform $a^{\gamma(\pi)}$ by $a^{\mathfrak{D}}$.

For each $\pi \in \Pi$, we extend the interpretation mapping $\mathcal{I} = \gamma(\pi)$ to concept terms via structural induction:

$$\begin{aligned} \top^{\mathcal{I}} &:= \Delta \\ \perp^{\mathcal{I}} &:= \emptyset \\ (\diamond_s C)^{\mathcal{I}} &:= \bigcup_{\pi' \in \sigma(s)} C^{\gamma(\pi')} \\ (\square_s C)^{\mathcal{I}} &:= \bigcap_{\pi' \in \sigma(s)} C^{\gamma(\pi')} \\ (C_1 \sqcap C_2)^{\mathcal{I}} &:= C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}} \\ (\exists R.C)^{\mathcal{I}} &:= \{ \delta \in \Delta \mid \langle \delta, \varepsilon \rangle \in R^{\mathcal{I}} \text{ for some } \varepsilon \in C^{\mathcal{I}} \} \\ (\exists R.\text{Self})^{\mathcal{I}} &:= \{ \delta \in \Delta \mid \langle \delta, \delta \rangle \in R^{\mathcal{I}} \} \end{aligned}$$

A role chain expression $\rho = R_1 \circ R_2 \circ \dots \circ R_n$ is interpreted as $\rho^{\mathcal{I}} := ((\dots (R_1^{\mathcal{I}} \circ R_2^{\mathcal{I}}) \circ \dots) \circ R_n^{\mathcal{I}})$, where, as usual, $R \circ U := \{ \langle x, z \rangle \mid \langle x, y \rangle \in R, \langle y, z \rangle \in U \}$.

Satisfaction of a statement by a DL standpoint structure \mathfrak{D} (and precisification π) is then defined as follows:

$$\begin{aligned} \mathfrak{D}, \pi \models C \sqsubseteq D &:\iff C^{\gamma(\pi)} \subseteq D^{\gamma(\pi)} \\ \mathfrak{D}, \pi \models \rho \sqsubseteq R &:\iff \rho^{\gamma(\pi)} \subseteq R^{\gamma(\pi)} \\ \mathfrak{D}, \pi \models C(a) &:\iff a^{\mathfrak{D}} \in C^{\gamma(\pi)} \\ \mathfrak{D}, \pi \models R(a, b) &:\iff \langle a^{\mathfrak{D}}, b^{\mathfrak{D}} \rangle \in R^{\gamma(\pi)} \\ \mathfrak{D}, \pi \models \neg\xi &:\iff \mathfrak{D}, \pi \not\models \xi \\ \mathfrak{D}, \pi \models \lambda_1 \wedge \dots \wedge \lambda_n &:\iff \mathfrak{D}, \pi \models \lambda_i \text{ for all } 1 \leq i \leq n \\ \mathfrak{D} \models \square_s \mu &:\iff \mathfrak{D}, \pi \models \mu \text{ for each } \pi \in \sigma(s) \\ \mathfrak{D} \models \diamond_s \mu &:\iff \mathfrak{D}, \pi \models \mu \text{ for some } \pi \in \sigma(s) \\ \mathfrak{D} \models s_1 \sqcap \dots \sqcap s_n \preceq s &:\iff \sigma(s_1) \cap \dots \cap \sigma(s_n) \subseteq \sigma(s) \end{aligned}$$

Finally, \mathfrak{D} is a *model* of a $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} (written $\mathfrak{D} \models \mathcal{K}$) iff it satisfies every statement in \mathcal{K} . As usual, we call \mathcal{K} *satisfiable* iff some \mathfrak{D} with $\mathfrak{D} \models \mathcal{K}$ exists. A $\mathbb{S}_{\mathcal{EL}^+}$ statement ψ is *entailed* by \mathcal{K} (written $\mathcal{K} \models \psi$) iff $\mathfrak{D} \models \psi$ holds for every model \mathfrak{D} of \mathcal{K} .

⁴As shown in our prior work (Gómez Álvarez, Rudolph, and Strass 2023), allowing for “empty standpoints” immediately incurs intractability, even for an otherwise empty vocabulary.

$$\begin{aligned}
\Diamond_s[\mu] &\longrightarrow \{v \preceq s, \Box_v[\mu]\} & (17) \\
\Box_s[\lambda_1 \wedge \dots \wedge \lambda_n] &\longrightarrow \{\Box_s[\lambda_1], \dots, \Box_s[\lambda_n]\} & (18) \\
\Box_s[\neg(C \sqsubseteq D)] &\longrightarrow \{\Box_s[A \sqsubseteq C], \Box_s[A \sqcap D \sqsubseteq \perp], \Box_s[\top \sqsubseteq \exists R'.A]\} & (19) \\
\Box_s[\neg C(a)] &\longrightarrow \{\Box_s[A(a)], \Box_s[A \sqcap C \sqsubseteq \perp]\} & (20) \\
\Box_s[\neg R(a, b)] &\longrightarrow \{\Box_s[A_a(a)], \Box_s[A_b(b)], \Box_s[A_a \sqcap \exists R.A_b \sqsubseteq \perp]\} & (21) \\
\Box_s[\neg(R_1 \circ \dots \circ R_n \sqsubseteq R)] &\longrightarrow \{\Box_s[\top \sqsubseteq \exists R'.A_a], \Box_s[A_a \sqcap \exists R.A_b \sqsubseteq \perp], \Box_s[A_a \sqsubseteq \exists R_1. \dots \exists R_n.A_b]\} & (22) \\
\neg(s_1 \cap \dots \cap s_n \preceq s) &\longrightarrow \{v \preceq s_1, \dots, v \preceq s_n, v \cap s \preceq 0\} & (23)
\end{aligned}$$

Figure 1: Normalisation rules for Phase 1. Therein, $u \in N_S \cup \{0\}$, the A, A_a, A_b denote fresh concept names, R' a fresh role name, and v a fresh standpoint name.

Normalisation Before we can present our deduction calculus for checking satisfiability of $\mathbb{S}_{\mathcal{EL}^+}$ knowledge bases, we need to introduce an appropriate normal form.

Definition 1 (Normal Form of $\mathbb{S}_{\mathcal{EL}^+}$ Knowledge Bases).

A knowledge base \mathcal{K} is in *normal form* iff it only contains statements of the following shapes:

- sharpening statements of the form $s \preceq s'$ and $s_1 \cap s_2 \preceq s'$ for $s, s', s_1, s_2 \in N_S$,
- modalised GCIs of the shape $\Box_s[C \sqsubseteq D]$, where
 - C can be of the form $A, \exists R.A$, or $A \sqcap A'$ with $A, A' \in N_C \cup \{\top\} \cup \{\exists R'.\text{Self} \mid R' \in N_R\}$, $R \in N_R$,
 - D can be of the form $B, \exists R.B, \Diamond_{s'}B$, or $\Box_{s'}B$ with $B \in N_C \cup \{\perp\} \cup \{\exists R'.\text{Self} \mid R' \in N_R\}$, $R \in N_R$, $s, s' \in N_S$, and
 - one of C, D is in $N_C \cup \{\top, \perp\} \cup \{\exists R.\text{Self} \mid R \in N_R\}$;
- modalised RIAs of the form $\Box_s[R_1 \sqsubseteq R_2]$ and $\Box_s[R_1 \circ R_2 \sqsubseteq R_3]$ with $R_1, R_2, R_3 \in N_R$;
- modalised assertions of the form $\Box_s[A(a)]$ or $\Box_s[R(a, b)]$ for $a, b \in N_I$, $A \in N_C$, and $R \in N_R$. \diamond

Note that complex/nested concepts can only occur on one side of a GCI and then must not nest deeper than one level.

For a given $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} , we can compute its normal form in two phases. In the first phase, we “break down” formulas into modalised axioms, effectively compiling away negation, and in the second phase we “break down” complex concepts occurring within these axioms.

Phase 1: Modalised Axioms We obtain the (outer) normal form of axioms by exhaustively applying the transformation rules depicted in Figure 1, where “rule application” means that the statement on the left-hand side is replaced with the set of statements on the right-hand side. This eliminates statements preceded by diamonds, modalised axiom sets, and negated axioms.

Phase 2: Restricted Concept Terms To obtain the (inner) normal forms of concept terms occurring in GCIs as well as the restricted forms of sharpening statements and role inclusion axioms, we use the rules displayed in Figure 2. The first three transformation rules are novel, the others were already proposed and formally justified in our earlier work (Gómez Álvarez, Rudolph, and Strass 2023).

A careful analysis yields that the overall transformation (Phase 1 + Phase 2) has the desired semantic and computational properties.

Lemma 1. Any $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} can be transformed into a $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K}' in normal form such that:

- \mathcal{K}' is a $\mathbb{S}_{\mathcal{EL}^+}$ -conservative extension of \mathcal{K} ,
- the size of \mathcal{K}' is at most linear in the size of \mathcal{K} , and
- the transformation can be computed in PTIME.

In particular, \mathcal{K}' being a $\mathbb{S}_{\mathcal{EL}}$ -conservative extension of \mathcal{K} means that \mathcal{K} and \mathcal{K}' are equisatisfiable.

Reasoning problems and reductions The deduction calculus we are going to present in the next section decides the fundamental reasoning task of satisfiability for $\mathbb{S}_{\mathcal{EL}^+}$:

Problem: $\mathbb{S}_{\mathcal{EL}^+}$ KNOWLEDGE BASE SATISFIABILITY
Input: $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} .
Output: YES, if \mathcal{K} has a model, NO otherwise.

This reasoning task is useful in itself, e.g. for knowledge engineers to check for grave modelling errors that turn the specified knowledge base globally inconsistent. From a user’s perspective, however, a more application-relevant problem is that of statement entailment, as it allows to “query” the specified knowledge for consequences:

Problem: $\mathbb{S}_{\mathcal{EL}^+}$ STATEMENT ENTAILMENT
Input: $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} , $\mathbb{S}_{\mathcal{EL}^+}$ statement ϕ .
Output: YES, if $\mathcal{K} \models \phi$, NO otherwise.

Typically, entailment $\mathcal{K} \models \phi$ can be (many-one-)reduced to unsatisfiability of $\mathcal{K} \cup \{\neg\phi\}$. This is not immediately possible in the case of $\mathbb{S}_{\mathcal{EL}^+}$ due to its restricted syntax: note that despite the possibility to negate single axioms or sharpening statements, the negation of monomials or formulae in general is not supported by the syntax of $\mathbb{S}_{\mathcal{EL}^+}$. However, it turns out that the a similar technique can be applied with some additional care.

It is clear that the straightforward reduction works for arbitrary modalised literals $\odot_s[\lambda]$, for negated formulas $\neg\odot_s[\mu]$, and for (possibly negated) sharpening statements,

$$\begin{aligned}
s_1 \cap \dots \cap s_n \preceq s &\longrightarrow \{s_1 \cap s_2 \preceq s', s' \cap s_3 \cap \dots \cap s_n \preceq s\} & (24) \\
s_1 \cap \dots \cap s_n \preceq \mathbf{0} &\longrightarrow \{\Box_{s_1}[\top \sqsubseteq A_1], \dots, \Box_{s_n}[\top \sqsubseteq A_n], \Box_*[A_1 \cap \dots \cap A_n \sqsubseteq \perp]\} & (25) \\
\Box_s[R_1 \circ \dots \circ R_n \sqsubseteq R] &\longrightarrow \{\Box_s[R_1 \circ R_2 \sqsubseteq R'], \Box_s[R' \circ R_3 \circ \dots \circ R_n \sqsubseteq R]\} & (26) \\
\Box_s[\bar{C}(a)] &\longrightarrow \{\Box_s[A(a)], \Box_s[A \sqsubseteq \bar{C}]\} & (27) \\
\Box_s[C \sqsubseteq \top] &\longrightarrow \emptyset & (28) \\
\Box_s[\perp \sqsubseteq D] &\longrightarrow \emptyset & (29) \\
\Box_s[B \sqsubseteq \exists R.\bar{C}] &\longrightarrow \{\Box_s[B \sqsubseteq \exists R.A], \Box_s[A \sqsubseteq \bar{C}]\} & (30) \\
\Box_s[B \sqsubseteq C \cap D] &\longrightarrow \{\Box_s[B \sqsubseteq A], \Box_s[A \sqsubseteq C], \Box_s[A \sqsubseteq D]\} & (31) \\
\Box_s[C \sqsubseteq \odot_u \bar{D}] &\longrightarrow \{\Box_s[C \sqsubseteq \odot_u A], \Box_s[A \sqsubseteq \bar{D}]\} & (32) \\
\Box_s[\exists R.\bar{C} \sqsubseteq D] &\longrightarrow \{\Box_s[\bar{C} \sqsubseteq A], \Box_s[\exists R.A \sqsubseteq D]\} & (33) \\
\Box_s[\bar{C} \cap D \sqsubseteq E] &\longrightarrow \{\Box_s[\bar{C} \sqsubseteq A], \Box_s[A \cap D \sqsubseteq E]\} & (34) \\
\Box_s[\diamond_u C \sqsubseteq D] &\longrightarrow \{\Box_u[C \sqsubseteq \Box_* A], \Box_s[A \sqsubseteq D]\} & (35) \\
\Box_s[\Box_u C \sqsubseteq D] &\longrightarrow \{v_0 \preceq u, v_1 \preceq u, \Box_u[C \sqsubseteq A], \Box_s[\diamond_{v_0} A \cap \diamond_{v_1} A \sqsubseteq D]\} & (36)
\end{aligned}$$

Figure 2: Normalisation rules for Phase 2. Therein, $u \in N_S$, \bar{C} and \bar{D} stand for complex concept terms not contained in $N_C \cup \{\top, \perp\} \cup \{\exists R.\text{Self} \mid R \in N_R\}$, whereas each occurrence of A (possibly with subscript) on a right-hand side denotes the introduction of a fresh concept name; each occurrence of R' on a right-hand side denotes the introduction of a fresh role name; likewise, v , v_0 , and v_1 denote the introduction of a fresh standpoint name. Rule (34) is applied modulo commutativity of \cap .

so what remains to be detailed is the reduction for modalised monomials. Consider $\odot_s \mu = \odot_s[\lambda_1 \wedge \dots \wedge \lambda_n]$. For $\odot_s = \Box_s$, we have that $\Box_s \mu$ is logically equivalent to $\Box_s[\lambda_1] \wedge \dots \wedge \Box_s[\lambda_n]$ and thus we can (Turing-)reduce checking entailment $\mathcal{K} \models \Box_s[\mu]$ to checking whether all of $\mathcal{K} \cup \{\neg \Box_s[\lambda_1]\}, \dots, \mathcal{K} \cup \{\neg \Box_s[\lambda_n]\}$ are unsatisfiable (which is still polynomial in \mathcal{K} and $\Box_s[\mu]$). Finally, for $\odot_s = \diamond_s$, we employ the idea underlying normalisation rule (17) to obtain the following:

Lemma 2. *Let \mathcal{K} be a $\mathbb{S}_{\mathcal{EL}+}$ knowledge base with normal form \mathcal{K}' and μ be a monomial. It holds that $\mathcal{K} \models \diamond_s[\mu]$ if and only if $\mathcal{K}' \models \Box_u[\mu]$ for some $u \in N_S$ with $\mathcal{K}' \models u \preceq s$.*

Proof sketch. The main idea is that for $\mathcal{K} \models \diamond_s[\mu]$ to hold there is a formula $\diamond_{s'}[\mu'] \in \mathcal{K}$ with $\mathcal{K} \models s' \preceq s$ and formulas $\Box_{s_1}[\mu_1], \dots, \Box_{s_m}[\mu_m] \in \mathcal{K}$ with $\mathcal{K} \models s' \preceq s_i$ for all $1 \leq i \leq m$ (where neither type of formula is strictly required, and $s' = s$ in case no $\diamond_{s'}$ formula is involved), such that $\{\mu', \mu_1, \dots, \mu_m\} \models \mu$. In case all relevant formulas are of the form $\Box_{s_i}[\mu_i]$, then $\mathcal{K} \models \Box_s[\mu]$ and the claim trivially holds (with $u = s$). In case some $\diamond_{s'}[\mu'] \in \mathcal{K}$ is involved, normalisation rule (17) will introduce the new standpoint name u that can serve as witness in the normalised KB \mathcal{K}' . \square

So to decide $\mathcal{K} \models \diamond_s[\mu]$, we normalise \mathcal{K} into \mathcal{K}' and then successively enumerate $s' \in N_S$ occurring in \mathcal{K}' for which $\mathcal{K}' \models s' \preceq s$ and test $\mathcal{K} \models \Box_{s'}[\mu]$ for each. In view of these considerations, we arrive at the following reducibility result.

Theorem 3. *There exists a PTIME Turing reduction from $\mathbb{S}_{\mathcal{EL}+}$ STATEMENT ENTAILMENT to $\mathbb{S}_{\mathcal{EL}+}$ KNOWLEDGE BASE SATISFIABILITY.*

Thus, every tractable decision procedure for satisfiability can be leveraged to construct a tractable entailment checker. Therefore, we will concentrate on a method for the former.

3 Refutation-Complete Deduction Calculus for Normalised KBs

In this section, we present the Hilbert-style deduction calculus for $\mathbb{S}_{\mathcal{EL}+}$. Premises and consequents of the calculus' deduction rules will be axioms in normal form with one notable exception: We allow for extended versions of modalised GCIs of the general shape

$$\Box_t[A \sqsubseteq \Box_s[B \Rightarrow C]], \quad (37)$$

the meaning of which should be intuitively clear, despite the fact that \Rightarrow is not a connective available in $\mathbb{S}_{\mathcal{EL}+}$. In terms of more expressive Standpoint DLs, such an axiom could be written $\Box_t[A \sqsubseteq \Box_s[\neg B \sqcup C]]$, but this would obfuscate the ‘‘Horn nature’’ of the statement. Note that the axiom can be expressed in $\mathbb{S}_{\mathcal{EL}+}$ by the two axioms $\Box_t[A \sqsubseteq \Box_s D]$ and $\Box_s[D \cap B \sqsubseteq C]$ using an auxiliary fresh concept D . Yet, for better treatment in the calculus, we need all the information ‘‘bundled’’ within one axiom. With this new axiom type in place, we dispense with axioms of the shapes $\Box_s[A \sqsubseteq B]$ and $\Box_s[A \sqsubseteq \Box_{s'} B]$, replacing them by $\Box_*[\top \sqsubseteq \Box_s[A \Rightarrow B]]$ and $\Box_s[A \sqsubseteq \Box_{s'}[\top \Rightarrow B]]$, respectively. Similarly, we will replace concept assertions of the form $\Box_s A(a)$ by $\Box_*[\{a\} \sqsubseteq \Box_s[\top \Rightarrow A]]$, where $\{a\}$ is understood as a ‘‘nominal concept’’, to be interpreted by the singleton set $\{a^{\mathfrak{D}}\}$ in the usual way.⁵ Then, it should be clear that these are equivalent axiom replacements.

As one final preprocessing step, we introduce, for every concept $\diamond_s B$ that occurs in the normalised KB and every

⁵Despite us using this convenient representation ‘‘under the hood’’, we emphasise that our calculus is not meant to be used with input knowledge bases with free use of nominals. In fact, we have shown that extending Standpoint \mathcal{EL} by nominal concepts leads to intractability (G3mez 3lvarez, Rudolph, and Strass 2023).

ABox individual a , a fresh standpoint symbol denoted $s[a, B]$ and a fresh concept name $P_{s,a,B}$ and add the following background axioms:

$$s[a, B] \preceq s \quad (38)$$

$$\Box_*[\{a\} \sqsubseteq \Box_s[B \Rightarrow P_{s,a,B}]] \quad (39)$$

$$\Box_*[P_{s,a,B} \sqsubseteq \Box_s[a, B][\top \Rightarrow B]] \quad (40)$$

Intuitively, the purpose of this conservative extension is that, whenever a is required to satisfy $\diamond_s B$, it will satisfy B in all $s[a, B]$ -precisifications, this way arranging for a concrete, “addressable” witness for $\diamond_s B(a)$.

Given an $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} , let $\mathcal{K}^{\text{prep}}$ denote its preprocessed variant, obtained through normalisation and the steps described above. Again note that $\mathcal{K}^{\text{prep}}$ can be computed in deterministic polynomial time. Now let \mathcal{K}^+ denote the set of axioms obtained from $\mathcal{K}^{\text{prep}}$ by saturating it under the deduction rules of Figure 3. We note that each axiom type has a bounded number of parameters, each of which can be instantiated by a polynomial number of elements (concepts, roles, individuals, standpoints) occurring in $\mathcal{K}^{\text{prep}}$. Consequently, the overall number of distinct inferrable axioms is polynomial and therefore the saturation process to obtain \mathcal{K}^+ runs in deterministic polynomial time. We will see in the next section that these observations give rise to a worst-case optimal Datalog implementation of the saturation procedure.

We next argue that the presented calculus has the desired properties. As usual, soundness of the calculus is easy to show and can be argued for each deduction rule separately by referring to the definition of the semantics. What remains to be shown is a particular type of completeness: Among the inferrable axioms, the particular intrinsically contradictory statement $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]]$ will play the pivotal role of indicating unsatisfiability of \mathcal{K} (also referred to as refutation). We will show that our calculus is *refutation-complete*, meaning that for any unsatisfiable $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} , we have that $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]] \in \mathcal{K}^+$. More concretely, we prove the contrapositive by establishing the existence of a model whenever $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]] \notin \mathcal{K}^+$. This model is canonical in a sense but, as opposed to canonical models of the \mathcal{EL} family, it will typically be infinite.

Canonical Model Construction Given a $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} with $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]] \notin \mathcal{K}^+$, we construct a model \mathfrak{D} of \mathcal{K} in an infinite process: we start from an initialised model \mathfrak{D}_0 and extend it (both by adding domain elements and precisifications) in a stepwise fashion, resulting in a “monotonic” sequence of models. The result of the process is the “limit” of this sequence, which can be expressed via an infinite union.

For the initialisation, we choose the standpoint structure $\mathfrak{D}_0 = \langle \Delta_0, \Pi_0, \sigma_0, \gamma_0 \rangle$ where:

- Δ_0 consists of one element δ_a for every individual name a mentioned in \mathcal{K} ;
- Π_0 consists of one precisification π_s for every standpoint s mentioned in \mathcal{K} (including $*$);
- σ_0 maps each standpoint s to $\{\pi_s\} \cup \{\pi_{s'} \mid s' \preceq s \in \mathcal{K}^+\}$;

- γ_0 maps each π_s to the description logic interpretation \mathcal{I} over Δ_0 , where
 - $a^{\mathcal{I}} = \delta_a$ for each individual name a ,
 - $A^{\mathcal{I}} = \{\delta_a \mid \Box_u[\{a\} \sqsubseteq \Box_s[\top \Rightarrow A]] \in \mathcal{K}^+, s \in \sigma_0^{-1}(\pi_s)\}$ for every concept name A , and
 - $R^{\mathcal{I}} = \{(\delta_a, \delta_b) \mid \Box_s[R(a, b)] \in \mathcal{K}^+, s \in \sigma_0^{-1}(\pi_s)\}$ for each role name R .

It can be shown that the obtained structure satisfies all axioms of \mathcal{K} except for those of the shape $\Box_s[E \sqsubseteq \exists R.F]$. This will also be the case for all structures $\mathfrak{D}_1, \mathfrak{D}_2, \dots$ subsequently produced. The sequence arises by iteratively adding more domain elements in order to satisfy more axioms of the type $\Box_s[E \sqsubseteq \exists R.F]$. Thereby, the concept and role memberships of pre-existing elements with respect to pre-existing standpoints will remain unchanged. This also justifies the definition of a labelling function Λ_π immutably assigning to every domain element δ a set of concepts, satisfied in π . For \mathfrak{D}_0 , we let Λ_π map elements δ_a according to

$$\Lambda_\pi(\delta_a) = \{C \mid \Box_u[\{a\} \sqsubseteq \Box_s[\top \Rightarrow C]] \in \mathcal{K}^+, s \in \sigma_0^{-1}(\pi)\}.$$

As discussed above, after having arrived at a structure $\mathfrak{D}_i = \langle \Delta_i, \Pi_i, \sigma_i, \gamma_i \rangle$, we inspect if \mathfrak{D}_i satisfies all axioms of the form $\Box_t[E \sqsubseteq \exists R.F]$. If so, \mathfrak{D}_i is a model of \mathcal{K} and we are done. Otherwise, we pick some $\delta^* \in \Delta_i$ and some $\pi^* \in \Pi_i$ with $\pi^* \in \sigma(t)$ and $\delta^* \in E^{\gamma(\pi^*)} \setminus (\exists R.F)^{\gamma(\pi^*)}$ for some (previously unsatisfied) axiom $\Box_t[E \sqsubseteq \exists R.F]$ from \mathcal{K} . Among the eligible pairs δ^*, π^* , we pick one for which the value $\min\{j \leq i \mid \delta^* \in \Delta_j\} + \min\{j \leq i \mid \pi^* \in \Pi_j\}$ is minimal; this ensures fairness in the sense that any axiom violation will ultimately be addressed. Given δ^* and π^* we now obtain $\mathfrak{D}_{i+1} = \langle \Delta_{i+1}, \Pi_{i+1}, \sigma_{i+1}, \gamma_{i+1} \rangle$ from $\mathfrak{D}_i = \langle \Delta_i, \Pi_i, \sigma_i, \gamma_i \rangle$ as follows:

- $\Delta_{i+1} = \Delta_i \cup \{\delta'\}$, where δ' is a fresh domain element;
- let $\text{Con}(F, \pi^*)$ denote the concepts subsumed by F under π^* , i.e., $\{A \mid \Box_u[\top \sqsubseteq \Box_s[F \Rightarrow A]] \in \mathcal{K}^+, s \in \sigma^{-1}(\pi^*)\}$
- Π_{i+1} is obtained from Π_i by adding a fresh precisification $\pi_{\delta', \diamond_s D}$ whenever there is some $C \in \text{Con}(F, \pi^*)$ with $\Box_t[C \sqsubseteq \diamond_s D] \in \mathcal{K}^+$ for some $t \in \sigma^{-1}(\pi^*)$
- let σ_{i+1} be such that $\sigma_{i+1}(s'') = \sigma_i(s'') \cup \{\pi_{\delta', \diamond_s D}\}$ if $s'' \in \{s\} \cup \{s' \mid s \preceq s' \in \mathcal{K}^+\}$ and $\sigma_{i+1}(s'') = \sigma_i(s'')$ otherwise.
- for $\pi \in \Pi_{i+1}$ we let $\Lambda_\pi(\delta')$ be

$$\text{Con}(F, \pi^*) \quad \text{if } \pi = \pi^*,$$

$$\bigcup_{G \in \text{Con}(F, \pi^*), s \in \sigma_{i+1}^{-1}(\pi), t \in \sigma_{i+1}^{-1}(\pi^*)} \{A \mid \Box_t[G \sqsubseteq \Box_s[\top \Rightarrow A]] \in \mathcal{K}^+\} \quad \text{if } \pi \in \Pi_i \setminus \{\pi^*\},$$

$$\bigcup_{G \in \text{Con}(F, \pi^*), s \in \sigma_{i+1}^{-1}(\pi), t \in \sigma_{i+1}^{-1}(\pi^*)} \{A \mid \Box_t[G \sqsubseteq \Box_s[D \Rightarrow A]] \in \mathcal{K}^+\} \quad \text{if } \pi = \pi_{\delta', \diamond_s D}.$$

- Let γ_{i+1} be the interpretation function defined as follows:
 - $a^{\gamma_{i+1}(\pi)} = a^{\gamma_i(\pi)} = \delta_a$ for each individual name a and each $\pi \in \Pi_{i+1}$.
 - for concept names A , we let $A^{\gamma_{i+1}(\pi)} = A^{\gamma_i(\pi)} \cup \{\delta'\}$ if $A \in \Lambda_\pi(\delta')$, and $A^{\gamma_{i+1}(\pi)} = A^{\gamma_i(\pi)}$ otherwise.

Tautologies

$$(T.1) \frac{}{s \preceq *} \quad (T.2) \frac{}{s \preceq s} \quad (T.3) \frac{}{\Box_*[\top \sqsubseteq \Box_*[C \Rightarrow C]]} \quad (T.4) \frac{}{\Box_*[\top \sqsubseteq \Box_*[C \Rightarrow \top]]} \quad (T.5) \frac{}{\Box_*[R \sqsubseteq R]}$$

Standpoint hierarchy rules (for all $s \in N_s$, ξ being any extended GCI, RIA, or role assertion)

$$(S.1) \frac{s \preceq s' \quad s' \preceq s''}{s \preceq s''} \quad (S.2) \frac{s \preceq s_1 \quad s \preceq s_2 \quad s_1 \cap s_2 \preceq s'}{s \preceq s'} \quad (S.3) \frac{\Box_{s'} \xi \quad s \preceq s'}{\Box_s \xi} \quad (S.4) \frac{\Box_t[C \sqsubseteq \Box_{s'}[D \Rightarrow E]] \quad s \preceq s'}{\Box_t[C \sqsubseteq \Box_s[D \Rightarrow E]]}$$

Internal inferences for extended GCIs

$$(I.1) \frac{\Box_s[C \sqsubseteq \Box_s[\top \Rightarrow D]]}{\Box_*[\top \sqsubseteq \Box_s[C \Rightarrow D]]} \quad (I.2) \frac{\Box_u[\top \sqsubseteq \Box_s[C \Rightarrow D]]}{\Box_*[\top \sqsubseteq \Box_s[C \Rightarrow D]]}$$

Role subsumptions

$$(R.1) \frac{\Box_s[R \sqsubseteq R''] \quad \Box_s[R'' \sqsubseteq R']}{\Box_s[R \sqsubseteq R']}$$

Forward chaining

$$(C.1) \frac{\Box_t[B \sqsubseteq \Box_s[C \Rightarrow D]] \quad \Box_t[B \sqsubseteq \Box_s[D \Rightarrow E]]}{\Box_t[B \sqsubseteq \Box_s[C \Rightarrow E]]} \quad (C.2) \frac{\Box_u[\top \sqsubseteq \Box_t[B \Rightarrow C]] \quad \Box_t[C \sqsubseteq \Box_s[D \Rightarrow E]]}{\Box_t[B \sqsubseteq \Box_s[D \Rightarrow E]]}$$

$$(C.3) \frac{\Box_u[\top \sqsubseteq \Box_t[C \Rightarrow D]] \quad \Box_t[D \sqsubseteq \Diamond_s E]}{\Box_t[C \sqsubseteq \Diamond_s E]} \quad (C.4) \frac{\Box_t[C \sqsubseteq \Diamond_s D] \quad \Box_t[C \sqsubseteq \Box_s[D \Rightarrow E]]}{\Box_t[C \sqsubseteq \Diamond_s E]}$$

Flattening of modalities

$$(F.1) \frac{\Box_t[C \sqsubseteq \Box_{s'}[\top \Rightarrow D]] \quad \Box_{s'}[D \sqsubseteq \Box_s[E \Rightarrow F]]}{\Box_t[C \sqsubseteq \Box_s[E \Rightarrow F]]} \quad (F.2) \frac{\Box_t[C \sqsubseteq \Box_{s'}[\top \Rightarrow D]] \quad \Box_{s'}[D \sqsubseteq \Diamond_s E]}{\Box_t[C \sqsubseteq \Diamond_s E]}$$

$$(F.3) \frac{\Box_t[C \sqsubseteq \Diamond_{s'} D] \quad \Box_{s'}[D \sqsubseteq \Box_s[E \Rightarrow F]]}{\Box_t[C \sqsubseteq \Box_s[E \Rightarrow F]]} \quad (F.4) \frac{\Box_t[C \sqsubseteq \Diamond_{s'} D] \quad \Box_{s'}[D \sqsubseteq \Diamond_s E]}{\Box_t[C \sqsubseteq \Diamond_s E]}$$

Inferences involving existential quantifiers and conjunction

$$(E.1) \frac{\Box_s[C \sqsubseteq \exists R.D] \quad \Box_u[\top \sqsubseteq \Box_s[D \Rightarrow E]] \quad \Box_s[R \sqsubseteq R']}{\Box_s[C \sqsubseteq \exists R'.E]} \quad (E.2) \frac{\Box_s[C \sqsubseteq \exists R_1.D] \quad \Box_s[D \sqsubseteq \exists R_2.E] \quad \Box_s[R_1 \circ R_2 \sqsubseteq R']}{\Box_s[C \sqsubseteq \exists R'.E]}$$

$$(E.3) \frac{\Box_s[C \sqsubseteq \exists R.D] \quad \Box_s[\exists R.D \sqsubseteq F]}{\Box_*[\top \sqsubseteq \Box_s[C \Rightarrow F]]} \quad (E.4) \frac{\Box_t[B \sqsubseteq \Box_s[C \Rightarrow C_1]] \quad \Box_t[B \sqsubseteq \Box_s[C \Rightarrow C_2]] \quad \Box_s[C_1 \cap C_2 \sqsubseteq D]}{\Box_t[B \sqsubseteq \Box_s[C \Rightarrow D]]}$$

Individual-based inferences

$$(A.1) \frac{\Box_u[\top \sqsubseteq \Box_s[B \Rightarrow C]]}{\Box_*[\{a\} \sqsubseteq \Box_s[B \Rightarrow C]]} \quad (A.2) \frac{\Box_u[\{a\} \sqsubseteq \Box_s[\top \Rightarrow C]]}{\Box_*[\top \sqsubseteq \Box_s[\{a\} \Rightarrow C]]} \quad (A.3) \frac{\Box_u[\{a\} \sqsubseteq \Box_s[B \Rightarrow C]]}{\Box_*[\{a\} \sqsubseteq \Box_s[B \Rightarrow C]]}$$

$$(A.4) \frac{\Box_s[R(a, b)] \quad \Box_s[R \sqsubseteq R']}{\Box_s[R'(a, b)]} \quad (A.5) \frac{\Box_s[R_1(a, b)] \quad \Box_s[R_2(b, c)] \quad \Box_s[R_1 \circ R_2 \sqsubseteq R']}{\Box_s[R'(a, c)]}$$

$$(A.6) \frac{\Box_s[R(a, b)] \quad \Box_u[\{b\} \sqsubseteq \Box_s[\top \Rightarrow B]]}{\Box_s[\{a\} \sqsubseteq \exists R.B]} \quad (A.7) \frac{\Box_s[R_1(a, b)] \quad \Box_s[\{b\} \sqsubseteq \exists R_2.C] \quad \Box_s[R_1 \circ R_2 \sqsubseteq R']}{\Box_s[\{a\} \sqsubseteq \exists R'.C]}$$

$$(A.8) \frac{\Box_s[R_1(a, b)] \quad \Box_u[\{b\} \sqsubseteq \Box_s[\top \Rightarrow B]] \quad \Box_s[B \sqsubseteq \exists R_2.C] \quad \Box_s[R_1 \circ R_2 \sqsubseteq R']}{\Box_s[\{a\} \sqsubseteq \exists R'.C]}$$

Interaction of self-loops with other statements

$$(L.1) \frac{\Box_u[\{a\} \sqsubseteq \Box_s[\top \Rightarrow \exists R.\text{Self}]]}{\Box_s[R(a, a)]} \quad (L.2) \frac{\Box_u[\top \sqsubseteq \Box_s[C \Rightarrow \exists R.\text{Self}]]}{\Box_s[C \sqsubseteq \exists R.C]} \quad (L.3) \frac{\Box_s[\exists R.D \sqsubseteq C]}{\Box_s[\exists R.\text{Self} \cap D \sqsubseteq C]}$$

$$(L.4) \frac{\Box_s[R(a, a)]}{\Box_*[\{a\} \sqsubseteq \Box_s[\top \Rightarrow \exists R.\text{Self}]]} \quad (L.5) \frac{\Box_s[R \sqsubseteq R']}{\Box_*[\top \sqsubseteq \Box_s[\exists R.\text{Self} \Rightarrow \exists R'.\text{Self}]]} \quad (L.6) \frac{\Box_s[R_1 \circ R_2 \sqsubseteq R']}{\Box_s[\exists R_1.\text{Self} \cap \exists R_2.\text{Self} \sqsubseteq \exists R'.\text{Self}]}$$

Backpropagation of \perp -inferences

$$(B.1) \frac{\Box_s[C \sqsubseteq \exists R.\perp]}{\Box_*[\top \sqsubseteq \Box_s[C \Rightarrow \perp]]} \quad (B.2) \frac{\Box_t[C \sqsubseteq \Box_s[\top \Rightarrow \perp]]}{\Box_*[\top \sqsubseteq \Box_t[C \Rightarrow \perp]]} \quad (B.3) \frac{\Box_t[C \sqsubseteq \Diamond_s \perp]}{\Box_*[\top \sqsubseteq \Box_t[C \Rightarrow \perp]]} \quad (B.4) \frac{\Box_u[\{a\} \sqsubseteq \Box_s[\top \Rightarrow \perp]]}{\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]]}$$

Figure 3: Deduction calculus for $\mathbb{S}_{\mathcal{EL}+}$

$$\begin{array}{ll}
\Box_s[C \sqsubseteq \Box_{s'}[D \Rightarrow E]] \rightsquigarrow \text{gci_nested}(s, C, s', D, E) & s_1 \cap s_2 \preceq s_3 \rightsquigarrow \text{sharper_intersection}(s_1, s_2, s_3) \\
\Box_s[C \sqcap D \sqsubseteq E] \rightsquigarrow \text{gci_conjunction_left}(s, C, D, E) & s_1 \preceq s_2 \rightsquigarrow \text{sharper}(s_1, s_2) \\
\Box_s[\exists R.C \sqsubseteq D] \rightsquigarrow \text{gci_existential_left}(s, R, C, D) & R_1 \circ R_2 \sqsubseteq R_3 \rightsquigarrow \text{ria3}(R_1, R_2, R_3) \\
\Box_s[C \sqsubseteq \exists R.D] \rightsquigarrow \text{gci_existential_right}(s, C, R, D) & R_1 \sqsubseteq R_2 \rightsquigarrow \text{ria2}(R_1, R_2) \\
\Box_s[C \sqsubseteq \Diamond_{s'} D] \rightsquigarrow \text{gci_diamond_right}(s, C, s', D) &
\end{array}$$

Figure 4: Datalog atoms used to represent normal-form statements of $\mathbb{S}_{\mathcal{EL}^+}$ as used in the deduction calculus. Therein, $s_1, s_2, s_3, s, s' \in N_S$, $C, D, E \in N_C$, and $R_1, R_2, R_3, R \in N_R$.

- for all role names T , we obtain $T^{\gamma_{i+1}(\pi)}$ essentially by performing a concurrent saturation process under all applicable RIAs, that is,

$$T^{\gamma_{i+1}(\pi)} = \bigcup_{k \in \mathbb{N}} [T^{\gamma_{i+1}(\pi)}]_k,$$

where we let $[T^{\gamma_{i+1}(\pi)}]_0 = \text{Self} \cup \text{Other}$, with

$$\text{Self} = \begin{cases} \{(\delta', \delta')\} & \text{if } \exists T. \text{Self} \in \Lambda_\pi(\delta'), \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\text{Other} = \begin{cases} \emptyset & \text{whenever } \pi \in \Pi_{i+1} \setminus \Pi_i \\ T^{\gamma_i(\pi)} & \text{for } \pi \in \Pi_i, \text{ if } T \neq R \text{ or } \pi \neq \pi^* \\ R^{\gamma_i(\pi^*)} \cup (\delta^*, \delta') & \text{if } T = R \text{ and } \pi = \pi^*, \end{cases}$$

and obtain

$$\begin{aligned}
[T^{\gamma_{i+1}(\pi)}]_{k+1} = & [T^{\gamma_{i+1}(\pi)}]_k \cup \bigcup_{s \in \sigma_{i+1}^{-1}(\pi), \Box_s[R_0 \sqsubseteq T] \in \mathcal{K}^+} [R_0^{\gamma_{i+1}(\pi)}]_k \\
& \cup \bigcup_{s \in \sigma_{i+1}^{-1}(\pi), \Box_s[R_1 \circ R_2 \sqsubseteq T] \in \mathcal{K}^+} [R_1^{\gamma_{i+1}(\pi)}]_k \circ [R_2^{\gamma_{i+1}(\pi)}]_k.
\end{aligned}$$

After producing the (potentially infinite) sequence $\mathcal{D}_0, \mathcal{D}_1, \dots$ we obtain the wanted model \mathcal{D} via

$$\mathcal{D} = \langle \Delta, \Pi, \sigma, \gamma \rangle = \langle \bigcup_i \Delta_i, \bigcup_i \Pi_i, \bigcup_i \sigma_i, \bigcup_i \gamma_i \rangle.$$

We then establish that the \mathcal{D} resulting from this construction indeed is a well-defined structure that satisfies all axioms of \mathcal{K} . To this end, an important observation is that for all domain elements $\delta \in \Delta$ and precisifications $\pi \in \Pi$ of \mathcal{D} , it holds that $C \in \Lambda_\pi(\delta)$ implies $\delta \in C^{\gamma(\pi)}$. Furthermore, we show that if $\perp \in \Lambda_\pi(\delta)$ were to hold for any $\delta \in \Delta$ and $\pi \in \Pi$ (which is the only way the model construction could possibly fail, by declaring an existing domain element to be contradictory), then this would necessarily imply $\mathcal{K}^+ \models \Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]]$. By virtue of these considerations, we arrive at the aspired result.

Theorem 4. *The deduction calculus displayed in Figure 3 is sound and refutation-complete for $\mathbb{S}_{\mathcal{EL}^+}$ knowledge bases.*

Then, together with our previous insights and Theorem 3, we can use Theorem 4 to establish tractability of the fundamental standard reasoning tasks in $\mathbb{S}_{\mathcal{EL}^+}$.

Corollary 5. $\mathbb{S}_{\mathcal{EL}^+}$ KNOWLEDGE BASE SATISFIABILITY and $\mathbb{S}_{\mathcal{EL}^+}$ STATEMENT ENTAILMENT are PTIME-complete.

Thereby, PTIME-hardness follows immediately from the PTIME-hardness of reasoning in plain \mathcal{EL} .

4 Datalog-Based Implementation

We have prototypically implemented our approach in the Datalog-based language SOUFFLÉ (Jordan, Scholz, and Subotić 2016). The prototype’s source code is available from our group’s github site at <https://github.com/cl-tud/standpoint-el-souffle-reasoner>. The implementation currently does not scale well, so optimising both calculus and implementation is an important issue for future work.

Calculus The calculus of Section 3 is implemented in the pure Datalog fragment of SOUFFLÉ’s input language. Following the common approach, as e.g. detailed by Krötzsch (2010), we introduce a predicate symbol for each possible (normal-form) formula shape as shown in Figure 4. Implementing the deduction calculus then boils down to writing Datalog rules for all deduction rules; the main rules of the calculus are shown in Figure 5 (p.9). For the axiom schemas (tautologies) we make use of helper predicates that keep track of the vocabulary; likewise, for nominals and self-loops we use binary predicates to translate back and forth between individual names/nominal concepts, and role names/self-loop concepts, respectively, hence treating nominals and self-loops as “ordinary” concept names.

Normalisation For obtaining the normal form of a given $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base in its full expressiveness, we employ several SOUFFLÉ features that are not strictly Datalog. For one, we use algebraic data types⁶ to define term-based encodings of all structured constructs involved in representing knowledge bases, such as concept terms, axioms, formulas, etc., where the base types “standpoint name”, “role name”, and “concept name” are subtypes of the built-in type symbol (i.e., string). More importantly, during normalisation we employ SOUFFLÉ’s built-in functor `cat`⁷ for concatenating strings to create unambiguous identifiers for newly introduced concept, role, and standpoint names.

5 Conclusion and Future Work

In this paper, we presented the knowledge representation formalism Standpoint \mathcal{EL}^+ , which extends the formerly proposed Standpoint \mathcal{EL} language (Gómez Álvarez, Rudolph, and Strass 2023) by a row of new modelling features: role chain axioms and self-loops, extended sharpening statements including standpoint disjointness, negated axioms,

⁶<https://souffle-lang.github.io/types#algebraic-data-types-adt>

⁷<https://souffle-lang.github.io/arguments#intrinsic-functor>


```

(T.1) sharper(s, *) :- is_sn(s).
(T.2) sharper(s, s) :- is_sn(s).
(T.3) gci_nested(*, T, *, c, c) :- is_cn(c).
(T.4) gci_nested(*, T, *, c, T) :- is_cn(c).
(T.5) ria2(*, r, r) :- is_rn(r).
(R.1) ria2(s, r1, r3) :- ria2(s, r1, r2), ria2(s, r2, r3).
(S.1) sharper(s1, s3) :- sharper(s1, s2), sharper(s2, s3).
(S.2) sharper(s, u) :- sharper(s, s1), sharper(s, s2), sharper_intersection(s1, s2, u).
(S.3) ria2(s, r1, r2) :- ria2(u, r1, r2), sharper(s, u).
(S.3) ria3(s, r1, r2, r3) :- ria3(u, r1, r2, r3), sharper(s, u).
(S.3) gci_nested(s, c, t, d, e) :- gci_nested(u, c, t, d, e), sharper(s, u).
(S.3) gci_dia_right(s, c, t, d) :- gci_dia_right(u, c, t, d), sharper(s, u).
(S.3) gci_ex_right(s, c, r, d) :- gci_ex_right(u, c, r, d), sharper(s, u).
(S.3) gci_ex_left(s, r, c, d) :- gci_ex_left(u, r, c, d), sharper(s, u).
(S.3) gci_con_left(s, c1, c2, d) :- gci_con_left(u, c1, c2, d), sharper(s, u).
(S.4) gci_nested(t, c, s, d, e) :- gci_nested(t, c, u, d, e), sharper(s, u).
(C.1) gci_nested(t, b, s, c, e) :- gci_nested(t, b, s, c, d), gci_nested(t, b, s, d, e).
(C.2) gci_nested(t, b, s, d, e) :- gci_nested(_, T, t, b, c), gci_nested(t, c, s, d, e).
(C.3) gci_dia_right(t, c, s, e) :- gci_nested(_, T, t, c, d), gci_dia_right(t, d, s, e).
(C.4) gci_dia_right(t, c, s, e) :- gci_dia_right(t, c, s, d), gci_nested(t, c, s, d, e).
(F.1) gci_nested(t, c, s, e, f) :- gci_nested(t, c, u, T, d), gci_nested(u, d, s, e, f).
(F.2) gci_dia_right(t, c, s, e) :- gci_nested(t, c, u, T, d), gci_dia_right(u, d, s, e).
(F.3) gci_nested(t, c, s, e, f) :- gci_dia_right(t, c, u, d), gci_nested(u, d, s, e, f).
(F.4) gci_dia_right(t, c, s, e) :- gci_dia_right(t, c, u, d), gci_dia_right(u, d, s, e).
(E.1) gci_ex_right(s, c, r2, e) :- gci_ex_right(s, c, r1, d), gci_nested(_, T, s, d, e), ria2(s, r1, r2).
(E.2) gci_ex_right(s, c, r3, e) :- gci_ex_right(s, c, r1, d), gci_ex_right(s, d, r2, e), ria3(s, r1, r2, r3).
(E.3) gci_nested(*, T, s, c, f) :- gci_ex_right(s, c, r, d), gci_ex_left(s, r, d, f).
(E.4) gci_nested(t, b, s, c, d) :- gci_nested(t, b, s, c, c1), gci_nested(t, b, s, c, c2), gci_con_left(s, c1, c2, d).
(I.1) gci_nested(*, T, s, c, d) :- gci_nested(s, c, s, T, d).
(I.2) gci_nested(*, T, s, c, d) :- gci_nested(_, T, s, c, d).
(B.1) gci_nested(*, T, s, c, ⊥) :- gci_ex_right(s, c, _, ⊥).
(B.2) gci_nested(*, T, t, c, ⊥) :- gci_nested(t, c, _, T, ⊥).
(B.3) gci_nested(*, T, s, c, ⊥) :- gci_dia_right(s, c, _, ⊥).
(B.4) gci_nested(*, T, *, T, ⊥) :- gci_nested(_, nom, _, T, ⊥), is_nom(nom).

```

Figure 5: Datalog implementation of the calculus excluding rules dealing with assertions and self-loops. Predicates with prefix “is_” declare the vocabulary: $C \in N_C \rightsquigarrow \text{is_cn}(C)$, $R \in N_R \rightsquigarrow \text{is_rn}(R)$, $s \in N_S \rightsquigarrow \text{is_sn}(s)$, and $a \in N_I \rightsquigarrow \text{is_nom}(\{a\})$. Note that deduction rule schema (S.3) needs multiple concrete instantiations. The symbols $*$, T , and \perp are used for readability here; in the actual implementation, we use proper (but globally fixed) Datalog constants.

and modalised axiom sets. We designed a deduction calculus that is sound and refutation-complete when applied to appropriately pre-processed $\mathbb{S}_{\mathcal{EL}+}$ knowledge bases. As preprocessing and exhaustive deduction rule application are shown to run in PTIME, we thereby established tractability of satisfiability checking of $\mathbb{S}_{\mathcal{EL}+}$ knowledge bases and – by virtue of a PTIME Turing reduction – also tractability of checking the entailment of $\mathbb{S}_{\mathcal{EL}+}$ statements from $\mathbb{S}_{\mathcal{EL}+}$ knowledge bases, notably also allowing negated statements.

We note that, if tractability is to be preserved, the options of further extending the expressivity of $\mathbb{S}_{\mathcal{EL}+}$ are limited. Clearly, any modelling feature that would turn the description logic \mathcal{EL} intractable – atomic negation, disjunction, cardinality restrictions, universal quantification as well as inverse or functional roles (Baader, Brandt, and Lutz 2005) – would also destroy tractability of $\mathbb{S}_{\mathcal{EL}+}$. But also the free use of nominal concepts, which is known to still warrant PTIME reasoning when added to \mathcal{EL} with role chain axioms and self-loops, has been shown to be computationally detrimental as soon as standpoints are involved. The same holds if one allows for the declaration of roles to be rigid or for a more liberal semantics that would admit empty standpoints (Gómez Álvarez, Rudolph, and Strass 2023).

Beyond the theoretical advancement, we also believe that the developed deduction calculus can pave the way to practical reasoner implementations by means of Datalog mater-

ialisation, a method already proven to be competitive for reasoning in lightweight description logics. In order to demonstrate the principled feasibility of this approach, we implemented a publicly available prototype in SOUFFLÉ.

There are numerous avenues for future work. While the calculus is adequate to show our theoretical results and demonstrate feasibility, we are confident that there is much room for improvement when it comes to optimisation. We expect that refactoring the set of deduction rules can significantly improve performance in practice. In Datalog terms, it would be beneficial to reduce the number and arity of the predicates involved, the number of variables per rule, and the number of alternative derivations of the same fact. These goals may be in conflict and it is typically not straightforward to find the optimal sweet spot. In this regard, realistic benchmarks can provide guidance. While no off-the-shelf standpoint ontologies exist yet, we expect that sensible test cases can be generated from linked open data, ontology alignment settings, or ontology repositories with versioning.

Likewise, the calculus can be analysed and improved in terms of more comprehensive completeness guarantees; in fact, we conjecture that it already yields all entailed “boxed” assertions and concept inclusions over concept names.

More generally, we will investigate standpoint extensions of other light- or heavyweight ontology languages regarding computational properties and efficient reasoning.

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A Proofs for Section 2

Normalisation We will next show Lemma 1. By introducing new standpoint, concept, and role names, any knowledge base \mathcal{K} can be turned into a normalised knowledge base \mathcal{K}' that is a conservative extension of \mathcal{K} , i.e., every model of \mathcal{K}' is also a model of \mathcal{K} , and every model of \mathcal{K} can be extended to a model of \mathcal{K}' by appropriately choosing the interpretations of the additional standpoint, concept, and role names.

To show that this transformation can be done in polynomial time, yielding a normalised KB \mathcal{K}' whose size is linear in the size of \mathcal{K} , we next define the size $\|\mathcal{K}\|$ of a knowledge base \mathcal{K} roughly as the number of symbols needed to write down \mathcal{K} , and define it formally as follows.

Definition 2. Let \mathcal{K} be a $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base. The *size* of \mathcal{K} , denoted $\|\mathcal{K}\|$, and its various constituents is defined inductively as follows:

$$\begin{aligned}
 \|\mathfrak{s}_1 \cap \dots \cap \mathfrak{s}_n \preceq \mathfrak{s}\| &:= n + 1 && \text{for } \mathfrak{s} \in N_S \cup \{\mathbf{0}\} \\
 \|\top\| &:= 1 \\
 \|\perp\| &:= 1 \\
 \|A\| &:= 1 && \text{for } A \in N_C \\
 \|C \sqcap D\| &:= 1 + \|C\| + \|D\| \\
 \|\exists R.C\| &:= 1 + \|C\| \\
 \|\exists R.\text{Self}\| &:= 1 \\
 \|\odot_s C\| &:= 1 + \|C\| \\
 \|C \sqsubseteq D\| &:= 1 + \|C\| + \|D\| \\
 \|R_1 \circ \dots \circ R_n \sqsubseteq R\| &:= n + 1 \\
 \|C(a)\| &:= 1 + \|C\| \\
 \|R(a_1, a_2)\| &:= 3 \\
 \|\neg \xi\| &:= 1 + \|\xi\| \\
 \|\lambda_1 \wedge \dots \wedge \lambda_n\| &:= \sum_{1 \leq i \leq n} \|\lambda_i\| \\
 \|\odot_s[\mu]\| &:= 1 + \|\mu\| \\
 \|\mathcal{K}\| &:= \sum_{\psi \in \mathcal{K}} \|\psi\|
 \end{aligned}$$

The subsequent proof is, for the most parts, an extension of our previous proof for a fragment of the language (Gómez Álvarez, Rudolph, and Strass 2023) where in this work we added the cases for the new constructors and phase 1 normalisation rules. We reproduce the whole proof here for coherence and reference.

Proof of Lemma 1. Let \mathcal{K} be a $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base. The statements in \mathcal{K} can be converted into normal form by exhaustively applying the replacement rules shown on page 5. In what follows, denote by \mathcal{K}' the result of exhaustive rule application to \mathcal{K} . To prove the lemma, we proceed to show the following:

1. Polynomial runtime and linear output size: Application of the normalisation rules terminates after at most a polynomial (in the size of \mathcal{K}) number of steps, and the size of the resulting KB \mathcal{K}' is at most linear in the size of \mathcal{K} .
2. Syntactic Correctness: \mathcal{K}' is in normal form according to Definition 1.
3. Semantic Correctness: \mathcal{K}' is a conservative extension of \mathcal{K} , more specifically:
 - (a) For every model \mathfrak{D} of \mathcal{K} there exists a DL standpoint structure \mathfrak{D}' that extends \mathfrak{D} (agrees with \mathfrak{D} on the vocabulary of \mathcal{K}) and that is a model of \mathcal{K}' .
 - (b) Every model \mathfrak{D}' of \mathcal{K}' is also a model of \mathcal{K} .
1. We first show that overall normalisation must terminate after at most $\|\mathcal{K}\|$ normalisation rule applications. The proof plan is as follows: For phase 1, we observe that application of a phase 1 rule (n) produces statements to which rule (n) is not applicable again. Rule (18) incurs a linear blowup in size for the monomial in question (a constant size increase for each literal in the monomial); all other phase 1 rules induce only a constant size increase per application, so the overall size increase of normalisation phase 1 is linear.

For phase 2, we concentrate on sets of GCIs (as they are the only places where nested concept terms can occur due to rule (27)) and denote \mathcal{K} by \mathcal{T} (for TBox) if only GCIs are involved, and proceed thus: We define the multiset $\mathcal{C}_{\mathcal{T}} : \text{SC}_{\mathcal{T}} \rightarrow \mathbb{N}$ that contains one copy for each occurrence of a concept term occurring in \mathcal{T} . We observe that for each complex concept \bar{C} causing some GCI not to be in normal form, a constant number of normalisation rule applications can be used to strictly decrease the cardinality of \bar{C} in $\mathcal{C}_{\mathcal{T}}$. Together with the fact that $\sum_{D \in \text{SC}_{\mathcal{T}}} \mathcal{C}_{\mathcal{T}}(D) \leq \|\mathcal{T}\|$, the claim then follows.

Define the set $\text{SC}_{\mathcal{T}}$ as the least set such that:

- If $\Box_s[C \sqsubseteq D] \in \mathcal{T}$, then $C, D \in \text{SC}_{\mathcal{T}}$;
- if $C \in \text{SC}_{\mathcal{T}}$, then for all subconcepts C' of C (written $C' \in \text{sub}(C)$) we have $C' \in \text{SC}_{\mathcal{T}}$.

Now for any TBox \mathcal{T}' (typically obtained from \mathcal{T} by applying zero or more normalisation rules), the multiset $\mathcal{C}_{\mathcal{T}'}: \text{SC}_{\mathcal{T}} \rightarrow \mathbb{N}$ is then as follows:

$$C \mapsto \sum_{\Box_s[D \sqsubseteq E] \in \mathcal{T}'} (c(C, D) + c(C, E))$$

where we define the concept-counting function $c: \text{SC}_{\mathcal{T}} \times \text{SC}_{\mathcal{T}} \rightarrow \mathbb{N}$ inductively:

$$c(C, D) := \begin{cases} 1 & \text{if } C = D, \\ c(C, E) & \text{if } C \neq D \text{ and } [D = \exists R.E \text{ or } D = \odot_s E] \\ c(C, E_1) + c(C, E_2) & \text{if } C \neq D \text{ and } D = E_1 \sqcap E_2 \\ 0 & \text{if } C \neq D \text{ and } D \in N_{\mathcal{C}} \cup \{\top, \perp\} \end{cases}$$

For example if $D = \top \sqcap \Diamond_s(\top \sqcap \exists R.(\top \sqcap \top))$, then $c(\top, D) = 4$ while $c(\top \sqcap \top, D) = 1$.

We next relate the overall cardinality (sum of number of occurrences) of $\mathcal{C}_{\mathcal{T}}$ to the size $\|\mathcal{T}\|$ on the basis that both can be represented as disjoint unions and we can therefore sum up the individual cardinalities.

Claim 1. We have $\sum_{C \in \text{SC}_{\mathcal{T}}} \mathcal{C}_{\mathcal{T}}(C) \leq \|\mathcal{T}\|$.

Proof of Claim. As $\|\mathcal{T}\| = \sum_{\tau \in \mathcal{T}} \|\tau\|$ and

$$\sum_{C \in \text{SC}_{\mathcal{T}}} \mathcal{C}_{\mathcal{T}}(C) = \sum_{C \in \text{SC}_{\mathcal{T}}} \sum_{\Box_s[D \sqsubseteq E] \in \mathcal{T}} (c(C, D) + c(C, E)) = \sum_{\Box_s[D \sqsubseteq E] \in \mathcal{T}} \sum_{C \in \text{SC}_{\mathcal{T}}} (c(C, D) + c(C, E))$$

it suffices to look at a single $\Box_s[D \sqsubseteq E] \in \mathcal{T}$ and show

$$\sum_{C \in \text{SC}_{\mathcal{T}}} (c(C, D) + c(C, E)) \leq \|\Box_s[D \sqsubseteq E]\|$$

which by

$$\sum_{C \in \text{SC}_{\mathcal{T}}} (c(C, D) + c(C, E)) = \sum_{D' \in \text{sub}(D)} c(D', D) + \sum_{E' \in \text{sub}(E)} c(E', E)$$

develops into

$$\sum_{D' \in \text{sub}(D)} c(D', D) + \sum_{E' \in \text{sub}(E)} c(E', E) \leq \|\Box_s[D \sqsubseteq E]\| = 1 + 1 + \|D\| + \|E\|$$

for which it suffices to show that for any concept $C \in \text{SC}_{\mathcal{T}}$, we have $\sum_{C' \in \text{sub}(C)} c(C', C) \leq \|C\|$, which we show by induction.

- The base case is clear, as $\sum_{C' \in \text{sub}(C)} c(C', C) = c(C, C) = 1 \leq 1 = \|C\|$ for any $C \in N_{\mathcal{C}} \cup \{\top, \perp\}$.
- $C = \exists R.D$:

$$\begin{aligned} & \sum_{C' \in \text{sub}(\exists R.D)} c(C', \exists R.D) \\ &= c(\exists R.D, \exists R.D) + \sum_{C' \in \text{sub}(D)} c(C', D) \\ &\stackrel{(IH)}{\leq} c(\exists R.D, \exists R.D) + \|D\| \\ &= 1 + \|D\| \\ &= \|\exists R.D\| \end{aligned}$$

- $C = \odot_s D$: Similar.
- $C = C_1 \sqcap C_2$:

$$\begin{aligned} & \sum_{C' \in \text{sub}(C_1 \sqcap C_2)} c(C', C_1 \sqcap C_2) \\ &= c(C_1 \sqcap C_2, C_1 \sqcap C_2) + \sum_{C' \in \text{sub}(C_1)} c(C', C_1) + \sum_{C' \in \text{sub}(C_2)} c(C', C_2) \\ &\stackrel{(IH)}{\leq} c(C_1 \sqcap C_2, C_1 \sqcap C_2) + \|C_1\| + \|C_2\| \\ &= 1 + \|C_1\| + \|C_2\| \\ &= \|C_1 \sqcap C_2\| \end{aligned}$$

This concludes the proof of Claim 1. \diamond

To prove an overall linear number of rule applications, we next show that for each complex concept \bar{C} whose occurrence in a GCI $\tau \in \mathcal{T}'$ causes τ not to be in normal form, there is a constant number of rule applications (that is, constant for all TBoxes) such that after rule application, the number of overall occurrences of \bar{C} has strictly decreased and additionally, any intermediate complex concepts introduced by the rule have been normalised in turn. To this end, we need to define some more notions. Let \mathcal{T} be a TBox and \mathcal{T}' be obtained by application of an arbitrary number of phase 2 normalisation rules. We say that a complex concept \bar{C} occurring in a GCI $\tau \in \mathcal{T}'$ is a *culprit for τ* iff τ is not in normal form because of \bar{C} ; a culprit \bar{C} is *top-level* for $\tau = \Box_s[D \sqsubseteq E]$ iff $\bar{C} = D$ or $\bar{C} = E$. We say that \mathcal{T}' is *faithful to \mathcal{T}* iff every culprit occurring in \mathcal{T}' already occurs in \mathcal{T} . In the proof below, we show that although normalisation rules sometimes introduce new culprits, those culprits will not lead to problems because each one can only cause a constant overhead.

Claim 2. *Let \mathcal{T} be a TBox and let \mathcal{T}' be obtained from \mathcal{T} by applying any number of rules from (30)–(36). Let \bar{C} be a top-level culprit for $\tau \in \mathcal{T}'$. Then there is a constant number of rule applications leading to a TBox \mathcal{T}'' that is faithful to \mathcal{T} and where $\mathcal{C}_{\mathcal{T}''}(\bar{C}) > \mathcal{C}_{\mathcal{T}'}(\bar{C})$.*

Proof of Claim. *We do a case distinction on the occurrence (left-hand vs. right-hand side) and form of \bar{C} . In every case, we will explicitly give the (constantly many) rules to apply to decrease the cardinality of \bar{C} . In most cases, it is easy to see that only one rule is needed, we show this exemplarily for one case and then concentrate on the two non-trivial cases.*

- $\bar{C} = \exists R.\bar{D}$ occurs on the right-hand side: Then we apply rule (30), removing one occurrence of \bar{C} . The resulting \mathcal{T}'' is faithful because the only newly introduced concept terms are $\exists R.A$ on a right-hand side and A on a left-hand side, and neither of these is a culprit.
- $\bar{C} = D_1 \sqcap D_2$ or $\bar{C} = \odot_s D$ and \bar{C} occurs on the right-hand side: Similar.
- $\bar{C} = \exists R.\bar{D}$ or $\bar{C} = \diamond_s D$ and \bar{C} occurs on the left-hand side: Exercise.
- $\bar{C} = D_1 \sqcap D_2$ occurs on the left-hand side: Denote $\tau = \Box_s[\bar{C} \sqsubseteq E] = \Box_s[D_1 \sqcap D_2 \sqsubseteq E]$. We apply rule (34) and obtain $\Box_s[D_1 \sqsubseteq A]$ and $\Box_s[A \sqcap D_2 \sqsubseteq E]$, thus removing once occurrence of \bar{C} . The latter rule contains the (new) culprit $A \sqcap D_2$ to which we can apply the same rule again (modulo commutativity) to obtain $\Box_s[D_2 \sqsubseteq A']$ and $\Box_s[A \sqcap A' \sqsubseteq E]$. The only remaining (potential) culprits are D_1 , D_2 , and E , whence the resulting \mathcal{T}'' is faithful to \mathcal{T} .
- $\bar{C} = \Box_u D$ occurs on the left-hand side: Denote $\tau = \Box_s[\Box_u D \sqsubseteq E]$. We apply rules (to underlined GCIs) as follows:

$$\begin{aligned}
& \left\{ \Box_s[\Box_u D \sqsubseteq E] \right\} \\
& \stackrel{(36)}{\rightsquigarrow} \left\{ \Box_u[D \sqsubseteq A_1], \Box_s[\diamond_{v_0} A_1 \sqcap \diamond_{v_1} A_1 \sqsubseteq E] \right\} \\
& \stackrel{(34)}{\rightsquigarrow} \left\{ \Box_u[D \sqsubseteq A_1], \Box_s[\diamond_{v_0} A_1 \sqsubseteq A_2], \Box_s[A_2 \sqcap \diamond_{v_1} A_1 \sqsubseteq E] \right\} \\
& \stackrel{(34)}{\rightsquigarrow} \left\{ \Box_u[D \sqsubseteq A_1], \Box_s[\diamond_{v_0} A_1 \sqsubseteq A_2], \Box_s[\diamond_{v_1} A_1 \sqsubseteq A_3], \Box_s[A_2 \sqcap A_3 \sqsubseteq E] \right\} \\
& \stackrel{(35)}{\rightsquigarrow} \left\{ \Box_u[D \sqsubseteq A_1], \Box_{v_0}[A_1 \sqsubseteq \Box_* A_4], \Box_s[A_4 \sqsubseteq A_2], \Box_s[\diamond_{v_1} A_1 \sqsubseteq A_3], \Box_s[A_2 \sqcap A_3 \sqsubseteq E] \right\} \\
& \stackrel{(35)}{\rightsquigarrow} \left\{ \Box_u[D \sqsubseteq A_1], \Box_{v_0}[A_1 \sqsubseteq \Box_* A_4], \Box_s[A_4 \sqsubseteq A_2], \Box_{v_1}[A_1 \sqsubseteq \Box_* A_5], \Box_s[A_5 \sqsubseteq A_3], \Box_s[A_2 \sqcap A_3 \sqsubseteq E] \right\}
\end{aligned}$$

It is easy to see that D and E are the only remaining (potential) culprits, and that one occurrence of $\bar{C} = \Box_u D$ has been removed. \diamond

Thus each culprit will be not only be removed eventually, but removing each single occurrence will take only a constant number of steps. Therefore, the number of rule applications is linear in $\sum_{C \in \text{SC}_{\mathcal{T}}} \mathcal{C}_{\mathcal{T}}(C)$. By Claim 1, the number of rule applications is linear in $\|\mathcal{T}\|$, thus linear in $\|\mathcal{K}\|$.

It remains to show that the overall increase in size is at most linear. We do this by showing that:

- for each single rule that is applied to nested concept terms and can potentially be applied recursively, the size increase caused by its application is constant;
- for each single rule that is applied only once to a statement (i.e., rule 25), the size increase is at most linear.

Together with the overall linear number of rule applications, it follows that the size of the resulting normalised TBox \mathcal{T}' is at most linear in the size of the original TBox \mathcal{T} .

Rule (24):

$$\begin{aligned}
& \|\Box_1 \sqcap \Box_2 \preceq s'\| + \|s' \sqcap \Box_3 \sqcap \dots \sqcap \Box_n \preceq s\| - \|\Box_1 \sqcap \dots \sqcap \Box_n \preceq s\| \\
& = 3 + n - (n + 1) \\
& = 2
\end{aligned}$$

Rule (25):

$$\begin{aligned} & \|\Box_{s_1}[\top \sqsubseteq A_1]\| + \dots + \|\Box_{s_n}[\top \sqsubseteq A_n]\| + \|\Box_*[A_1 \sqcap \dots \sqcap A_n \sqsubseteq \perp]\| - \|\mathbf{s}_1 \sqcap \dots \sqcap \mathbf{s}_n \preceq \mathbf{0}\| \\ &= 4n + (n + 3) - (n + 1) \\ &= 4n + 2 \end{aligned}$$

Rule (26):

$$\begin{aligned} & \|\Box_s[R_1 \circ R_2 \sqsubseteq R']\| + \|\Box_s[R' \circ R_3 \circ \dots \circ R_n \sqsubseteq R]\| - \|\Box_s[R_1 \circ \dots \circ R_n \sqsubseteq R]\| \\ &= 4 + (n + 1) - (n + 2) \\ &= 3 \end{aligned}$$

Rule (27):

$$\begin{aligned} & \|\Box_s[A(a)]\| + \|\Box_s[A \sqsubseteq \bar{C}]\| - (\|\Box_s[\bar{C}(a)]\|) \\ &= 3 + 3 + \|\bar{C}\| - (1 + \|\bar{C}\| + 1) \\ &4 \end{aligned}$$

Rule (28): Clear.

Rule (29): Clear.

Rule (30):

$$\begin{aligned} & \|\Box_s[B \sqsubseteq \exists R.A]\| + \|\Box_s[A \sqsubseteq \bar{C}]\| - \|\Box_s[B \sqsubseteq \exists R.\bar{C}]\| \\ &= 1 + 1 + \|B\| + 2 + 1 + 1 + 1 + \|\bar{C}\| - (1 + 1 + \|B\| + 1 + \|\bar{C}\|) \\ &= 4 \end{aligned}$$

Rule (31):

$$\begin{aligned} & \|\Box_s[B \sqsubseteq A]\| + \|\Box_s[A \sqsubseteq C]\| + \|\Box_s[A \sqsubseteq D]\| - \|\Box_s[B \sqsubseteq C \sqcap D]\| \\ &= (1 + 1 + \|B\| + 1) + (1 + 1 + 1 + \|C\|) + (1 + 1 + 1 + \|D\|) - (1 + 1 + \|B\| + 1 + \|C\| + \|D\|) \\ &= 6 \end{aligned}$$

Rule (32):

$$\begin{aligned} & \|\Box_s[C \sqsubseteq \odot_u A]\| + \|\Box_s[A \sqsubseteq \bar{D}]\| - \|\Box_s[C \sqsubseteq \odot_u \bar{D}]\| \\ &= 1 + 1 + \|C\| + 2 + 1 + 1 + 1 + \|\bar{D}\| - (1 + 1 + \|C\| + 1 + \|\bar{D}\|) \\ &= 4 \end{aligned}$$

Rule (33):

$$\begin{aligned} & \|\Box_s[\bar{C} \sqsubseteq A]\| + \|\Box_s[\exists R.A \sqsubseteq D]\| - \|\exists R.\bar{C} \sqsubseteq D\| \\ &= 1 + 1 + \|\bar{C}\| + 1 + 1 + 1 + 2 + \|D\| - (1 + 1 + \|\bar{C}\| + \|D\|) \\ &= 5 \end{aligned}$$

Rule (34):

$$\begin{aligned} & \|\Box_s[\bar{C} \sqsubseteq A]\| + \|\Box_s[A \sqcap D \sqsubseteq E]\| - \|\Box_s[\bar{C} \sqcap D \sqsubseteq E]\| \\ &= 1 + 1 + \|\bar{C}\| + 1 + 1 + 1 + 1 + 1 + \|D\| + \|E\| - (1 + 1 + 1 + \|\bar{C}\| + \|D\| + \|E\|) \\ &= 4 \end{aligned}$$

Rule (35):

$$\begin{aligned} & \|\Box_u[C \sqsubseteq \Box_* A]\| + \|\Box_s[A \sqsubseteq D]\| - \|\Box_s[\diamond_u C \sqsubseteq D]\| \\ &= 1 + 1 + \|C\| + 2 + 1 + 1 + 1 + \|D\| - (1 + 1 + 1 + \|C\| + \|D\|) \\ &= 4 \end{aligned}$$

Rule (36):

$$\begin{aligned} & \|\mathbf{v}_0 \preceq \mathbf{u}\| + \|\mathbf{v}_1 \preceq \mathbf{u}\| + \|\Box_u[C \sqsubseteq A]\| + \|\Box_s[\diamond_{v_0} A \sqcap \diamond_{v_1} A \sqsubseteq D]\| - \|\Box_s[\Box_u C \sqsubseteq D]\| \\ &= 2 + 2 + 1 + 1 + \|C\| + 1 + 1 + 1 + 2 + 2 + \|D\| - (1 + 1 + 1 + \|C\| + \|D\|) \\ &= 10 \end{aligned}$$

2. Assume that \mathcal{K}' is the result of exhaustively applying the normalisation rules (17)–(36) to \mathcal{K} . The proof is by contradiction. Assume that a GCI $\odot_s[C \sqsubseteq D]$ is not in normal form; we do a case distinction on the possible reasons for this where in each case it will turn out that at least one of the rules is applicable in contradiction to the presumption.

- $\odot_s \neq \square_s$: Then $\odot_s = \diamond_s$ and Rule (17) is applicable.
- D is of the form $\exists R.E$ with $E \notin \text{BC}_{\mathcal{K}} \cup \{\perp\}$: Then Rule (30) is applicable.
- D is of the form $E_1 \sqcap E_2$: Then Rule (31) is applicable.
- D is of the form $\odot_{s'}E$ with $E \notin \text{BC}_{\mathcal{K}} \cup \{\perp\}$: Then Rule (32) is applicable.
- $D = \top$ or $C = \perp$: Then Rule (28) or Rule (29) is applicable.
- C is of the form $\exists R.B$ with $B \notin \text{BC}_{\mathcal{K}}$: Then Rule (33) is applicable.
- C is of the form $B_1 \sqcap B_2$ with $\{B_1, B_2\} \not\subseteq \text{BC}_{\mathcal{K}}$: Then Rule (34) is applicable.
- C is of the form $\diamond_{s'}B$ with $B \notin \text{BC}_{\mathcal{K}}$: Then Rule (35) is applicable.
- C is of the form $\square_{s'}B$ with $B \notin \text{BC}_{\mathcal{K}}$: Then Rule (36) is applicable.

3. We show correctness of each rule. Correctness of the overall normalisation process follows by induction on the number of rule applications. We slightly adapt the notation to denote by $\mathcal{K}' = \langle \mathcal{S}, \mathcal{T}', \mathcal{A} \rangle$ the KB that results from application of a single rule and do a case distinction on the rules. In each case, assume $\mathfrak{D} = \langle \Delta, \Pi, \gamma, \sigma \rangle$ with $\mathfrak{D} \models \mathcal{K}$ and denote $\mathfrak{D}' = \langle \Delta', \Pi', \gamma', \sigma' \rangle$ in case the components differ from those of \mathfrak{D} . (In most cases, we only need to show $\mathfrak{D}' \models \mathcal{T}'$ and therefore do not mention the other KB components.) We start with phase 1 rules.

Rule (17) (a) Let $\mathfrak{D} \models \diamond_s[\mu]$. Then there is a $\pi \in \sigma(s)$ such that $\mathfrak{D}, \pi \models \mu$. Define \mathfrak{D}' from \mathfrak{D} by $\sigma'(u) := \{\pi\}$. It follows that $\mathfrak{D}' \models u \preceq s$ and $\mathfrak{D}' \models \square_u[\mu]$.

(b) Let $\mathfrak{D}' \models \mathcal{K}'$ and consider any $\pi \in \sigma'(u)$ (which exists due to standpoint-non-emptiness). Clearly $\mathfrak{D}' \models \mathcal{S}'$ shows $\pi \in \sigma'(u) \subseteq \sigma'(s)$, whence $\mathfrak{D}, \pi \models \mu$ witnesses that $\mathfrak{D}' \models \diamond_s[\mu]$.

Rule (18) (a) Let $\mathfrak{D} \models \square_s[\lambda_1 \wedge \dots \wedge \lambda_n]$. Then for every $\pi \in \sigma(s)$ and $1 \leq i \leq n$, we have $\mathfrak{D}, \pi \models \lambda_i$. Thus also $\mathfrak{D} \models \square_s[\lambda_i]$ for all $1 \leq i \leq n$.

(b) Let $\mathfrak{D}' \models \square_s[\lambda_i]$ for all $1 \leq i \leq n$. Then for all $\pi \in \sigma'(s)$ and all $1 \leq i \leq n$, we have $\mathfrak{D}, \pi \models \lambda_i$. Therefore, for all $\pi \in \sigma'(s)$ we have $\mathfrak{D}, \pi \models \lambda_1 \wedge \dots \wedge \lambda_n$ and thus $\mathfrak{D} \models \square_s[\lambda_1 \wedge \dots \wedge \lambda_n]$.

Rule (19) (a) Let $\mathfrak{D} \models \square_s[\neg(C \sqsubseteq D)]$, that is, for all $\pi \in \sigma(s)$ assume $\mathfrak{D}, \pi \not\models C \sqsubseteq D$. Then, in every $\pi \in \sigma(s)$ there is some $\delta_\pi \in \Delta$ such that $\delta_\pi \in C^{\gamma(\pi)} \setminus D^{\gamma(\pi)}$. Define \mathfrak{D}' from \mathfrak{D} as follows: For every $\pi \in \sigma(s)$ set $A^{\gamma'(\pi)} := \{\delta_\pi\}$ and $R^{\gamma'(\pi)} := \{\langle \varepsilon, \delta_\pi \rangle \mid \varepsilon \in \Delta\}$. Now for every $\pi \in \sigma'(s) = \sigma(s)$: We have $\delta_\pi \in A^{\gamma'(\pi)}$ and $\delta_\pi \in C^{\gamma'(\pi)}$ whence $\mathfrak{D}' \models \square_s[A \sqsubseteq C]$. We have $\delta_\pi \notin D^{\gamma'(\pi)}$ whence $\mathfrak{D}' \models \square_s[A \sqcap D \sqsubseteq \perp]$. We have $\varepsilon \in \Delta$ implies $\langle \varepsilon, \delta_\pi \rangle \in R^{\gamma'(\pi)}$ and $\delta_\pi \in A^{\gamma'(\pi)}$ whence $\mathfrak{D}' \models \square_s[\top \sqsubseteq \exists R'.A]$.

(b) Let $\mathfrak{D}' \models \square_s[A \sqsubseteq C] \wedge \square_s[A \sqcap D \sqsubseteq \perp] \wedge \square_s[\top \sqsubseteq \exists R'.A]$ and let $\pi' \in \sigma'(s)$ be arbitrary. Then by $\mathfrak{D}' \models \top \sqsubseteq \exists R'.A$ and $\Delta' \neq \emptyset$ we get $A^{\gamma'(\pi')} \neq \emptyset$. Thus assume $\delta' \in A^{\gamma'(\pi')}$. By $\mathfrak{D}' \models \square_s[A \sqsubseteq C]$ we get $\delta' \in C^{\gamma'(\pi')}$. By $\mathfrak{D}' \models \square_s[A \sqcap D \sqsubseteq \perp]$ we get $\delta' \notin D^{\gamma'(\pi')}$. Therefore, for every $\pi' \in \sigma'(s)$, we find a $\delta' \in C^{\gamma'(\pi')} \setminus D^{\gamma'(\pi')}$; we conclude that $\mathfrak{D}' \models \square_s[\neg(C \sqsubseteq D)]$.

Rule (20) (a) Let $\mathfrak{D} \models \square_s[\neg C(a)]$, that is, for every $\pi \in \sigma(s)$ assume $a^{\mathfrak{D}} \notin C^{\gamma(\pi)}$. Define \mathfrak{D}' from \mathfrak{D} as follows: For every $\pi \in \sigma(s)$, set $A^{\gamma'(\pi)} := \{a^{\mathfrak{D}}\}$. Then clearly $\mathfrak{D}' \models \square_s[A(a)]$, and due to the above also $\mathfrak{D}' \models \square_s[A \sqcap C \sqsubseteq \perp]$.

(b) Let $\mathfrak{D}' \models \square_s[A \sqcap C \sqsubseteq \perp] \wedge \square_s[A(a)]$. Then in every $\pi' \in \sigma'(s)$, we have $a^{\mathfrak{D}'} \in A^{\gamma'(\pi')}$ and $A^{\gamma'(\pi')} \cap C^{\gamma'(\pi')} = \emptyset$, that is, $a^{\mathfrak{D}'} \notin C^{\gamma'(\pi')}$.

Rule (21) (a) Let $\mathfrak{D} \models \square_s[\neg R(a, b)]$, that is, for every $\pi \in \sigma(s)$ assume $\mathfrak{D}, \pi \not\models R(a, b)$ (i.e. $\langle a^{\mathfrak{D}}, b^{\mathfrak{D}} \rangle \notin R^{\gamma(\pi)}$). Define \mathfrak{D}' from \mathfrak{D} as follows: For every $\pi \in \sigma(s)$, set $A_a^{\gamma'(\pi)} := \{a^{\mathfrak{D}}\}$ and $A_b^{\gamma'(\pi)} := \{b^{\mathfrak{D}}\}$. We then have $\mathfrak{D}' \models \square_s[A_a(a)]$, $\mathfrak{D}' \models \square_s[A_b(b)]$, and $a^{\mathfrak{D}} \notin (\exists R.A_b)^{\gamma'(\pi)}$.

(b) Let $\mathfrak{D}' \models \square_s[A_b(b)] \wedge \square_s[A_a \sqcap \exists R.A_b \sqsubseteq \perp] \wedge \square_s[A_a(a)]$. Then for every $\pi' \in \sigma'(s)$: $b^{\mathfrak{D}'} \in A_b^{\gamma'(\pi')}$, $a^{\mathfrak{D}'} \in A_a^{\gamma'(\pi')}$, and for all $\delta \in A_a^{\gamma'(\pi')}$ we find $\delta \notin (\exists R.A_b)^{\gamma'(\pi')}$, thus in particular $a^{\mathfrak{D}'} \notin (\exists R.A_b)^{\gamma'(\pi')}$ and $\langle a^{\mathfrak{D}'}, b^{\mathfrak{D}'} \rangle \notin R^{\gamma'(\pi')}$.

Rule (22) (a) Let $\mathfrak{D} \models \square_s[\neg(R_1 \circ \dots \circ R_n \sqsubseteq R)]$, that is, in every $\pi \in \sigma(s)$ there exist $\delta_0, \delta_1, \dots, \delta_n \in \Delta$ such that for all $1 \leq i \leq n$, $\langle \delta_{i-1}, \delta_i \rangle \in R_i^{\gamma(\pi)}$, but $\langle \delta_0, \delta_n \rangle \notin R^{\gamma(\pi)}$. Define \mathfrak{D}' from \mathfrak{D} as follows: For every $\pi \in \sigma(s)$, set $A_a^{\gamma'(\pi)} := \{\delta_0\}$, $A_b^{\gamma'(\pi)} := \{\delta_n\}$, and $R^{\gamma'(\pi)} := \{\langle \varepsilon, \delta_0 \rangle \mid \varepsilon \in \Delta\}$. It follows by construction that $\mathfrak{D}' \models \square_s[\top \sqsubseteq \exists R'.A_a]$, and $a^{\mathfrak{D}'} \notin (\exists R.A_b)^{\gamma'(\pi)}$ whence $\mathfrak{D} \models \square_s[A_a \sqcap \exists R.A_b \sqsubseteq \perp]$. We can also show for each $\pi \in \sigma'(s) = \sigma(s)$ and for all $1 \leq i \leq n$ (by induction on i), that we have $\delta_{i-1} \in (\exists R_i \dots \exists R_n.A_b)^{\gamma'(\pi)}$, whence in particular $\mathfrak{D}' \models \square_s[A_a \sqsubseteq \exists R_1 \dots \exists R_n.A_b]$.

(b) Let $\mathfrak{D}' \models \square_s[\top \sqsubseteq \exists R'.A_a] \wedge \square_s[A_a \sqcap \exists R.A_b \sqsubseteq \perp] \wedge \square_s[A_a \sqsubseteq \exists R_1 \dots \exists R_n.A_b]$ and consider any $\pi \in \sigma'(s)$. From $\Delta' \neq \emptyset$ and $\mathfrak{D}' \models \square_s[\top \sqsubseteq \exists R'.A_a]$ we get $A_a^{\gamma'(\pi)} \neq \emptyset$, say $\delta_0 \in A_a^{\gamma'(\pi)}$. Therefore, from $\mathfrak{D}' \models \square_s[A_a \sqsubseteq \exists R_1 \dots \exists R_n.A_b]$ we get that there exist $\delta_1, \dots, \delta_n$ such that $\langle \delta_0, \delta_1 \rangle \in R_1^{\gamma'(\pi)}$, ..., and $\langle \delta_{n-1}, \delta_n \rangle \in R_n^{\gamma'(\pi)}$ where $\delta_n \in A_b^{\gamma'(\pi)}$. By

$\mathfrak{D}' \models \Box_s[A_a \cap \exists R.A_b \sqsubseteq \perp]$ we get that $\langle \delta_0, \delta_n \rangle \notin R^{\gamma'(\pi)}$. Therefore, for any $\pi \in \sigma'(s)$ we find $\langle \delta_0, \delta_n \rangle \in (R_1 \circ \dots \circ R_n)^{\gamma'(\pi)} \setminus R^{\gamma'(\pi)}$.

Rule (23): (a) Let $\mathfrak{D} \models \neg(s_1 \cap \dots \cap s_n \preceq s)$. Then there is a $\pi^* \in \sigma(s_1) \cap \dots \cap \sigma(s_n) \setminus \sigma(s)$. Construct \mathfrak{D}' from \mathfrak{D} by setting $\sigma'(u) := \{\pi^*\}$. It is clear that $\sigma'(u) \subseteq \sigma'(s_i)$ for all $1 \leq i \leq n$; furthermore $\pi^* \notin \sigma(s)$ shows that $\mathfrak{D}' \models u \cap s \preceq \mathbf{0}$. (b) Let $\mathfrak{D}' \models u \preceq s_1 \wedge \dots \wedge u \preceq s_n \wedge u \cap s \preceq \mathbf{0}$. By non-emptiness of $\sigma'(u)$, there is some $\pi_* \in \sigma'(u)$. Clearly $\pi_* \in \sigma'(s_i)$ for all $1 \leq i \leq n$, whence $\pi_* \in \sigma'(s_1) \cap \dots \cap \sigma'(s_n)$. By $\mathfrak{D}' \models u \cap s \preceq \mathbf{0}$ we furthermore get $\pi_* \notin \sigma'(s)$.

We conclude with correctness of phase 2 rules.

Rule (24): (a) Straightforward, we interpret $\sigma'(s') := (\sigma(s_1) \cap \sigma(s_2)) \cup \{\pi_s\}$ for some arbitrary $\pi_s \in \sigma(s)$.

(b) Clear.

Rule (25): (a) Let $\mathfrak{D} \models s_1 \cap \dots \cap s_n \preceq \mathbf{0}$. It is clear that in \mathfrak{D}' we must define, for all $1 \leq i \leq n$ and $\pi \in \sigma(s_i)$, $A_i^{\gamma'(\pi)} := \Delta$, to satisfy the axioms of the form $\Box_{s_i}[\top \sqsubseteq A_i]$. Assume for the sake of obtaining a contradiction that $\mathfrak{D}' \not\models \Box_*[A_1 \cap \dots \cap A_n \sqsubseteq \perp]$. Then there is a $\pi \in \Pi$ and some $\delta \in \Delta' = \Delta$ such that $\delta \in A_i^{\gamma'(\pi)}$ for all $1 \leq i \leq n$. But by construction this means that $\pi \in \sigma(s_1) \cap \dots \cap \sigma(s_n)$, a contradiction to the presumption. Thus $\mathfrak{D}' \models \Box_*[A_1 \cap \dots \cap A_n \sqsubseteq \perp]$. (b) Let $\mathfrak{D}' \models \Box_{s_1}[\top \sqsubseteq A_1] \wedge \dots \wedge \Box_{s_n}[\top \sqsubseteq A_n] \wedge \Box_*[A_1 \cap \dots \cap A_n \sqsubseteq \perp]$. The proof works as in direction (a).

Rule (26): (a) Straightforward, we interpret $R^{\gamma'(\pi)} := R_1^{\gamma'(\pi)} \circ R_2^{\gamma'(\pi)}$.

(b) Clear.

Rule (27): (a) Let $\mathfrak{D} \models \Box_s[\bar{C}(a)]$. In \mathfrak{D}' we define $A^{\gamma'(\pi)} := \bar{C}^{\gamma'(\pi)}$ and $\mathfrak{D}' \models \Box_s[A(a)] \wedge \Box_s[A \sqsubseteq \bar{C}]$ follows.

(b) Clear.

Rules (28) and (29): Clear, as any DL interpretation satisfies $C \sqsubseteq \top$ and $\perp \sqsubseteq D$.

Rule (30): (a) Let $\mathfrak{D} \models \Box_s[B \sqsubseteq \exists R.\bar{C}]$. Define \mathfrak{D}' from \mathfrak{D} as follows: For any $\pi \in \sigma(s)$, set $A^{\gamma'(\pi)} := \bar{C}^{\gamma'(\pi)}$. It follows by definition that $\mathfrak{D}' \models \Box_s[B \sqsubseteq \exists R.A]$ and $\mathfrak{D}' \models \Box_s[A \sqsubseteq \bar{C}]$.

(b) Let $\mathfrak{D}' \models \mathcal{T}'$ and consider any $\pi \in \sigma'(s)$ and $\delta \in B^{\gamma'(\pi)}$. By $\mathfrak{D}' \models \Box_s[B \sqsubseteq \exists R.A]$ we get $\delta \in (\exists R.A)^{\gamma'(\pi)}$, whence there is an $\varepsilon \in \Delta'$ with $\langle \delta, \varepsilon \rangle \in r^{\gamma'(\pi)}$ and $\varepsilon \in A^{\gamma'(\pi)}$. By $\mathfrak{D}' \models \Box_s[A \sqsubseteq \bar{C}]$, we get $\varepsilon \in \bar{C}^{\gamma'(\pi)}$ and ultimately $\delta \in (\exists R.\bar{C})^{\gamma'(\pi)}$.

Rule (31): (a) Let $\mathfrak{D} \models \Box_s[B \sqsubseteq C \cap D]$ and define \mathfrak{D}' from \mathfrak{D} by setting, for each $\pi \in \sigma(s)$, $A^{\gamma'(\pi)} := B^{\gamma'(\pi)}$. Then by definition we get $\mathfrak{D}' \models \Box_s[B \sqsubseteq A]$, as well as $\mathfrak{D}' \models \Box_s[A \sqsubseteq C]$ and $\mathfrak{D}' \models \Box_s[A \sqsubseteq D]$.

(b) Let $\mathfrak{D}' \models \mathcal{T}'$ and consider any $\pi \in \sigma'(s)$ and $\delta \in B^{\gamma'(\pi)}$. By $\mathcal{T}' \models \Box_s[B \sqsubseteq A]$ we get $\delta \in A^{\gamma'(\pi)}$. By $\mathcal{T}' \models \Box_s[A \sqsubseteq C]$ and $\mathcal{T}' \models \Box_s[A \sqsubseteq D]$ we get $\delta \in C^{\gamma'(\pi)}$ and $\delta \in D^{\gamma'(\pi)}$, respectively. In combination, $\delta \in (C \cap D)^{\gamma'(\pi)}$.

Rule (32): (a) Let $\mathfrak{D} \models \Box_s[C \sqsubseteq \odot_u \bar{D}]$ and define \mathfrak{D}' from \mathfrak{D} by setting, for each $\pi \in \sigma(s)$, $A^{\gamma'(\pi)} := \bar{D}^{\gamma'(\pi)}$. It follows directly that $\mathfrak{D}' \models \Box_s[C \sqsubseteq \odot_u A]$ and $\mathfrak{D}' \models \Box_s[A \sqsubseteq \bar{D}]$.

(b) Let $\mathfrak{D}' \models \mathcal{T}'$ and consider any $\pi \in \sigma'(s)$ and $\delta \in C^{\gamma'(\pi)}$. From $\mathfrak{D}' \models \Box_s[C \sqsubseteq \odot_u A]$ we get $\delta \in (\odot_u A)^{\gamma'(\pi)}$. Thus, for some (all) $\pi' \in \sigma'(u)$, we get $\delta \in A^{\gamma'(\pi')}$; and in turn, by $\mathfrak{D}' \models \Box_s[A \sqsubseteq \bar{D}]$, for some (all) $\pi' \in \sigma'(u)$, we get $\delta \in \bar{D}^{\gamma'(\pi')}$.

Rule (33): (a) Let $\mathfrak{D} \models \Box_s[\exists R.\bar{C} \sqsubseteq D]$. Define \mathfrak{D}' from \mathfrak{D} as follows: For any $\pi \in \sigma(s)$, set $A^{\gamma'(\pi)} := \bar{C}^{\gamma'(\pi)}$. It follows from this definition that $\mathfrak{D}' \models \Box_s[\bar{C} \sqsubseteq A]$ and $\mathfrak{D}' \models \Box_s[\exists R.A \sqsubseteq D]$.

(b) Let $\mathfrak{D}' \models \mathcal{T}'$ and consider $\pi \in \sigma'(s)$ and any $\delta \in \exists R.\bar{C}^{\gamma'(\pi)}$. There is thus a $\varepsilon \in \Delta$ with $\langle \delta, \varepsilon \rangle \in r^{\gamma'(\pi)}$ and $\varepsilon \in \bar{C}^{\gamma'(\pi)}$. By $\mathfrak{D}' \models \Box_s[\bar{C} \sqsubseteq A]$ we get $\varepsilon \in A^{\gamma'(\pi)}$, and in turn $\delta \in (\exists R.A)^{\gamma'(\pi)}$. By $\mathfrak{D}' \models \Box_s[\exists R.A \sqsubseteq D]$, we get $\delta \in D^{\gamma'(\pi)}$.

Rule (34): (a) Let $\mathfrak{D} \models \Box_s[\bar{C} \cap D \sqsubseteq E]$. Define \mathfrak{D}' from \mathfrak{D} as follows: For any $\pi \in \sigma(s)$, set $A^{\gamma'(\pi)} := \bar{C}^{\gamma'(\pi)}$. It follows from this definition that $\mathfrak{D}' \models \Box_s[\bar{C} \sqsubseteq A]$ and $\mathfrak{D}' \models \Box_s[A \cap D \sqsubseteq E]$.

(b) Let $\mathfrak{D}' \models \mathcal{T}'$ and consider $\pi \in \sigma'(s)$ and $\delta \in \bar{C}^{\gamma'(\pi)} \cap D^{\gamma'(\pi)}$. By $\mathfrak{D}' \models \Box_s[\bar{C} \sqsubseteq A]$ we get $\delta \in A^{\gamma'(\pi)}$, and by $\mathfrak{D}' \models \Box_s[A \cap D \sqsubseteq E]$ and $\delta \in D^{\gamma'(\pi)}$ we get $\delta \in E^{\gamma'(\pi)}$.

Rule (35): (a) Let $\mathfrak{D} \models \Box_s[\diamond_u C \sqsubseteq D]$ and define \mathfrak{D}' from \mathfrak{D} by setting $A^{\gamma'(\pi)} := (\diamond_u C)^{\gamma'(\pi)}$ for every $\pi \in \Pi$.

- $\mathfrak{D}' \models \Box_u[C \sqsubseteq \Box_* A]$: Let $\pi \in \sigma'(u) = \sigma(u)$ and $\delta \in C^{\gamma'(\pi)} = C^{\gamma'(\pi)}$. Then by definition, we have $\delta \in A^{\gamma'(\pi)}$ for all $\pi \in \Pi$, whence $\delta \in (\Box_* A)^{\gamma'(\pi)}$.

- $\mathfrak{D}' \models \Box_s[A \sqsubseteq D]$: Let $\pi \in \sigma'(s) = \sigma(s)$ and $\delta \in A^{\gamma'(\pi)}$. Then by definition, $\delta \in (\diamond_u C)^{\gamma'(\pi)}$ and by the presumption that $\mathfrak{D} \models \Box_s[\diamond_u C \sqsubseteq D]$ we get $\delta \in D^{\gamma'(\pi)} = D^{\gamma'(\pi)}$.

(b) Let $\mathfrak{D}' \models \mathcal{T}'$ and consider $\pi \in \sigma'(s)$ and $\delta \in (\diamond_u C)^{\gamma'(\pi)}$. Then there is a $\pi' \in \sigma'(u)$ such that $\delta \in C^{\gamma'(\pi')}$. Thus $\delta \in (\Box_* A)^{\gamma'(\pi')}$, that is, $\delta \in \bigcap_{\pi'' \in \Pi} A^{\gamma'(\pi')}$. Thus in particular for $\pi \in \sigma'(s)$, we get $\delta \in A^{\gamma'(\pi)}$ and by $\mathfrak{D}' \models \Box_s[A \sqsubseteq D]$ we obtain $\delta \in D^{\gamma'(\pi)}$.

Rule (36): (a) Let $\mathfrak{D} \models \Box_s[\Box_u C \sqsubseteq D]$. We define $\mathfrak{D}' = \langle \Delta', \Pi', \sigma', \gamma' \rangle$ as follows:

- Δ' consists of two copies δ'_0, δ'_1 of every function $\delta': \sigma(u) \times \Delta \rightarrow \Delta$.

- Π' consists of all functions $\pi': \sigma(u) \times \Delta \rightarrow \Pi$ plus two extra, distinct copies of the particular function $\pi'_{\text{diag}} := \{(\pi, \delta) \mapsto \pi\}$, denoted π'_{v_0} and π'_{v_1} .
- $\sigma'(v_b) := \{\pi'_{v_b}\}$ for each $b \in \{0, 1\}$ and, for all other s ,

$$\sigma'(s) := \left\{ \pi' \in \Pi' \mid \bigcup_{\substack{\pi \in \sigma(u), \\ \delta \in \Delta}} \{\pi'(\pi, \delta)\} \subseteq \sigma(s) \right\}$$

For every $\pi' \in \Pi'$, the DL interpretation $\gamma'(\pi')$ over Δ' is such that

$$a^{\gamma'(\pi')} := \{(\pi, \delta) \mapsto \delta_a\}_0 \quad \text{for } a \in N_I$$

where $\delta_a \in \Delta$ denotes the domain element for which $\delta_a = a^{\gamma(\pi)}$ for all $\pi \in \Pi$.

Further, for $\pi' \in \Pi' \setminus \{\pi'_{v_0}, \pi'_{v_1}\}$, the DL interpretation $\gamma'(\pi')$ over Δ' is such that

$$\begin{aligned} A^{\gamma'(\pi')} &:= \left\{ \delta'_b \in \Delta' \mid \delta'_b(\pi, \delta) \in A^{\gamma(\pi'(\pi, \delta))} \text{ for all } \pi \in \sigma(u), \delta \in \Delta \right\} && \text{for } A \in N_C \\ R^{\gamma'(\pi')} &:= \left\{ \langle \delta'_b, \varepsilon'_b \rangle \in \Delta' \times \Delta' \mid \langle \delta'_b(\pi, \delta), \varepsilon'_b(\pi, \delta) \rangle \in R^{\gamma(\pi'(\pi, \delta))} \text{ for all } \pi \in \sigma(u), \delta \in \Delta \right\} && \text{for } R \in N_R \end{aligned}$$

and $\gamma'(\pi'_{v_b})$ assigns as follows:

$$\begin{aligned} A^{\gamma'(\pi'_{v_b})} &:= \left(\{\delta'_b \in \Delta'\} \cap A^{\gamma'(\pi'_{\text{diag}})} \right) \cup \left\{ \delta'_{1-b} \in \Delta' \mid \bigcup_{\substack{\pi \in \sigma(u), \\ \delta \in \Delta}} \{\delta'_{1-b}(\pi, \delta)\} \subseteq \bigcap_{\pi \in \sigma(u)} A^{\gamma(\pi)} \right\} && \text{for } A \in N_C \\ R^{\gamma'(\pi'_{v_b})} &:= \left(\{\langle \delta'_b, \varepsilon'_b \rangle \in \Delta' \times \Delta'\} \cap R^{\gamma'(\pi'_{\text{diag}})} \right) \\ &\cup \left\{ \langle \delta'_{1-b}, \varepsilon'_{1-b} \rangle \in \Delta' \times \Delta' \mid \bigcup_{\substack{\pi \in \sigma(u), \\ \delta \in \Delta}} \{\langle \delta'_{1-b}(\pi, \delta), \varepsilon'_{1-b}(\pi, \delta) \rangle\} \subseteq \bigcap_{\pi \in \sigma(u)} R^{\gamma(\pi)} \right\} \\ &\cup \left\{ \langle \delta'_{1-b}, \varepsilon'_b \rangle \in \Delta' \times \Delta' \mid \langle \zeta'_b, \varepsilon'_b \rangle \in R^{\gamma'(\pi'_{\text{diag}})} \text{ for some } \zeta'_b \approx \delta'_{1-b} \right\} && \text{for } R \in N_R \end{aligned}$$

where $\zeta'_b \approx \delta'_{1-b}$ holds iff for every $\pi \in \sigma(u)$ we have $\bigcup_{\delta \in \Delta} \{\zeta'_b(\pi, \delta)\} = \bigcup_{\substack{\pi_* \in \sigma(u), \\ \delta_* \in \Delta}} \{\delta'_{1-b}(\pi_*, \delta_*)\}$. Finally for the fresh concept name A introduced by the rule, define $A^{\gamma'(\pi')} := C^{\gamma'(\pi')}$ for all $\pi' \in \sigma'(u)$.

We now show that $\mathcal{D}' \models \mathcal{K}'$. For this, we start out with some useful observations.

Claim 3. For every concept E , $\delta' \in \Delta'$, and $\pi' \in \Pi' \setminus \{\pi'_{v_0}, \pi'_{v_1}\}$:

$$\delta' \in E^{\gamma'(\pi')} \iff \left(\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in E^{\gamma(\pi'(\pi, \delta))} \right)$$

Proof of Claim. We use structural induction on E .

- $E = A \in \text{BC}_{\mathcal{K}}$: By definition.
- $E = E_1 \sqcap E_2$:

$$\begin{aligned} &\delta' \in (E_1 \sqcap E_2)^{\gamma'(\pi')} \\ \iff &\delta' \in E_1^{\gamma'(\pi')} \cap E_2^{\gamma'(\pi')} \\ \iff &\delta' \in E_1^{\gamma'(\pi')} \text{ and } \delta' \in E_2^{\gamma'(\pi')} \\ \stackrel{\text{(IH)}}{\iff} &\left[\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in E_1^{\gamma(\pi'(\pi, \delta))} \right] \text{ and } \left[\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in E_2^{\gamma(\pi'(\pi, \delta))} \right] \\ \iff &\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \left[\delta'(\pi, \delta) \in E_1^{\gamma(\pi'(\pi, \delta))} \text{ and } \delta'(\pi, \delta) \in E_2^{\gamma(\pi'(\pi, \delta))} \right] \\ \iff &\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in E_1^{\gamma(\pi'(\pi, \delta))} \cap E_2^{\gamma(\pi'(\pi, \delta))} \\ \iff &\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in (E_1 \sqcap E_2)^{\gamma(\pi'(\pi, \delta))} \end{aligned}$$

- $E = \exists R.B$:

$$\begin{aligned}
& \delta' \in \exists R.B^{\gamma'(\pi')} \\
& \iff \exists \varepsilon' \in \Delta' : \left[\langle \delta', \varepsilon' \rangle \in R^{\gamma'(\pi')} \text{ and } \varepsilon' \in B^{\gamma'(\pi')} \right] \\
& \stackrel{(IH)}{\iff} \exists \varepsilon' \in \Delta' : \left[\langle \delta', \varepsilon' \rangle \in R^{\gamma'(\pi')} \text{ and } \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \varepsilon'(\pi, \delta) \in B^{\gamma(\pi'(\pi, \delta))} \right] \\
& \iff \exists \varepsilon' \in \Delta' : \left[\left[\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \langle \delta'(\pi, \delta), \varepsilon'(\pi, \delta) \rangle \in R^{\gamma(\pi'(\pi, \delta))} \right] \text{ and } \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \varepsilon'(\pi, \delta) \in B^{\gamma(\pi'(\pi, \delta))} \right] \\
& \iff \exists \varepsilon' \in \Delta' : \left[\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \left[\langle \delta'(\pi, \delta), \varepsilon'(\pi, \delta) \rangle \in R^{\gamma(\pi'(\pi, \delta))} \text{ and } \varepsilon'(\pi, \delta) \in B^{\gamma(\pi'(\pi, \delta))} \right] \right] \\
& \stackrel{\dagger}{\iff} \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \exists \varepsilon \in \Delta : \left[\langle \delta'(\pi, \delta), \varepsilon \rangle \in R^{\gamma(\pi'(\pi, \delta))} \text{ and } \varepsilon \in B^{\gamma(\pi'(\pi, \delta))} \right] \\
& \stackrel{\ddagger}{\iff} \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \exists \varepsilon \in \Delta : \left[\langle \delta'(\pi, \delta), \varepsilon \rangle \in R^{\gamma(\pi'(\pi, \delta))} \text{ and } \varepsilon \in B^{\gamma(\pi'(\pi, \delta))} \right] \\
& \iff \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in (\exists R.B)^{\gamma(\pi'(\pi, \delta))}
\end{aligned}$$

Note that for the “ \Leftarrow ” direction, there is always an $\varepsilon' \in \Delta'$ such that for all $\pi \in \sigma(\mathbf{u})$ and $\delta \in \Delta$ we find $\varepsilon'(\pi, \delta) = \varepsilon$ for the appropriate $\varepsilon \in \Delta$ (step \dagger), and thus this ε' is in this sense independent of concrete $\pi \in \sigma(\mathbf{u})$ and $\delta \in \Delta$ (step \ddagger).

- $E = \diamond_{s'} B$:

$$\begin{aligned}
& \delta' \in (\diamond_{s'} B)^{\gamma'(\pi')} \\
& \iff \exists \pi'' \in \sigma'(s') : \delta' \in B^{\gamma'(\pi'')} \\
& \stackrel{(IH)}{\iff} \exists \pi'' \in \sigma'(s') : \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in B^{\gamma(\pi''(\pi, \delta))} \\
& \iff \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \exists \pi'' \in \sigma'(s') : \delta'(\pi, \delta) \in B^{\gamma(\pi''(\pi, \delta))} \\
& \iff \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \exists \pi_{s'} \in \sigma(s') : \delta'(\pi, \delta) \in B^{\gamma(\pi_{s'})} \\
& \iff \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in (\diamond_{s'} B)^{\gamma(\pi'(\pi, \delta))}
\end{aligned}$$

Again, for “ \Leftarrow ”, we find a $\pi'' \in \sigma'(s')$ for which always $\pi''(\pi, \delta) = \pi_{s'}$ as desired.

- $E = \square_{s'} B$:

$$\begin{aligned}
& \delta' \in (\square_{s'} B)^{\gamma'(\pi')} \\
& \iff \forall \pi'' \in \sigma'(s') : \delta' \in B^{\gamma'(\pi'')} \\
& \stackrel{(IH)}{\iff} \forall \pi'' \in \sigma'(s') : \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in B^{\gamma(\pi''(\pi, \delta))} \\
& \iff \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi'' \in \sigma'(s') : \delta'(\pi, \delta) \in B^{\gamma(\pi''(\pi, \delta))} \\
& \stackrel{\dagger}{\iff} \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_{s'} \in \sigma(s') : \delta'(\pi, \delta) \in B^{\gamma(\pi_{s'})} \\
& \iff \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in (\square_{s'} B)^{\gamma(\pi'(\pi, \delta))}
\end{aligned}$$

The equivalence marked \dagger holds because for every $\pi_{s'} \in \sigma(s')$ we can find a $\pi'' \in \sigma'(s')$ such that $\pi''(\pi, \delta) = \pi_{s'}$ for all $\pi \in \sigma(\mathbf{u})$ and $\delta \in \Delta$ (e.g. the constant function $\{(\pi, \delta) \mapsto \pi_{s'}\}$); conversely, for every $\pi'' \in \sigma'(s')$, $\pi \in \sigma(\mathbf{u})$, and $\delta \in \Delta$, we have $\pi''(\pi, \delta) \in \sigma(s')$ by definition. \diamond

The claim above does not consider the newly introduced precisifications. They are however covered by the next claim.

Claim 4. For every concept E , $\delta' \in \Delta'$, and $\pi' \in \{\pi'_{v_0}, \pi'_{v_1}\}$:

- $\delta' \in E^{\gamma'(\pi')} \implies \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in E^{\gamma(\pi'(\pi, \delta))}$
- $\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \delta'(\pi, \delta) \in E^{\gamma(\pi'(\pi_*, \delta))} \implies \delta' \in E^{\gamma'(\pi')}$

Proof of Claim. We use structural induction on E .

- $E = A \in \text{BC}_K$: Assume $\pi' = \pi'_{v_0}$ (the remaining case is symmetric). For direction (3a) we have:

$$\begin{aligned}
& \delta'_b \in A^{\gamma'(\pi'_{v_0})} \text{ for some } b \in \{0, 1\} \\
\iff & \delta'_0 \in A^{\gamma'(\pi'_{\text{diag}})} \text{ or } \bigcup_{\substack{\pi \in \sigma(\mathbf{u}), \\ \delta \in \Delta}} \{\delta'_1(\pi, \delta)\} \subseteq \bigcap_{\pi \in \sigma(\mathbf{u})} A^{\gamma(\pi)} \\
\iff & \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'_0(\pi, \delta) \in A^{\gamma(\pi'_{\text{diag}}(\pi, \delta))} \text{ or } \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \delta'_1(\pi, \delta) \in A^{\gamma(\pi_*)} \\
\iff & \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'_0(\pi, \delta) \in A^{\gamma(\pi)} \text{ or } \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \delta'_1(\pi, \delta) \in A^{\gamma(\pi_*)} \\
\implies & \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'_0(\pi, \delta) \in A^{\gamma(\pi)} \text{ or } \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'_1(\pi, \delta) \in A^{\gamma(\pi)} \\
\iff & \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'_b(\pi, \delta) \in A^{\gamma(\pi)} \text{ for some } b \in \{0, 1\} \\
\iff & \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'_b(\pi, \delta) \in A^{\gamma(\pi'_{\text{diag}}(\pi, \delta))} \text{ for some } b \in \{0, 1\} \\
\iff & \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'_b(\pi, \delta) \in A^{\gamma(\pi'_{v_0}(\pi, \delta))} \text{ for some } b \in \{0, 1\}
\end{aligned}$$

In direction (3b), we have:

$$\begin{aligned}
& \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \delta'_b(\pi, \delta) \in E^{\gamma(\pi'_*(\pi_*, \delta))} \text{ for some } b \in \{0, 1\} \\
\iff & \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \delta'_0(\pi, \delta) \in E^{\gamma(\pi'_*(\pi_*, \delta))} \text{ or } \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \delta'_1(\pi, \delta) \in E^{\gamma(\pi'_*(\pi_*, \delta))} \\
\implies & \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'_0(\pi, \delta) \in A^{\gamma(\pi)} \text{ or } \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \delta'_1(\pi, \delta) \in A^{\gamma(\pi_*)}
\end{aligned}$$

In the last line, we continue with the equivalences from direction (3a) above the line with “ \implies ”.

- $E = \exists R.B$: We show the claim for $\pi' = \pi'_{v_0}$, as the other case is symmetric. In direction (3a), let $\delta'_b \in (\exists R.B)^{\gamma'(\pi')}$. By definition of the semantics, there is a $\varepsilon'_c \in \Delta'$ such that $\langle \delta'_b, \varepsilon'_c \rangle \in R^{\gamma'(\pi')}$ and $\varepsilon'_c \in B^{\gamma'(\pi')}$. Applying the induction hypothesis to $\varepsilon'_c \in B^{\gamma'(\pi')}$ yields that

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \varepsilon'_c(\pi, \delta) \in B^{\gamma(\pi'(\pi, \delta))} = B^{\gamma(\pi)} \quad (41)$$

By definition of $R^{\gamma'(\pi')}$, we have three cases for $\langle \delta'_b, \varepsilon'_c \rangle \in R^{\gamma'(\pi')}$, in each of which we show the claim.

- (a) $\langle \delta'_0, \varepsilon'_0 \rangle \in R^{\gamma'(\pi'_{\text{diag}})}$. That is, $\langle \delta'_0(\pi, \delta), \varepsilon'_0(\pi, \delta) \rangle \in R^{\gamma(\pi)}$ for all $\pi \in \sigma(\mathbf{u})$ and $\delta \in \Delta$. Together with Equation (41), we obtain

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \left(\langle \delta'_0(\pi, \delta), \varepsilon'_0(\pi, \delta) \rangle \in R^{\gamma(\pi)} \text{ and } \varepsilon'_c(\pi, \delta) \in B^{\gamma(\pi)} \right)$$

Now since ε'_0 and ε'_1 are two distinct copies of the same function, this means that

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \left(\langle \delta'_0(\pi, \delta), \varepsilon'_0(\pi, \delta) \rangle \in R^{\gamma(\pi)} \text{ and } \varepsilon'_0(\pi, \delta) \in B^{\gamma(\pi)} \right)$$

Thus

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \exists \varepsilon \in \Delta : \left(\langle \delta'_0(\pi, \delta), \varepsilon \rangle \in R^{\gamma(\pi)} \text{ and } \varepsilon \in B^{\gamma(\pi)} \right)$$

namely we set $\varepsilon := \varepsilon'(\pi, \delta)$ in each case. By the definition of the semantics we get

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \delta'(\pi, \delta) \in (\exists R.B)^{\gamma(\pi)}$$

which, by $\pi' = \pi'_{v_0} = \pi'_{\text{diag}}$ means

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \delta'(\pi, \delta) \in (\exists R.B)^{\gamma(\pi'(\pi, \delta))}$$

- (b) $\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \langle \delta'_1(\pi, \delta), \varepsilon'_1(\pi, \delta) \rangle \in R^{\gamma(\pi_*)}$. In combination with Equation (41), we get

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \left(\langle \delta'_1(\pi, \delta), \varepsilon'_1(\pi, \delta) \rangle \in R^{\gamma(\pi_*)} \text{ and } \varepsilon'_c(\pi, \delta) \in B^{\gamma(\pi)} \right)$$

which, by virtue of ε'_0 and ε'_1 being the same function, means

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \left(\langle \delta'_1(\pi, \delta), \varepsilon'_1(\pi, \delta) \rangle \in R^{\gamma(\pi_*)} \text{ and } \varepsilon'_1(\pi, \delta) \in B^{\gamma(\pi)} \right)$$

which in particular implies

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \left(\langle \delta'_1(\pi, \delta), \varepsilon'_1(\pi, \delta) \rangle \in R^{\gamma(\pi)} \text{ and } \varepsilon'_1(\pi, \delta) \in B^{\gamma(\pi)} \right)$$

which implies the claim as in the case above.

(c) there is some $\zeta'_0 \in \Delta'$ with $\zeta'_0 \approx \delta'_1$ and $\langle \zeta'_0, \varepsilon'_0 \rangle \in R^{\gamma'(\pi'_{\text{diag}})}$. The latter property implies

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \langle \zeta'_0(\pi, \delta), \varepsilon'_0(\pi, \delta) \rangle \in R^{\gamma(\pi)}$$

In turn, from $\zeta'_0 \approx \delta'_0$ we obtain

$$\forall \pi \in \sigma(\mathbf{u}) : \bigcup_{\delta \in \Delta} \{\zeta'_0(\pi, \delta)\} = \bigcup_{\substack{\pi_* \in \sigma(\mathbf{u}), \\ \delta_* \in \Delta}} \{\delta'_1(\pi_*, \delta_*)\}$$

This means in particular that

$$\forall \pi, \pi_* \in \sigma(\mathbf{u}) : \forall \delta_* \in \Delta : \exists \delta_+ \in \Delta : \delta'_1(\pi_*, \delta_*) = \zeta'_0(\pi, \delta_+)$$

Thus

$$\forall \pi, \pi_* \in \sigma(\mathbf{u}) : \forall \delta_* \in \Delta : \exists \delta_+ \in \Delta : \langle \delta'_1(\pi_*, \delta_*), \varepsilon'_0(\pi, \delta_+) \rangle \in R^{\gamma(\pi)}$$

which we combine with Equation (41) as usual to obtain

$$\forall \pi, \pi_* \in \sigma(\mathbf{u}) : \forall \delta_* \in \Delta : \exists \delta_+ \in \Delta : \left(\langle \delta'_1(\pi_*, \delta_*), \varepsilon'_0(\pi, \delta_+) \rangle \in R^{\gamma(\pi)} \text{ and } \varepsilon'_c(\pi, \delta_+) \in B^{\gamma(\pi)} \right)$$

which in particular implies

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \exists \delta_+ \in \Delta : \left(\langle \delta'_1(\pi, \delta), \varepsilon'_0(\pi, \delta_+) \rangle \in R^{\gamma(\pi)} \text{ and } \varepsilon'_0(\pi, \delta_+) \in B^{\gamma(\pi)} \right)$$

whence we get

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \delta'_1(\pi, \delta) \in (\exists R.B)^{\gamma(\pi)}$$

In direction (3b), assume that

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \delta'_0(\pi, \delta) \in (\exists R.B)^{\gamma(\pi'_{v_0}, \delta)}$$

By the definition of the semantics, we obtain

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \exists \varepsilon \in \Delta : \left(\langle \delta'_0(\pi, \delta), \varepsilon \rangle \in R^{\gamma(\pi'_{v_0}, \delta)} \text{ and } \varepsilon_{\pi, \delta} \in B^{\gamma(\pi'_{v_0}, \delta)} \right)$$

Since $\pi' = \pi'_{v_0} = \pi'_{\text{diag}}$, this means that

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \exists \varepsilon_{\pi, \delta} \in \Delta : \left(\langle \delta'(\pi, \delta), \varepsilon \rangle \in R^{\gamma(\pi_*)} \text{ and } \varepsilon \in B^{\gamma(\pi_*)} \right)$$

Since Δ' contains every function $\zeta' : \sigma(\mathbf{u}) \times \Delta \rightarrow \Delta$, there in particular exists a function $\varepsilon'_0 \in \Delta'$ such that $\varepsilon'_0(\pi, \delta) = \varepsilon_{\pi, \delta}$ for all $\pi \in \sigma(\mathbf{u})$ and $\delta \in \Delta$. Thus

$$\exists \varepsilon'_0 \in \Delta' : \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \left(\langle \delta'_0(\pi, \delta), \varepsilon'_0(\pi, \delta) \rangle \in R^{\gamma(\pi_*)} \text{ and } \varepsilon'_0(\pi, \delta) \in B^{\gamma(\pi_*)} \right)$$

which means

$$\left(\exists \varepsilon'_0 \in \Delta' : \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \langle \delta'_0(\pi, \delta), \varepsilon'_0(\pi, \delta) \rangle \in R^{\gamma(\pi_*)} \right) \text{ and} \\ \left(\exists \varepsilon'_0 \in \Delta' : \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \varepsilon'_0(\pi, \delta) \in B^{\gamma(\pi_*)} \right)$$

where we can apply the induction hypothesis to the second line to obtain

$$\left(\exists \varepsilon'_0 \in \Delta' : \forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \langle \delta'_0(\pi, \delta), \varepsilon'_0(\pi, \delta) \rangle \in R^{\gamma(\pi_*)} \right) \text{ and}$$

$$\varepsilon'_0 \in B^{\gamma'(\pi'_{v_0})}$$

which in particular means that

$$\exists \varepsilon'_0 \in \Delta' : \left(\left(\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \langle \delta'_0(\pi, \delta), \varepsilon'_0(\pi, \delta) \rangle \in R^{\gamma(\pi)} \right) \text{ and } \varepsilon'_0 \in B^{\gamma'(\pi'_{v_0})} \right)$$

which (employing among other things that $\pi'_{v_0}(\pi, \delta) = \pi$ for all $\pi \in \sigma(\mathbf{u})$ and $\delta \in \Delta$) implies

$$\exists \varepsilon'_0 \in \Delta' : \left(\langle \delta'_0, \varepsilon'_0 \rangle \in R^{\gamma'(\pi'_{v_0})} \text{ and } \varepsilon' \in B^{\gamma'(\pi'_{v_0})} \right)$$

that is, $\delta'_0 \in (\exists R.B)^{\gamma'(\pi'_{v_0})}$.

- $E = \diamond_{s'} B$: In direction (3a), assume $\delta' \in (\diamond_{s'} B)^{\gamma'(\pi')}$. Thus there exists some $\pi'' \in \sigma'(s')$ such that $\delta' \in B^{\gamma'(\pi'')}$. The induction hypothesis yields

$$\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in B^{\gamma(\pi''(\pi, \delta))}$$

Thus for all $\pi \in \sigma(u)$ and $\delta \in \Delta$ there exists a $\pi''_{\pi, \delta} \in \sigma(s')$, namely $\pi''_{\pi, \delta} := \pi''(\pi, \delta)$, such that $\delta'(\pi, \delta) \in B^{\gamma(\pi''_{\pi, \delta})}$. This directly yields $\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in (\diamond_{s'} B)^{\gamma'(\pi')}$.

In direction (3b), assume

$$\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \delta'(\pi, \delta) \in (\diamond_{s'} B)^{\gamma(\pi''_{v_0}(\pi_*, \delta))}$$

This means that

$$\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \exists \pi''_{\pi_*, \delta} \in \sigma(s') : \delta'(\pi, \delta) \in B^{\gamma(\pi''_{\pi_*, \delta})}$$

Choose $\pi'_{s'} \in \sigma'(s')$ such that for all $\pi_* \in \sigma(u)$ and $\delta \in \Delta$, we have $\pi'_{s'}(\pi_*, \delta) = \pi''_{\pi_*, \delta}$. Thus we obtain

$$\exists \pi'_{s'} \in \sigma'(s') : \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \delta'(\pi, \delta) \in B^{\gamma(\pi'_{s'}(\pi_*, \delta))}$$

We apply the induction hypothesis and get $\exists \pi'_{s'} \in \Delta' : \delta' \in B^{\gamma'(\pi'_{s'})}$, that is, $\delta' \in (\diamond_{s'} B)^{\gamma'(\pi')}$.

- $E = \square_{s'} B$: In direction (3a), we have

$$\begin{aligned} & \delta' \in (\square_{s'} B)^{\gamma'(\pi')} \\ \implies & \forall \pi'' \in \sigma'(s') : \delta' \in B^{\gamma'(\pi'')} \\ \stackrel{(IH)}{\implies} & \forall \pi'' \in \sigma'(s') : \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in B^{\gamma(\pi''(\pi, \delta))} \\ \implies & \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi'' \in \sigma'(s') : \delta'(\pi, \delta) \in B^{\gamma(\pi''(\pi, \delta))} \\ \stackrel{\dagger}{\implies} & \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_{s'} \in \sigma(s') : \delta'(\pi, \delta) \in B^{\gamma(\pi_{s'})} \\ \implies & \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in (\square_{s'} B)^{\gamma(\pi)} \end{aligned}$$

and for the converse direction, (3b), we consider

$$\begin{aligned} & \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \delta'(\pi, \delta) \in (\square_{s'} B)^{\gamma(\pi''(\pi_*, \delta))} \\ \implies & \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \forall \pi_{s'} \in \sigma(s') : \delta'(\pi, \delta) \in B^{\gamma(\pi_{s'})} \\ \stackrel{\dagger}{\implies} & \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \forall \pi'' \in \sigma'(s') : \delta'(\pi, \delta) \in B^{\gamma(\pi''(\pi, \delta))} \\ \implies & \forall \pi'' \in \sigma'(s') : \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \delta'(\pi, \delta) \in B^{\gamma(\pi''(\pi, \delta))} \\ \stackrel{(IH)}{\implies} & \forall \pi'' \in \sigma'(s') : \delta' \in B^{\gamma'(\pi'')} \\ \implies & \delta' \in (\square_{s'} B)^{\gamma'(\pi')} \end{aligned}$$

Here, in both directions the implications marked \dagger can be justified just like in the proof of Claim 3. ◇

The two previous claims can be combined to obtain the following “global” property:

Claim 5. For every concept E , $\delta' \in \Delta'$, and $\pi' \in \Pi'$:

$$\delta' \in E^{\gamma'(\pi')} \implies \left(\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in E^{\gamma(\pi''(\pi, \delta))} \right)$$

Proof of Claim. Follows from direction “ \implies ” of Claim 3 and Item 3a of Claim 4. ◇

For the new precisifications, we sometimes also have to pay attention to which copy of the functions in Δ' we are currently considering, especially for the main case we will need below, namely where both copies are contained in the same concept’s extension in both precisifications.

Claim 6. For every concept E , function $\delta' : \sigma(u) \times \Delta \rightarrow \Delta$, and $b \in \{0, 1\}$:

$$\delta'_b \in E^{\pi'_{v_b}} \implies \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'(\pi, \delta) \in E^{\gamma(\pi)} \quad (42)$$

$$\delta'_{1-b} \in E^{\pi'_{v_b}} \implies \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \delta'(\pi, \delta) \in E^{\gamma(\pi_*)} \quad (43)$$

Proof of Claim. We use structural induction on E .

• $E = A \in \text{BC}_K$: By definition.

• $E = \exists R.B$:

(42): Let $\delta'_b \in (\exists R.B)^{\gamma'(\pi'_{v_b})}$ for some $b \in \{0, 1\}$. By definition of the semantics, there is an $\varepsilon'_c \in \Delta'$ such that $\langle \delta'_b, \varepsilon'_c \rangle \in R^{\gamma'(\pi'_{v_b})}$ and $\varepsilon'_c \in B^{\gamma'(\pi'_{v_b})}$. By definition of $R^{\gamma'(\pi'_{v_b})}$ the only possible reason for $\langle \delta'_b, \varepsilon'_c \rangle \in R^{\gamma'(\pi'_{v_b})}$ is that $b = c$ and $\langle \delta'_b, \varepsilon'_c \rangle \in R^{\gamma'(\pi'_{\text{diag}})}$, that is,

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \langle \delta'_b(\pi, \delta), \varepsilon'_c(\pi, \delta) \rangle \in R^{\gamma(\pi)}$$

Due to $b = c$, we can apply the induction hypothesis of (42) to $\varepsilon'_c \in B^{\gamma'(\pi'_{v_b})}$ and obtain

$$\forall \pi \in \mathbf{u} : \forall \delta \in \Delta : \varepsilon'_c(\pi, \delta) \in B^{\gamma(\pi)}$$

Togher with the above, we obtain

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \left(\langle \delta'_b(\pi, \delta), \varepsilon'_c(\pi, \delta) \rangle \in R^{\gamma(\pi)} \text{ and } \varepsilon'_c(\pi, \delta) \in B^{\gamma(\pi)} \right)$$

Thus

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \exists \varepsilon_{\pi, \delta} \in \Delta : \left(\langle \delta'_b(\pi, \delta), \varepsilon_{\pi, \delta} \rangle \in R^{\gamma(\pi)} \text{ and } \varepsilon_{\pi, \delta} \in B^{\gamma(\pi)} \right)$$

namely we set $\varepsilon_{\pi, \delta} := \varepsilon'_c(\pi, \delta)$ in each case. By the definition of the semantics we get

$$\forall \pi \in \sigma : \forall \delta \in \Delta : \delta'(\pi, \delta) \in (\exists R.B)^{\gamma(\pi)}$$

(43): Let $\delta'_{1-b} \in (\exists R.B)^{\gamma'(\pi'_{v_b})}$. By the definition of the semantics, there exists some $\varepsilon'_c \in \Delta'$ such that $\langle \delta'_{1-b}, \varepsilon'_c \rangle \in R^{\gamma'(\pi'_{v_b})}$ and $\varepsilon'_c \in B^{\gamma'(\pi'_{v_b})}$. According to the definition of $R^{\gamma'(\pi'_{v_b})}$, there are two possible cases:

(a) $c = 1 - b$ and $\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \langle \delta'_{1-b}(\pi, \delta), \varepsilon'_c(\pi, \delta) \rangle \in R^{\gamma(\pi_*)}$. We can apply the induction hypothesis of (43) to the fact that $\varepsilon'_c = \varepsilon'_{1-b} \in B^{\gamma'(\pi'_{v_b})}$ and obtain

$$\forall \pi \in \mathbf{u} : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \varepsilon'_c(\pi, \delta) \in B^{\gamma(\pi_*)}$$

In combination with the presumption of this case, we obtain

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \left(\langle \delta'_b(\pi, \delta), \varepsilon'_c(\pi, \delta) \rangle \in R^{\gamma(\pi_*)} \text{ and } \varepsilon'_c(\pi, \delta) \in B^{\gamma(\pi_*)} \right)$$

thus proving the claim.

(b) $c = b$ and there is some $\zeta'_b \in \Delta'$ with $\zeta'_b \approx \delta'_{1-b}$ and $\langle \zeta'_b, \varepsilon'_c \rangle \in R^{\gamma'(\pi'_{\text{diag}})}$. The latter property implies

$$\forall \pi_* \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \langle \zeta'_b(\pi_*, \delta), \varepsilon'_c(\pi_*, \delta) \rangle \in R^{\gamma(\pi_*)}$$

In turn, from $\zeta'_b \approx \delta'_{1-b}$ we obtain

$$\forall \pi_* \in \sigma(\mathbf{u}) : \bigcup_{\delta \in \Delta} \{ \zeta'_b(\pi_*, \delta) \} = \bigcup_{\substack{\pi \in \sigma(\mathbf{u}), \\ \delta \in \Delta}} \{ \delta'_{1-b}(\pi, \delta) \}$$

This means in particular that

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \exists \delta_+ \in \Delta : \zeta'_b(\pi_*, \delta_+) = \delta'_{1-b}(\pi, \delta)$$

whence

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \exists \delta_+ \in \Delta : \langle \delta'_{1-b}(\pi, \delta), \varepsilon'_c(\pi_*, \delta_+) \rangle \in R^{\gamma(\pi_*)}$$

Now $b = c$ means that we can apply the induction hypothesis of (42) to $\varepsilon'_c = \varepsilon'_b \in B^{\gamma'(\pi'_{v_b})}$, from which we get

$$\forall \pi_* \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \varepsilon'_b(\pi_*, \delta) \in B^{\gamma(\pi_*)}$$

which we combine with the line above as usual to obtain

$$\forall \pi, \pi_* \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \exists \delta_+ \in \Delta : \left(\langle \delta'_{1-b}(\pi, \delta), \varepsilon'_c(\pi_*, \delta_+) \rangle \in R^{\gamma(\pi_*)} \text{ and } \varepsilon'_c(\pi_*, \delta_+) \in B^{\gamma(\pi_*)} \right)$$

which in particular implies

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \exists \varepsilon \in \Delta : \left(\langle \delta'_{1-b}(\pi, \delta), \varepsilon \rangle \in R^{\gamma(\pi_*)} \text{ and } \varepsilon \in B^{\gamma(\pi_*)} \right)$$

namely in each case $\varepsilon := \varepsilon'_c(\pi_*, \delta_+)$, whence we get

$$\forall \pi \in \sigma(\mathbf{u}) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(\mathbf{u}) : \delta'_{1-b}(\pi, \delta) \in (\exists R.B)^{\gamma(\pi_*)}$$

thus proving the claim.

- $E = \diamond_{s'} B$: Let $\pi' \in \{\pi'_{v_0}, \pi'_{v_1}\}$ and consider $\delta'_b \in (\diamond_{s'} B)^{\gamma'(\pi')}$. Then there is some $\pi'' \in \sigma'(s')$ such that $\delta'_{1-b} \in B^{\gamma'(\pi')}$. By Claim 5 we obtain

$$\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'_b(\pi, \delta) \in B^{\gamma(\pi'', \delta)}$$

whence for all $\pi \in \sigma(u)$ and $\delta \in \Delta$ we can set $\pi_{\pi, \delta} := \pi''(\pi, \delta)$ to rewrite this into

$$\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \exists \pi_{\pi, \delta} \in \sigma(s') : \delta'_b(\pi, \delta) \in B^{\gamma(\pi_{\pi, \delta})}$$

and we conclude the claim:

$$\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \delta'_b(\pi, \delta) \in (\diamond_{s'} B)^{\gamma(\pi_*)}$$

- $E = \square_{s'} B$: Let $\pi' \in \{\pi'_{v_0}, \pi'_{v_1}\}$ and consider $\delta'_b \in (\square_{s'} B)^{\gamma'(\pi')}$. Then for all $\pi'' \in \sigma'(s')$, we have $\delta'_b \in B^{\gamma'(\pi')}$, which by Claim 5 means that

$$\forall \pi'' \in \sigma'(s') : \forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'_b(\pi, \delta) \in B^{\gamma(\pi'', \delta)}$$

Let $\pi_{s'} \in \sigma(s')$ be arbitrary. Consider the function $\pi'_{s'} = \{(\pi, \delta) \mapsto \pi_{s'}\}$, for which we have $\pi'_{s'} \in \sigma'(s')$ by definition and thus also

$$\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \delta'_b(\pi, \delta) \in B^{\gamma(\pi'_{s'}(\pi, \delta))} = B^{\gamma(\pi_{s'})}$$

Since $\pi_{s'}$ was arbitrarily chosen, we get

$$\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_{s'} \in \sigma(s') : \delta'_b(\pi, \delta) \in B^{\gamma(\pi_{s'})}$$

which shows the claim. \diamond

We now continue the main proof, showing that $\mathfrak{D}' \models \mathcal{K}'$.

- $\mathfrak{D}' \models \square_u [C \sqsubseteq A]$: By definition.
- $\mathfrak{D}' \models \square_s [\diamond_{v_0} A \sqcap \diamond_{v_1} A \sqsubseteq D]$: Let $\pi' \in \sigma'(s)$ and $\delta'_b \in (\diamond_{v_0} A \sqcap \diamond_{v_1} A)^{\gamma'(\pi')}$ for some $b \in \{0, 1\}$. We have to show $\delta'_b \in D^{\gamma'(\pi')}$. By definition of \mathfrak{D}' , we conclude that $\delta'_b \in A^{\gamma'(\pi'_{v_0})} \cap A^{\gamma'(\pi'_{v_1})}$; we furthermore have $\pi'_{v_0}, \pi'_{v_1} \in \sigma'(u)$ and $A^{\gamma'(\pi'_{v_y})} = C^{\gamma'(\pi'_{v_y})}$ for all $y \in \{0, 1\}$; thus it follows that $\delta'_b \in C^{\gamma'(\pi'_{v_0})} \cap C^{\gamma'(\pi'_{v_1})}$. So in any case we obtain that $\delta'_b \in C^{\gamma'(\pi_{v_1-b})}$, whence by Claim 6 we get

$$\forall \pi \in \sigma(u) : \forall \delta \in \Delta : \forall \pi_* \in \sigma(u) : \delta'(\pi, \delta) \in C^{\gamma(\pi_*)}$$

Hence, for all $\pi \in \sigma(u)$ and $\delta \in \Delta$ we have $\delta'(\pi, \delta) \in (\square_u C)^{\gamma(\pi)}$. From $\mathfrak{D} \models \square_s [\square_u C \sqsubseteq D]$ it therefore follows that $\delta'(\pi_*, \delta) \in D^{\gamma(\pi)}$ for all $\pi_* \in \sigma(u)$, $\delta \in \Delta$, and $\pi \in \sigma(s)$. Now by $\pi' \in \sigma'(s)$ we obtain that for any $\pi_* \in \sigma(u)$ and $\delta \in \Delta$ we get $\pi'(\pi_*, \delta) \in \sigma(s)$. Thus it follows that $\delta'(\pi, \delta) \in D^{\gamma(\pi'(\pi_*, \delta))}$ for all $\pi, \pi_* \in \sigma(u)$ and $\delta \in \Delta$. By Claim 3/Claim 4 above, we get $\delta' \in D^{\gamma'(\pi')}$.

- $\mathfrak{D}' \models \tau$ for all axioms $\tau \in \mathcal{K}' \setminus \{\square_s [C \sqsubseteq A], \square_s [\diamond_{v_0} A \sqcap \diamond_{v_1} A \sqsubseteq D]\}$: We do a case distinction on the possible forms of axioms.
 - $\tau = \square_{s'} [E \sqsubseteq F]$. By presumption $\mathfrak{D} \models \mathcal{K}$ we get that for every $\pi \in \sigma(s')$ we have $E^{\gamma(\pi)} \subseteq F^{\gamma(\pi)}$. Let $\pi' \in \sigma'(s')$ and $\delta' \in E^{\gamma'(\pi')}$. From Claim 5 we get that $\delta'(\pi, \delta) \in E^{\gamma(\pi'(\pi, \delta))}$ for all $\pi \in \sigma(u)$ and $\delta \in \Delta$. From $\pi' \in \sigma'(s')$ we get that $\pi'(\pi_*, \delta) \in \sigma(s')$ for all $\pi_* \in \sigma(u)$ and $\delta \in \Delta$. It follows that $E^{\gamma(\pi'(\pi_*, \delta))} \subseteq F^{\gamma(\pi'(\pi_*, \delta))}$ for all $\pi_* \in \sigma(u)$ and $\delta \in \Delta$. In particular, $\delta'(\pi, \delta) \in F^{\gamma(\pi'(\pi_*, \delta))}$ for all $\pi, \pi_* \in \sigma(u)$ and $\delta \in \Delta$. Therefore, Claim 3/Claim 4 yields $\delta' \in F^{\gamma'(\pi')}$.
 - $\tau = \square_{s'} [R_1 \circ \dots \circ R_n \sqsubseteq R]$. Let $\pi' \in \sigma'(s')$ and $\delta'_0, \delta'_1, \dots, \delta'_n \in \Delta'$ with $\langle \delta'_{i-1}, \delta'_i \rangle \in R_i^{\gamma'(\pi')}$ for all $1 \leq i \leq n$. Then by definition, for all $1 \leq i \leq n$ we get $\langle \delta'_{i-1}(\pi, \delta), \delta'_i(\pi, \delta) \rangle \in R_i^{\gamma(\pi')}$ for all $\pi \in \sigma(u)$ and $\delta \in \Delta$. Since by presumption $\mathfrak{D} \models \square_{s'} [R_1 \circ \dots \circ R_n \sqsubseteq R]$, we get $\langle \delta'_0(\pi, \delta), \delta'_n(\pi, \delta) \rangle \in R^{\gamma(\pi'(\pi, \delta))}$ for all $\pi' \in \sigma'(s')$, $\pi \in \sigma(u)$, and $\delta \in \Delta$ (since $\pi' \in \sigma'(s')$ implies that $\pi'(\pi_*, \delta) \in \sigma(s')$ for all $\pi_* \in \sigma(u)$ and $\delta \in \Delta$). Thus overall, we get $\langle \delta'_0, \delta'_n \rangle \in R^{\gamma'(\pi')}$; since $\pi' \in \sigma'(s')$ was arbitrary, we get $\mathfrak{D}' \models R_1 \circ \dots \circ R_n \sqsubseteq R$.
- $\mathfrak{D}' \models \alpha$ for all assertions $\alpha \in \mathcal{K}$: We do a case distinction on the possible forms of assertions.
 - $\alpha = \square_{s'} [B(a)]$: It follows that $B \in \text{BC}_{\mathcal{K}}$ since \mathcal{K} is in normal form. From $\mathfrak{D} \models \mathcal{A}$ we get that for any $\pi \in \sigma(s')$ we have $a^{\gamma(\pi)} = \delta_a \in B^{\gamma(\pi)}$. Let $\pi' \in \sigma'(s')$. By definition, we get that $a^{\gamma'(\pi')}(\pi, \delta) = \delta_a$ for all $\delta \in \Delta$ and all $\pi \in \Pi$ and in particular for all $\pi \in \sigma(u)$. From $\pi' \in \sigma'(s')$ we get that for any $\pi \in \sigma(u)$ and $\delta \in \Delta$ we find $\pi'(\pi, \delta) \in \sigma(s')$. It thus follows that $a^{\gamma'(\pi')}(\pi, \delta) = \delta_a \in B^{\gamma(\pi'(\pi, \delta))}$ for all $\pi \in \sigma(u)$ and $\delta \in \Delta$. Consequently, $a^{\gamma'(\pi')} \in B^{\gamma'(\pi')}$.

- $\alpha = \Box_{s'}[R(a, c)]$: From $\mathfrak{D} \models \mathcal{A}$ it follows that for every $\pi \in \sigma(s')$ we have $\langle a^{\gamma(\pi)}, c^{\gamma(\pi)} \rangle \in R^{\gamma(\pi)}$. Let $\pi' \in \sigma'(s')$. As above, we get $a^{\gamma'(\pi')}(\pi, \delta) = \delta_a$ and $c^{\gamma'(\pi')}(\pi, \delta) = \delta_c$ for every $\delta \in \Delta$ and $\pi \in \sigma(u)$. From $\pi' \in \sigma'(s')$, we get $\pi'(\pi, \delta) \in \sigma(s')$ for every $\delta \in \Delta$ and $\pi \in \sigma(u)$. Thus $\langle a^{\gamma'(\pi')}(\pi, \delta), a^{\gamma'(\pi')}(\pi, \delta) \rangle \in R^{\gamma(\pi'(\pi, \delta))}$ for all $\delta \in \Delta$ and $\pi \in \sigma(u)$ whence $\langle a^{\gamma'(\pi')}, a^{\gamma'(\pi')} \rangle \in R^{\gamma'(\pi')}$.
- $\mathfrak{D}' \models \zeta$ for all sharpening statements $\zeta \in \mathcal{K}$: Let $s_1 \cap \dots \cap s_n \preceq s' \in \mathcal{K}$ and $\pi' \in \sigma'(s_1) \cap \dots \cap \sigma'(s_n)$. Then by definition of σ' , we get $\bigcup_{\pi \in \sigma(u), \delta \in \Delta} \{\pi'(\pi, \delta)\} \subseteq \sigma(s_i)$ for all $1 \leq i \leq n$. Since $\mathfrak{D} \models s_1 \cap \dots \cap s_n \preceq s'$, we get that for all $\pi \in \sigma(u)$ and $\delta \in \Delta$ we have $\pi'(\pi, \delta) \in \sigma(s')$. By definition of σ' this means that $\pi' \in \sigma'(s')$.
- (b) Let $\mathfrak{D}' \models \mathcal{T}'$ and consider $\pi' \in \sigma'(s)$ and $\delta' \in (\Box_u C)^{\gamma'(\pi')}$. Since $\pi'_{v_0}, \pi'_{v_1} \in \sigma'(u)$, we thus get $\delta' \in C^{\gamma'(\pi'_{v_0})} \cap C^{\gamma'(\pi'_{v_1})}$. By $\mathfrak{D}' \models \Box_u[C \sqsubseteq A]$, we obtain that $\delta' \in A^{\gamma'(\pi'_{v_0})} \cap A^{\gamma'(\pi'_{v_1})}$; that is, $\delta' \in (\Diamond_{v_0} A \sqcap \Diamond_{v_1} A)^{\gamma'(\pi')}$. Since also $\mathfrak{D}' \models \Box_s[\Diamond_{v_0} A \sqcap \Diamond_{v_1} A \sqsubseteq D]$, it follows that $\delta' \in D^{\gamma'(\pi')}$. \square

B Proofs for Section 3

Theorem 6 (Soundness). *If a $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} is satisfiable, there is a non-contradictory calculus derivation for knowledge base \mathcal{K} .*

Proof. Suppose \mathcal{K} is satisfiable. Then there is a model $\mathfrak{D} = \langle \Delta, \Pi, \sigma, \gamma \rangle$ such that $\mathfrak{D} \models \mathcal{K}$.

To show soundness it suffices to show that for each rule $\Gamma \mid \theta$, we have $\mathfrak{D} \models \theta$ if $\mathfrak{D} \models \Gamma$. This can be seen easily for both axioms and rules, and hence limit ourselves to show some examples.

• Axioms

(T.1) By the definition of σ , for all $s \in \text{ST}_{\mathcal{K}}$ we have $\sigma(s) \subseteq \sigma(*)$. Hence, $\mathfrak{D} \models s \preceq *$ as desired.

(T.3) Let $\pi \in \Pi$ be a precisification and let $\delta \in \Delta$ be a domain element. Since $B \Rightarrow C \equiv \neg B \sqcup C$, if $\delta \in C^{\gamma(\pi)}$ then clearly $\delta \in (C \Rightarrow C)^{\gamma(\pi)}$. Similarly, if $\delta \notin C^{\gamma(\pi)}$ then also $\delta \in (C \Rightarrow C)^{\gamma(\pi)}$. Thus clearly $(C \Rightarrow C)^{\gamma(\pi)} = \Delta$, and hence $\mathfrak{D} \models \top \sqsubseteq \Box_*[C \Rightarrow C]$ as desired.

• Rules

(S.1) Assume that $\mathfrak{D} \models s \preceq s'$ and $\mathfrak{D} \models s' \preceq s''$. Then we have that $\sigma(s) \subseteq \sigma(s')$ and $\sigma(s') \subseteq \sigma(s'')$, thus $\sigma(s) \subseteq \sigma(s'')$ and consequently $\mathfrak{D} \models s \preceq s''$ as desired.

(F.4) Assume that $\Box_t[C \sqsubseteq \Diamond_{s'} D]$ and $\Box_{s'}[D \sqsubseteq \Diamond_s E]$. If $\mathfrak{D} \models \Box_t[C \sqsubseteq \Diamond_{s'} D]$, then for all $\pi \in \sigma(t)$ if $\delta \in C^{\gamma(\pi)}$ then there is some $\pi' \in \sigma(s')$ such that $\delta \in D^{\gamma(\pi')}$. Moreover, since $\mathfrak{D} \models \Box_{s'}[D \sqsubseteq \Diamond_s E]$ and $\delta \in D^{\gamma(\pi')}$, there is some $\pi'' \in \sigma(s)$ such that $\delta \in E^{\gamma(\pi'')}$, and thus $\mathfrak{D} \models \Box_t[C \sqsubseteq \Diamond_s E]$ as desired. \square

Theorem 7 (Completeness). *If there is a non-contradictory calculus derivation for a $\mathbb{S}_{\mathcal{EL}^+}$ knowledge base \mathcal{K} , then \mathcal{K} is satisfiable.*

Model construction

Given a Standpoint \mathcal{EL} knowledge base \mathcal{K} in normal form, let \mathcal{K}^+ denote the set of axioms obtained by saturating \mathcal{K} under the above deduction rules. Assuming \mathcal{K}^+ does not contain $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]]$, we construct a model of \mathcal{K} in an infinite process: we start from an initialized model and extend it (both by adding domain elements and precisifications) in a stepwise fashion, resulting in a “monotonic” sequence of models. The result of the process is the “limit” of this sequence, which can be expressed via an infinite union.

For the initialization, we choose the standpoint structure $\mathfrak{D}_0 = \langle \Delta_0, \Pi_0, \sigma_0, \gamma_0 \rangle$ where:

- Δ_0 consists of one element δ_a for every individual name a mentioned in \mathcal{K} ;
- Π_0 consists of one precisification π_s for every standpoint s mentioned in \mathcal{K} (including $*$);
- σ_0 maps each standpoint s to the set $\{\pi_s\} \cup \{\pi_{s'} \mid s' \prec s \in \mathcal{K}^+\}$;
- γ_0 maps each π_s to the description logic interpretation \mathcal{I} over Δ_0 , where
 - $a^{\mathcal{I}} = \delta_a$ for each individual name a .
 - $A^{\mathcal{I}} = \{\delta_a \mid \Box_u[\{a\} \sqsubseteq \Box_s[\top \Rightarrow A]] \in \mathcal{K}^+, s \in \sigma_0^{-1}(\pi_s)\}$
 - $R^{\mathcal{I}} = \{(\delta_a, \delta_b) \mid \Box_s[R(a, b)] \in \mathcal{K}^+, s \in \sigma_0^{-1}(\pi_s)\}$ for each role name R ;

Note that the obtained structure satisfies all axioms of \mathcal{K} except for those of the shape $\Box_s[E \sqsubseteq \exists R.F]$. This will also be the case for all structures $\mathfrak{D}_1, \mathfrak{D}_2, \dots$ produced in the following.

Moreover, we define $\Lambda_\pi(\delta_a) = \{C \mid \Box_u[\{a\} \sqsubseteq \Box_s[\top \Rightarrow C]] \in \mathcal{K}^+, s \in \sigma_0^{-1}(\pi)\}$.

Given such a structure $\mathfrak{D}_i = \langle \Delta_i, \Pi_i, \sigma_i, \gamma_i \rangle$, check if it satisfies all axioms of the form $\Box_t[E \sqsubseteq \exists R.F]$. If so, \mathfrak{D}_i is a model of \mathcal{K} and we are done. Otherwise, we pick some $\delta^* \in \Delta_i$ and some $\pi^* \in \Pi_i$ with $\pi^* \in \sigma(t)$ for which $\delta^* \in E^{\gamma(\pi^*)} \setminus (\exists R.F)^{\gamma(\pi^*)}$ for some (previously unsatisfied) axiom $\Box_t[E \sqsubseteq \exists R.F]$ from \mathcal{K} . Among the admissible pairs δ^*, π^* , we pick one that is minimal wrt. $\min\{j \leq i \mid \delta^* \in \Delta_j\} + \min\{j \leq i \mid \pi^* \in \Pi_j\}$.

Then obtain $\mathfrak{D}_{i+1} = \langle \Delta_{i+1}, \Pi_{i+1}, \sigma_{i+1}, \gamma_{i+1} \rangle$ from $\mathfrak{D}_i = \langle \Delta_i, \Pi_i, \sigma_i, \gamma_i \rangle$ as follows:

- $\Delta_{i+1} = \Delta_i \cup \{\delta'\}$, where δ' is a fresh domain element;
- let $\text{Con}(F, \pi^*) = \{A \mid \Box_u[\top \sqsubseteq \Box_s[F \Rightarrow A]] \in \mathcal{K}^+, s \in \sigma^{-1}(\pi^*)\}$
- Π_{i+1} is obtained from Π_i by adding a fresh precisification $\pi_{\delta', \diamond_s D}$ whenever there is some $C \in \text{Con}(F, \pi^*)$ with $\Box_t[C \sqsubseteq \diamond_s D] \in \mathcal{K}^+$ for some $t \in \sigma^{-1}(\pi^*)$
- let σ_{i+1} be such that $\sigma_{i+1}(s'') = \sigma_i(s'') \cup \{\pi_{\delta', \diamond_s D}\}$ if $s'' \in \{s\} \cup \{s' \mid s \prec s' \in \mathcal{K}^+\}$ and $\sigma_{i+1}(s'') = \sigma_i(s'')$ otherwise.
- for $\pi \in \Pi_{i+1}$ we let

$$\Lambda_\pi(\delta') := \begin{cases} \text{Con}(F, \pi^*) & \text{if } \pi = \pi^* \\ \bigcup_{G \in \text{Con}(F, \pi^*), s \in \sigma_{i+1}^{-1}(\pi), t \in \sigma_{i+1}^{-1}(\pi^*)} \{A \mid \Box_t[G \sqsubseteq \Box_s[\top \Rightarrow A]] \in \mathcal{K}^+\} & \text{if } \pi \in \Pi_i \setminus \{\pi^*\} \\ \bigcup_{G \in \text{Con}(F, \pi^*), s \in \sigma_{i+1}^{-1}(\pi), t \in \sigma_{i+1}^{-1}(\pi^*)} \{A \mid \Box_t[G \sqsubseteq \Box_s[D \Rightarrow A]] \in \mathcal{K}^+\} & \text{if } \pi = \pi_{\delta', \diamond_s D} \in \Pi_{i+1} \setminus \Pi_i \end{cases}$$

- Let γ_{i+1} be the interpretation function defined as follows:
 - $a^{\gamma_{i+1}(\pi)} = a^{\gamma_i(\pi)} = \delta_a$ for each individual name a and each $\pi \in \Pi_{i+1}$.
 - for concept names A , we let $A^{\gamma_{i+1}(\pi)} = A^{\gamma_i(\pi)} \cup \{\delta'\}$ if $A \in \Lambda_\pi(\delta')$, and $A^{\gamma_{i+1}(\pi)} = A^{\gamma_i(\pi)}$ otherwise.
 - for all role names T , we obtain $T^{\gamma_{i+1}(\pi)}$ through a concurrent saturation process, that is, $T^{\gamma_{i+1}(\pi)} = \bigcup_{k \in \mathbb{N}} [T^{\gamma_{i+1}(\pi)}]_k$, where we let $[T^{\gamma_{i+1}(\pi)}]_0 = \text{Self} \cup \text{Other}$, with

$$\text{Self} = \begin{cases} \{(\delta', \delta')\} & \text{if } \exists T. \text{Self} \in \Lambda_\pi(\delta'), \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\text{Other} = \begin{cases} \emptyset & \text{whenever } \pi \in \Pi_{i+1} \setminus \Pi_i \\ T^{\gamma_i(\pi)} & \text{for } \pi \in \Pi_i \text{ whenever } T \neq R \text{ or } \pi \neq \pi^* \\ R^{\gamma_i(\pi^*)} \cup (\delta^*, \delta') & \text{if } T = R \text{ and } \pi = \pi^* \end{cases}$$

Moreover, obtain

$$[T^{\gamma_{i+1}(\pi)}]_{k+1} = [T^{\gamma_{i+1}(\pi)}]_k \cup \bigcup_{\substack{s \in \sigma_{i+1}^{-1}(\pi) \\ \Box_s[R_0 \sqsubseteq T] \in \mathcal{K}^+}} [R_0^{\gamma_{i+1}(\pi)}]_k \cup \bigcup_{\substack{s \in \sigma_{i+1}^{-1}(\pi) \\ \Box_s[R_1 \circ R_2 \sqsubseteq T] \in \mathcal{K}^+}} [R_1^{\gamma_{i+1}(\pi)}]_k \circ [R_2^{\gamma_{i+1}(\pi)}]_k$$

After producing the (potentially infinite) sequence $\mathfrak{D}_0, \mathfrak{D}_1, \dots$ we obtain the wanted model \mathfrak{D} via

$$\mathfrak{D} = \langle \bigcup_i \Delta_i, \bigcup_i \Pi_i, \bigcup_i \sigma_i, \bigcup_i \gamma_i \rangle$$

Lemma 8. Let \mathfrak{D}_i be as described by the model construction, and let $\pi \in \Pi_i$. Then, there is a standpoint $s_\pi \in \sigma_i^{-1}(\pi)$ such that for all $s \in \sigma_i^{-1}(\pi)$ we have $s_\pi \prec s \in \mathcal{K}^+$.

Proof. This follows easily by the construction of σ . First, consider the case of \mathfrak{D}_0 . By definition, $\sigma_0(s) = \{\pi_s\} \cup \{\pi_{s'} \mid s' \prec s \in \mathcal{K}^+\}$, and since by T.2 we have $s \prec s \in \mathcal{K}^+$, hence the claim holds for \mathfrak{D}_0 . Now we assume that the claim holds for \mathfrak{D}_i and we show that it also holds for \mathfrak{D}_{i+1} . Recall that $\sigma_{i+1}(s) = \sigma_i(s) \cup \{\pi_{\delta', \diamond_{s'} D}\}$ if $s \in \{s''\} \cup \{s' \mid s \prec s' \in \mathcal{K}^+\}$ and $\sigma_{i+1}(s) = \sigma_i(s)$ otherwise. Again by T.2 we have $s'' \prec s'' \in \mathcal{K}^+$, hence the claim holds for \mathfrak{D}_{i+1} and thus for \mathfrak{D} as desired. \square

Lemma 9. Let \mathfrak{D}_i be as described by the model construction, and let $\pi \in \Pi_i$. Then, if $\phi \in \mathcal{K}^+$, $\pi \in \sigma_i(s)$ and \square_s is a symbol in ϕ , then the formula ϕ^π replacing some occurrences of \square_s by \square_{s_π} is also in the saturated set, $\phi^\pi \in \mathcal{K}^+$.

Proof. By Lemma 8 and $\pi \in \sigma_i(s)$, we have that $s_\pi \prec s \in \mathcal{K}^+$. Then it is easy to see that for ϕ having any formula shape with a \square_s , we can apply rule S.3 and rule S.4 and obtain precisely that each formula $\phi^\pi \in \mathcal{K}^+$, with ϕ^π being any formula resulting from replacing one or more occurrences of \square_s by \square_{s_π} . \square

Lemma 10. Let \mathfrak{D}_i be as described by the model construction. Then $A \in \Lambda_\pi(\delta)$ iff $\delta \in A^{\gamma_i(\pi)}$ and $\exists R.\text{Self} \in \Lambda_\pi(\delta)$ iff $(\delta, \delta) \in R^{\gamma_i(\pi)}$.

Proof. For the concept names the case is trivial since the definition coincides. Thus, we show the case of self-loops by induction on the model construction.

Base Case : Consider $\exists R.\text{Self} \in \Lambda_\pi(\delta)$ of \mathfrak{D}_0 . (\Rightarrow) If $\exists R.\text{Self} \in \Lambda_\pi(\delta)$ then $\square_u[\{a\} \sqsubseteq \square_s[\top \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$ for $\pi \in \sigma_0(s)$. Then we obtain $\square_s[R(a, a)] \in \mathcal{K}^+$ by rule L.1, and thus by construction $(\delta, \delta) \in R^{\gamma_0(\pi)}$. (\Leftarrow) If $(\delta, \delta) \in R^{\gamma_0(\pi)}$ then by construction we have some $\square_s[R(a, a)] \in \mathcal{K}^+$ with $\pi \in \sigma_0(s)$. Then by the rule L.4 we obtain $\square_u[\{a\} \sqsubseteq \square_s[\top \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$ and hence $\exists R.\text{Self} \in \Lambda_\pi(\delta)$ as desired.

Inductive Step : By the inductive hypothesis, assume that the statement holds for \mathfrak{D}_i and we show that it also holds for \mathfrak{D}_{i+1} . Consider $\exists R.\text{Self} \in \Lambda_\pi(\delta')$ of \mathfrak{D}_{i+1} . (\Rightarrow) If $\exists R.\text{Self} \in \Lambda_\pi(\delta')$ then by construction $(\delta', \delta') \in R^{\gamma_{i+1}(\pi)}$, and we notice that for $\delta \in \Delta_i \setminus \Delta_{i+1}$ the same holds by induction. (\Leftarrow) This is the harder case. We show that if $(\delta', \delta') \in R^{\gamma_{i+1}(\pi)}$ then $\exists R.\text{Self} \in \Lambda_\pi(\delta')$. We observe the saturation process that defines $R^{\gamma_{i+1}(\pi)}$ and we notice that for every pair $(\delta_1, \delta_2) \in [R^{\gamma_{i+1}(\pi)}]_{k+1}$ and $(\delta_1, \delta_2) \notin [R^{\gamma_{i+1}(\pi)}]_k$, we have $\delta' = \delta_2$. With regards to self-loops, we will ext show that, since δ' is fresh, relations of the shape $(\delta', \delta') \in [R^{\gamma_{i+1}(\pi)}]_{k+1}$ can exclusively be triggered by relations $(\delta', \delta') \in [R^{\gamma_{i+1}(\pi)}]_0$, i.e. elements in the set *Self*.

- By definition if $(\delta', \delta') \in [R^{\gamma_{i+1}(\pi)}]_0$ then necessarily $\exists R.\text{Self} \in \Lambda_\pi(\delta')$ since δ' is fresh.
- Let $(\delta', \delta') \notin [R^{\gamma_{i+1}(\pi)}]_k$ and $(\delta', \delta') \in [R^{\gamma_{i+1}(\pi)}]_{k+1}$. Then either

Case 1 $(\delta', \delta') \in \{[R_0^{\gamma_{i+1}(\pi)}]_k \mid \square_{s'}[R_0 \sqsubseteq R] \in \mathcal{K}^+, s' \in \sigma_{i+1}^{-1}(\pi)\}$ Thus, by Lemma 9, there must be some $\square_{s_\pi}[R_0 \sqsubseteq R] \in \mathcal{K}^+$ such that $(\delta', \delta') \in [R_0^{\gamma_{i+1}(\pi)}]_k$.

Case 1.1 Assume $\pi = \pi^*$ and $\pi \in \sigma(s)$

- (1) $\square_u[\top \sqsubseteq \square_{s_\pi}[F \Rightarrow \exists R_0.\text{Self}]] \in \mathcal{K}^+$, by the assumption that $\exists R_0.\text{Self} \in \Lambda_\pi(\delta')$
- (2) $\square_{s_\pi}[R_0 \sqsubseteq R] \in \mathcal{K}^+$ by the assumption
- (3) $\square_*[\top \sqsubseteq \square_{s_\pi}[\exists R_0.\text{Self} \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (2) and rule L.5
- (4) $\square_u[\top \sqsubseteq \square_{s_\pi}[F \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (1), (3) and rule C.1, thus $\exists R.\text{Self} \in \Lambda_\pi(\delta')$ as desired

Case 1.2 Assume $\pi \in \Pi_i \setminus \{\pi^*\}$ and $\pi \in \sigma(s)$. Since $\exists R_0.\text{Self} \in \Lambda_\pi(\delta')$ then

- (1) $\square_{s_{\pi^*}}[G \sqsubseteq \square_{s_\pi}[\top \Rightarrow \exists R_0.\text{Self}]] \in \mathcal{K}^+$ by Lemma 9 for some $G \in \text{Con}(F, \pi^*)$
- (2) $\square_{s_\pi}[R_0 \sqsubseteq R] \in \mathcal{K}^+$ by the assumption
- (3) $\square_*[\top \sqsubseteq \square_{s_\pi}[\exists R_0.\text{Self} \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (2) and rule L.5
- (4) $\square_*[\top \sqsubseteq \square_*[G \Rightarrow \top]] \in \mathcal{K}^+$ by axiom T.4
- (5) $\square_*[G \sqsubseteq \square_{s_\pi}[\exists R_0.\text{Self} \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$ by (3), (4) and rule C.2
- (6) $\square_{s_{\pi^*}}[G \sqsubseteq \square_{s_\pi}[\exists R_0.\text{Self} \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$ by (7) and Lemma 9
- (7) $\square_{s_{\pi^*}}[G \sqsubseteq \square_{s_\pi}[\top \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (1), (8) and rule C.1, thus $\exists R.\text{Self} \in \Lambda_\pi(\delta')$ as desired

Case 1.3 Assume $\pi = \pi_{\delta', \diamond_{s'} E}$ and $\pi \in \sigma(s)$. Since $\exists R_0.\text{Self} \in \Lambda_\pi(\delta')$ then

- (1) $\square_{s_{\pi^*}}[G \sqsubseteq \square_{s_\pi}[E \Rightarrow \exists R_0.\text{Self}]] \in \mathcal{K}^+$ by Lemma 9 for some $G \in \text{Con}(F, \pi^*)$
- (2) $\square_{s_\pi}[R_0 \sqsubseteq R] \in \mathcal{K}^+$ by the assumption
- (3) $\square_*[\top \sqsubseteq \square_{s_\pi}[\exists R_0.\text{Self} \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (2) and rule L.5
- (4) $\square_*[\top \sqsubseteq \square_*[G \Rightarrow \top]] \in \mathcal{K}^+$ by axiom T.4
- (5) $\square_*[G \sqsubseteq \square_{s_\pi}[\exists R_0.\text{Self} \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$ by (3), (4) and rule C.2

(6) $\Box_{S_{\pi^*}} [G \sqsubseteq \Box_{S_{\pi}} [\exists R_0.\text{Self} \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$ by (7) and Lemma 9

(7) $\Box_{S_{\pi^*}} [G \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (1), (8) and rule C.1, thus $\exists R.\text{Self} \in \Lambda_{\pi}(\delta')$ as desired

Case 2 $(\delta', \delta') \in \{[R_1^{\gamma_{i+1}(\pi)}]_k \circ [R_2^{\gamma_{i+1}(\pi)}]_k \mid \Box_{s'} [R_1 \circ R_2 \sqsubseteq R] \in \mathcal{K}^+, s' \in \sigma_{i+1}^{-1}(\pi)\}$. Thus, by Lemma 9, there must be some $\Box_{S_{\pi}} [R_1 \circ R_2 \sqsubseteq R] \in \mathcal{K}^+$ such that $(\delta', \delta') \in [R_1^{\gamma_{i+1}(\pi)}]_k$ and $(\delta', \delta') \in [R_2^{\gamma_{i+1}(\pi)}]_k$.

Case 2.1 Assume $\pi = \pi^*$ and $\pi \in \sigma(s)$

(1) $\Box_* [\top \sqsubseteq \Box_{S_{\pi}} [F \Rightarrow \exists R_1.\text{Self}]] \in \mathcal{K}^+$, by the assumption that $\exists R_1.\text{Self} \in \Lambda_{\pi}(\delta')$ and rule I.2

(2) $\Box_* [\top \sqsubseteq \Box_{S_{\pi}} [F \Rightarrow \exists R_2.\text{Self}]] \in \mathcal{K}^+$, by the assumption that $\exists R_2.\text{Self} \in \Lambda_{\pi}(\delta')$ and rule I.2

(3) $\Box_{S_{\pi}} [R_1 \circ R_2 \sqsubseteq R] \in \mathcal{K}^+$ by the assumption

(4) $\Box_{S_{\pi}} [\exists R_1.\text{Self} \sqcap \exists R_2.\text{Self} \Rightarrow \exists R.\text{Self}] \in \mathcal{K}^+$, by (3) and rule L.6

(5) $\Box_* [\top \sqsubseteq \Box_{S_{\pi}} [F \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (1), (2), (4) and rule E.4, thus $\exists R.\text{Self} \in \Lambda_{\pi}(\delta')$ as desired

Case 2.2 Assume $\pi \in \Pi_i \setminus \{\pi^*\}$ and $\pi \in \sigma(s)$. Since $\exists R_0.\text{Self} \in \Lambda_{\pi}(\delta')$ then

(1) $\Box_{S_{\pi^*}} [G_1 \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow \exists R_1.\text{Self}]] \in \mathcal{K}^+$ by Lemma 9 for some $G \in \text{Con}(F, \pi^*)$

(2) $\Box_{S_{\pi^*}} [G_2 \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow \exists R_2.\text{Self}]] \in \mathcal{K}^+$ by Lemma 9 for some $G \in \text{Con}(F, \pi^*)$

(3) $\Box_{S_{\pi}} [R_1 \circ R_2 \sqsubseteq R] \in \mathcal{K}^+$ by the assumption

(4) $\Box_{S_{\pi}} [\exists R_1.\text{Self} \sqcap \exists R_2.\text{Self} \Rightarrow \exists R.\text{Self}] \in \mathcal{K}^+$, by (3) and rule L.6

(5) $\Box_{u_1} [\top \sqsubseteq \Box_{S_{\pi^*}} [F \Rightarrow G_1]] \in \mathcal{K}^+$ by the construction of $\text{Con}(F, \pi^*)$ and Lemma 9

(6) $\Box_{u_2} [\top \sqsubseteq \Box_{S_{\pi^*}} [F \Rightarrow G_2]] \in \mathcal{K}^+$ by the construction of $\text{Con}(F, \pi^*)$ and Lemma 9

(7) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow \exists R_1.\text{Self}]] \in \mathcal{K}^+$ by (1), (5) and rule C.2

(8) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow \exists R_2.\text{Self}]] \in \mathcal{K}^+$ by (2), (6) and rule C.2

(9) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (4), (7), (8) and rule E.4, thus $\exists R.\text{Self} \in \Lambda_{\pi}(\delta')$ as desired

Case 2.3 Assume $\pi = \pi_{\delta', \diamond, s, E}$ and $\pi \in \sigma(s)$. Since $\exists R_0.\text{Self} \in \Lambda_{\pi}(\delta')$ then

(1) $\Box_{S_{\pi^*}} [G_1 \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow \exists R_1.\text{Self}]] \in \mathcal{K}^+$ by Lemma 9 for some $G \in \text{Con}(F, \pi^*)$

(2) $\Box_{S_{\pi^*}} [G_2 \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow \exists R_2.\text{Self}]] \in \mathcal{K}^+$ by Lemma 9 for some $G \in \text{Con}(F, \pi^*)$

(3) $\Box_{S_{\pi}} [R_1 \circ R_2 \sqsubseteq R] \in \mathcal{K}^+$ by the assumption

(4) $\Box_{S_{\pi}} [\exists R_1.\text{Self} \sqcap \exists R_2.\text{Self} \Rightarrow \exists R.\text{Self}] \in \mathcal{K}^+$, by (3) and rule L.6

(5) $\Box_{u_1} [\top \sqsubseteq \Box_{S_{\pi^*}} [F \Rightarrow G_1]] \in \mathcal{K}^+$ by the construction of $\text{Con}(F, \pi^*)$ and Lemma 9

(6) $\Box_{u_2} [\top \sqsubseteq \Box_{S_{\pi^*}} [F \Rightarrow G_2]] \in \mathcal{K}^+$ by the construction of $\text{Con}(F, \pi^*)$ and Lemma 9

(7) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow \exists R_1.\text{Self}]] \in \mathcal{K}^+$ by (1), (5) and rule C.2

(8) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow \exists R_2.\text{Self}]] \in \mathcal{K}^+$ by (2), (6) and rule C.2

(9) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (4), (7), (8) and rule E.4, thus $\exists R.\text{Self} \in \Lambda_{\pi}(\delta')$ as desired \square

Completeness proof. In order to show completeness, we will show that if the model construction fails, then necessarily the saturated knowledge base contains the inconsistency axiom, $\Box_* [\top \sqsubseteq \Box_* [\top \Rightarrow \perp]] \in \mathcal{K}^+$. First, we will use the previously constructed model to show that for all $\phi \in \mathcal{K}$ then $\mathfrak{D} \models \phi$. This is proved by induction for all types of axioms allowed in the normal form. By Lemma 10, we use $C \in \Lambda_{\pi}(\delta)$ in place of $\delta \in C^{\gamma_i(\pi)}$ for C a concept name and of $(\delta, \delta) \in R^{\gamma_i(\pi)}$ when C is of the form $\exists R.\text{Self}$.

Claim 7. Let \mathcal{K} be a $\mathbb{S}_{\mathcal{E}\mathcal{L}^+}$ knowledge base and let \mathfrak{D} be the model obtained by the model construction. Then for all $\phi \in \mathcal{K}$ we have $\mathfrak{D} \models \phi$.

$\phi = s \preceq s'$ Assume $s \preceq s' \in \mathcal{K}$. Then it is clear from the definition of σ_0 that by construction $\mathfrak{D}_0 \models s \preceq s'$. Now, we assume that $\mathfrak{D}_i \models s \preceq s'$ and show that $\mathfrak{D}_{i+1} \models s \preceq s'$.

1. Notice that we have $\sigma_i(s) \subseteq \sigma_i(s')$ and σ_{i+1} is such that for all $\pi \in \Pi_i$ and standpoint s'' , $\pi \in \sigma_{i+1}(s'')$ if $\pi \in \sigma_i(s'')$.
2. So we must only show that if $\pi \in \Pi_{i+1} \setminus \Pi_i$ and $\pi \in \sigma_{i+1}(s)$ then we must have $\pi \in \sigma_{i+1}(s')$.
3. By the construction of σ_{i+1} , if $\pi \in \Pi_{i+1} \setminus \Pi_i$, then there is a standpoint s'' for which π has been created such that either $s = s''$ or $s \prec s'' \in \mathcal{K}^+$.
4. If $s = s''$, then $\pi \in \sigma_{i+1}(s)$ as desired, from the definition of σ_{i+1} and from $s \preceq s' \in \mathcal{K}^+$.
5. Otherwise, since \mathcal{K}^+ is saturated under the deduction rules and by the assumption, we have $s \preceq s' \in \mathcal{K}^+$, then by the non-applicability of the rule S.1, we must also have $s' \preceq s'' \in \mathcal{K}^+$, which again by the definition of σ_{i+1} implies that $\pi \in \sigma_{i+1}(s')$ as desired.
6. Thus, necessarily $\sigma_{i+1}(s) \subseteq \sigma_{i+1}(s')$ and $\mathfrak{D}_{i+1} \models s \preceq s'$.

7. Finally, from the definition of \mathfrak{D} we obtain $\mathfrak{D} \models s \preceq s'$ as desired.

$\phi = s_1 \cap s_2 \preceq s'$ Assume $s_1 \cap s_2 \preceq s' \in \mathcal{K}$. Let us first show that $\mathfrak{D}_0 \models s_1 \cap s_2 \preceq s'$.

Case 1: Assume that $\sigma_0(s_1) \cap \sigma_0(s_2) = \emptyset$. Then by construction there is no s such that $s \prec s_1 \in \mathcal{K}^+$ and $s \prec s_2 \in \mathcal{K}^+$, and $\mathfrak{D}_0 \models s_1 \cap s_2 \preceq s'$ is trivially satisfied.

Case 2: Assume that $\sigma_0(s_1) \cap \sigma_0(s_2) \neq \emptyset$. Then, by the construction of σ_0 , if $\pi \in \sigma_0(s_1) \cap \sigma_0(s_2)$ then there must be some s such that $\pi \in \sigma_0(s)$ and both $s \prec s_1 \in \mathcal{K}^+$ and $s \prec s_2 \in \mathcal{K}^+$, thus $\pi \in \sigma_0(s_1)$ and $\pi \in \sigma_0(s_2)$. Then, by the non-applicability of the rules S.1 and S.2, also $s \prec s' \in \mathcal{K}^+$, and thus also $\pi \in \sigma_0(s')$. Hence, $\mathfrak{D}_0 \models s_1 \cap s_2 \preceq s'$ as desired.

We now assume that $\mathfrak{D}_i \models s_1 \cap s_2 \preceq s'$ and, in a similar way, we show that then $\mathfrak{D}_{i+1} \models s_1 \cap s_2 \preceq s'$.

Case 1: Assume that $\sigma_{i+1}(s_1) \cap \sigma_{i+1}(s_2) = \emptyset$. Then by construction there is no s such that $s \prec s_1 \in \mathcal{K}^+$ and $s \prec s_2 \in \mathcal{K}^+$, and $\mathfrak{D}_{i+1} \models s_1 \cap s_2 \preceq s'$ is trivially satisfied.

Case 2: Assume that $\sigma_{i+1}(s_1) \cap \sigma_{i+1}(s_2) \neq \emptyset$. Then, by the construction of σ_{i+1} , if $\pi \in \sigma_{i+1}(s_1) \cap \sigma_{i+1}(s_2)$ then either (a) $\pi \in \Pi_{i+1} \cap \Pi_i$ and thus $\pi \in \sigma_i(s')$ by the inductive hypothesis, giving us $\pi \in \sigma_{i+1}(s')$ as required or (b) $\pi \in \Pi_{i+1} \setminus \Pi_i$, and hence there must be some s such that $\pi \in \sigma_{i+1}(s)$ and both $s \prec s_1 \in \mathcal{K}^+$ and $s \prec s_2 \in \mathcal{K}^+$, thus $\pi \in \sigma_{i+1}(s_1)$ and $\pi \in \sigma_0(s_2)$. Then, again by the non-applicability of the rules S.1 and S.2, also $s \prec s' \in \mathcal{K}^+$, and thus also $\pi \in \sigma_{i+1}(s')$. Hence, $\mathfrak{D}_{i+1} \models s_1 \cap s_2 \preceq s'$ as desired.

Finally, from the definition of \mathfrak{D} we obtain $\mathfrak{D} \models s_1 \cap s_2 \preceq s'$ as desired.

$\phi = \Box_s[R \sqsubseteq R']$ Assume $\Box_s[R \sqsubseteq R'] \in \mathcal{K}^+$.

Base Case : First, let us show that $\mathfrak{D}_0 \models \Box_s[R \sqsubseteq R']$. We let $\pi \in \sigma_0(s)$ and $(\delta, \delta') \in R^{\gamma_0(\pi)}$.

(1) $\Box_{s_\pi}[R(a, b)] \in \mathcal{K}^+$ by construction and Lemma 9

(2) $\Box_{s_\pi}[R \sqsubseteq R'] \in \mathcal{K}^+$ by assumption and Lemma 9

(3) $\Box_{s_\pi}[R'(a, b)] \in \mathcal{K}^+$ by (1), (2) and rule A.4

From this we obtain that if $\pi \in \sigma_0(s)$ and $(\delta, \delta') \in R^{\gamma_0(\pi)}$ then $(\delta, \delta') \in R'^{\gamma_0(\pi)}$, thus $\mathfrak{D}_0, \pi \models R \sqsubseteq R'$ for all $\pi \in \sigma_0(s)$ and hence $\mathfrak{D}_0 \models \Box_s[R \sqsubseteq R']$ by the semantics.

Inductive Step : Now, we assume that $\mathfrak{D}_i \models \Box_s[R \sqsubseteq R']$ and show that $\mathfrak{D}_{i+1} \models \Box_s[R \sqsubseteq R']$. We must show that if $(\delta, \delta') \in R^{\gamma_{i+1}(\pi)}$ then $(\delta, \delta') \in R'^{\gamma_{i+1}(\pi)}$ for all $\pi \in \sigma_{i+1}(s)$. From the concurrent saturation process that determines $R'^{\gamma_{i+1}(\pi)}$, we precisely have that $(\delta, \delta') \in [R'^{\gamma_{i+1}(\pi)}]_{k+1}$ if $(\delta, \delta') \in [R^{\gamma_{i+1}(\pi)}]_k$, $\Box_s[R \sqsubseteq R'] \in \mathcal{K}^+$ and $\pi \in \sigma(s)$, and hence it easily follows that $\mathfrak{D}_{i+1} \models \Box_s[R \sqsubseteq R']$ as desired.

$\phi = \Box_s[R_1 \circ R_2 \sqsubseteq R']$ Assume $\Box_s[R_1 \circ R_2 \sqsubseteq R'] \in \mathcal{K}$.

Base Case : First, let us show that $\mathfrak{D}_0 \models \Box_s[R_1 \circ R_2 \sqsubseteq R']$. We let $\pi \in \sigma_0(s)$, $(\delta, \delta') \in R_1^{\gamma_0(\pi)}$ and $(\delta', \delta'') \in R_2^{\gamma_0(\pi)}$.

(1) $\Box_{s_\pi}(R_1 \circ R_2 \sqsubseteq R') \in \mathcal{K}^+$ and $\pi \in \sigma_0(s)$ by assumption and Lemma 9

(2) $\Box_{s_\pi}[R(a, b)] \in \mathcal{K}^+$ by construction and Lemma 9

(3) $\Box_{s_\pi}[R(b, c)] \in \mathcal{K}^+$ by construction and Lemma 9

(4) $\Box_{s_\pi}[R'(a, c)] \in \mathcal{K}^+$ by (1), (2), (3) and rule A.5

From this we obtain that if $\pi \in \sigma_0(s)$, $(\delta, \delta') \in R_1^{\gamma_0(\pi)}$ and $(\delta', \delta'') \in R_2^{\gamma_0(\pi)}$ then $(\delta, \delta'') \in R'^{\gamma_0(\pi)}$. Thus $\mathfrak{D}_0, \pi \models R_1 \circ R_2 \sqsubseteq R'$ for all $\pi \in \sigma_0(s)$ and hence $\mathfrak{D}_0 \models \Box_s[R_1 \circ R_2 \sqsubseteq R']$ by the semantics.

Inductive Step : Now, we assume that $\mathfrak{D}_i \models \Box_s[R_1 \circ R_2 \sqsubseteq R']$ and show that $\mathfrak{D}_{i+1} \models \Box_s[R_1 \circ R_2 \sqsubseteq R']$. We must show that if $(\delta, \delta') \in R_1^{\gamma_{i+1}(\pi)}$ and $(\delta', \delta'') \in R_2^{\gamma_{i+1}(\pi)}$ then $(\delta, \delta'') \in R'^{\gamma_{i+1}(\pi)}$ for all $\pi \in \sigma_{i+1}(s)$. From the concurrent saturation process that determines $R_1^{\gamma_{i+1}(\pi)}$, $R_2^{\gamma_{i+1}(\pi)}$ and $R'^{\gamma_{i+1}(\pi)}$ and the assumption, we have that for some $k < m$ $(\delta, \delta') \in [R_1^{\gamma_{i+1}(\pi)}]_k$ and for some $l < m$ $(\delta', \delta'') \in [R_2^{\gamma_{i+1}(\pi)}]_l$, thus also $(\delta, \delta') \in [R_1^{\gamma_{i+1}(\pi)}]_{m-1}$ and $(\delta', \delta'') \in [R_2^{\gamma_{i+1}(\pi)}]_{m-1}$. Hence we also have $(\delta, \delta'') \in [R'^{\gamma_{i+1}(\pi)}]_m$ and finally $(\delta, \delta'') \in R'^{\gamma_{i+1}(\pi)}$ as desired.

$\phi = \Box_s[C \sqsubseteq D]$ Assume $\Box_s[C \sqsubseteq D] \in \mathcal{K}$ (i.e. $\Box_u[\top \sqsubseteq \Box_s[C \Rightarrow D]] \in \mathcal{K}^+$ and thus also $\Box_*[\top \sqsubseteq \Box_s[C \Rightarrow D]] \in \mathcal{K}^+$ due to rule I.2).

Base Case : First, let us show that $\mathfrak{D}_0 \models \Box_s[C \sqsubseteq D]$. We let $\pi \in \sigma_0(s)$ and $C \in \Lambda_\pi(\delta)$.

(1) $\Box_*[\top \sqsubseteq \Box_s[C \Rightarrow D]] \in \mathcal{K}^+$, by assumption

(2) $\Box_u[\{a\} \sqsubseteq \Box_{s_\pi}[\top \Rightarrow C]] \in \mathcal{K}^+$, by construction, Lemma 9 and the assumption that $C \in \Lambda_\pi(\delta)$

- (3) $\Box_*[\top \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$ by (1) and Lemma 9
- (4) $\Box_*[\{a\} \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$ by (3) and rule A.1
- (5) $\Box_u[\{a\} \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$ by (4) and rule S.4
- (6) $\Box_u[\{a\} \sqsubseteq \Box_{S_\pi}[\top \Rightarrow D]] \in \mathcal{K}^+$ by (2), (5) rule C.1, and thus $D \in \Lambda_\pi(\delta)$ as required

From this we obtain that if $\pi \in \sigma_0(s)$ and $C \in \Lambda_\pi(\delta)$ then $D \in \Lambda_\pi(\delta)$, thus $\mathfrak{D}_0, \pi \models C \sqsubseteq D$ for all $\pi \in \sigma_0(s)$ and hence $\mathfrak{D}_0 \models \Box_s[C \sqsubseteq D]$ by the semantics.

Inductive Step : Now, we assume that $\mathfrak{D}_i \models \Box_s[C \sqsubseteq D]$ and show that $\mathfrak{D}_{i+1} \models \Box_s[C \sqsubseteq D]$. Consider $\delta \in \Delta_{i+1} \setminus \Delta_i$ (the case of $\delta \in \Delta_i$ is trivial). Thus, δ has been introduced to satisfy some axiom $\Box_t[E \sqsubseteq \exists R.F] \in \mathcal{K}^+$ with $t \in \sigma_{i+1}(\pi^*)$. We must show that if $C \in \Lambda_\pi(\delta)$ then $D \in \Lambda_\pi(\delta)$ for all $\pi \in \sigma_{i+1}(s)$. We assume $C \in \Lambda_\pi(\delta)$ and consider the three cases.

Case 1 : Assume $\pi = \pi^*$ and $\pi \in \sigma(s)$. Since $C \in \Lambda_\pi(\delta)$ and by rule I.2 and Lemma 9, we have $\Box_*[\top \sqsubseteq \Box_{S_\pi}[F \Rightarrow C]] \in \mathcal{K}^+$. By assumption $\Box_*[\top \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$. Then by rule C.1 we obtain $\Box_*[\top \sqsubseteq \Box_{S_\pi}[F \Rightarrow D]] \in \mathcal{K}^+$, and thus $D \in \Lambda_\pi(\delta')$ as required.

Case 2 : Assume $\pi \in \Pi_i \setminus \{\pi^*\}$ and $\pi \in \sigma(s)$. Since $C \in \Lambda_\pi(\delta')$ then for some $G \in \text{Con}(F, \pi^*)$ there is $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C]] \in \mathcal{K}^+$ by Lemma 9. By assumption, $\Box_*[\top \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$.

- (1) $\Box_*[\top \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$, by assumption
- (2) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C]] \in \mathcal{K}^+$, by assumption
- (3) $\Box_{S_\pi}[\top \sqsubseteq \Box_*[C \Rightarrow \top]] \in \mathcal{K}^+$, by axiom T.4 and rule S.3
- (4) $\Box_*[C \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$, by (1), (3) and axiom C.2
- (5) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$, by (2), (4) and axiom F.1
- (6) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[\top \Rightarrow D]] \in \mathcal{K}^+$, by (2), (5) and axiom C.1. Thus we obtain $D \in \Lambda_\pi(\delta)$ as desired

Case 3 : Assume $\pi = \pi_{\delta', \diamond_s E}$ and $\pi \in \sigma(s)$. Since $C \in \Lambda_\pi(\delta')$ then for some $G \in \text{Con}(F, \pi^*)$ by Lemma 9 there is $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[E \Rightarrow C]] \in \mathcal{K}^+$. Recall that, by assumption, $\Box_*[\top \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$.

- (1) $\Box_*[\top \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$, by assumption
- (2) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[E \Rightarrow C]] \in \mathcal{K}^+$, by assumption
- (3) $\Box_*[\top \sqsubseteq \Box_{S_{\pi^*}}[G \Rightarrow \top]] \in \mathcal{K}^+$, by axiom T.4 and S.4
- (4) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$, by (1), (3) and axiom C.2
- (5) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[E \Rightarrow D]] \in \mathcal{K}^+$, by (2), (4) and axiom C.1. Thus we obtain $D \in \Lambda_\pi(\delta)$ as desired

$\phi = \Box_s[C_1 \sqcap C_2 \sqsubseteq D]$ Let us now show that if $\Box_s[C_1 \sqcap C_2 \sqsubseteq D] \in \mathcal{K}$ then $\mathfrak{D} \models \Box_s[C_1 \sqcap C_2 \sqsubseteq D]$.

Base Case : First, let us show that $\mathfrak{D}_0 \models \Box_s[C_1 \sqcap C_2 \sqsubseteq D]$. Let $\pi \in \Pi$, $C_1 \in \Lambda_\pi(\delta)$, $C_2 \in \Lambda_\pi(\delta)$ and $\delta = a^{\gamma_0}$. We need to show that $D \in \Lambda_\pi(\delta)$

- (1) $\Box_*[\{a\} \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C_1]] \in \mathcal{K}^+$ by construction, rule A.3, and Lemma 9
- (2) $\Box_*[\{a\} \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C_2]] \in \mathcal{K}^+$ by construction, rule A.3, and Lemma 9
- (3) $\Box_{S_\pi}[C_1 \sqcap C_2 \sqsubseteq D]$ by assumption and Lemma 9
- (4) $\Box_*[\{a\} \sqsubseteq \Box_{S_\pi}[\top \Rightarrow D]] \in \mathcal{K}^+$ by (1), (2), (3) and rule E.4, thus we obtain $D \in \Lambda_\pi(\delta)$ as desired

Inductive Step : Now, we assume that $\mathfrak{D}_i \models \Box_s[C_1 \sqcap C_2 \sqsubseteq D]$ and show that $\mathfrak{D}_{i+1} \models \Box_s[C_1 \sqcap C_2 \sqsubseteq D]$. Consider $\delta \in \Delta_{i+1} \setminus \Delta_i$ (the case of $\delta \in \Delta_i$ is trivial). Thus, δ has been introduced to satisfy some axiom $\Box_t[E \sqsubseteq \exists R.F] \in \mathcal{K}^+$ with $t \in \sigma_{i+1}(\pi^*)$. We must show that if $C_1 \in \Lambda_\pi(\delta)$ and $C_2 \in \Lambda_\pi(\delta)$ then $D \in \Lambda_\pi(\delta)$ for all $\pi \in \sigma_{i+1}(s)$. We assume $C_1, C_2 \in \Lambda_\pi(\delta)$ and consider the three cases.

Case 1 : Assume $\pi = \pi^*$ and $\pi \in \sigma_{i+1}(s)$. Since $C_1, C_2 \in \Lambda_\pi(\delta)$ then (by construction and rule A.3) $\Box_*[\top \sqsubseteq \Box_{S_\pi}[F \Rightarrow C_1]]$, $\Box_*[\top \sqsubseteq \Box_{S_\pi}[F \Rightarrow C_2]] \in \mathcal{K}^+$ by Lemma 9 and also $\Box_{S_\pi}[C_1 \sqcap C_2 \sqsubseteq D] \in \mathcal{K}^+$. Then by rule E.4 we obtain $\Box_*[\top \sqsubseteq \Box_{S_\pi}[F \Rightarrow D]] \in \mathcal{K}^+$, and thus $D \in \Lambda_\pi(\delta')$ as required.

Case 2 : Assume $\pi \in \Pi_i \setminus \{\pi^*\}$ and $\pi \in \sigma_{i+1}(s)$. Since $C_1, C_2 \in \Lambda_\pi(\delta')$ then for some $G_1, G_2 \in \text{Con}(F, \pi^*)$ and by Lemma 9, there are $\Box_{S_{\pi^*}}[G_1 \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C_1]]$, $\Box_{S_{\pi^*}}[G_2 \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C_2]] \in \mathcal{K}^+$:

- (1) $\Box_{S_\pi}[C_1 \sqcap C_2 \sqsubseteq D] \in \mathcal{K}^+$,
- (2) $\Box_{S_{\pi^*}}[G_1 \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C_1]] \in \mathcal{K}^+$
- (3) $\Box_{S_{\pi^*}}[G_2 \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C_2]] \in \mathcal{K}^+$
- (4) $\Box_{u_1}[\top \sqsubseteq \Box_{S_{\pi^*}}[F \Rightarrow G_1]] \in \mathcal{K}^+$ by the construction of $\text{Con}(F, \pi^*)$ and Lemma 9
- (5) $\Box_{u_2}[\top \sqsubseteq \Box_{S_{\pi^*}}[F \Rightarrow G_2]] \in \mathcal{K}^+$ by the construction of $\text{Con}(F, \pi^*)$ and Lemma 9

- (6) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow C_1]] \in \mathcal{K}^+$ by (2), (4) and rule C.2
- (7) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow C_2]] \in \mathcal{K}^+$ by (3), (5) and rule C.2
- (8) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow D]] \in \mathcal{K}^+$ by (1), (6), (7) and rule E.4, thus $D \in \Lambda_{\pi}(\delta')$ as desired

Case 3 : Assume $\pi = \pi_{\delta', \diamond_{s_1} E} \in \Pi_{i+1} \setminus \Pi_i$ and $\pi \in \sigma_{i+1}(s)$. Since $C_1, C_2 \in \Lambda_{\pi}(\delta')$ then for some $G_1, G_2 \in \text{Con}(F, \pi^*)$ and by Lemma 9, there are $\Box_{S_{\pi^*}} [G_1 \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow C_1]], \Box_{S_{\pi^*}} [G_2 \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow C_2]] \in \mathcal{K}^+$

- (1) $\Box_{S_{\pi}} [C_1 \sqcap C_2 \sqsubseteq D] \in \mathcal{K}^+$,
- (2) $\Box_{S_{\pi^*}} [G_1 \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow C_1]] \in \mathcal{K}^+$
- (3) $\Box_{S_{\pi^*}} [G_2 \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow C_2]] \in \mathcal{K}^+$
- (4) $\Box_{u_1} [\top \sqsubseteq \Box_{S_{\pi^*}} [F \Rightarrow G_1]] \in \mathcal{K}^+$ by the construction of $\text{Con}(F, \pi^*)$ and Lemma 9
- (5) $\Box_{u_2} [\top \sqsubseteq \Box_{S_{\pi^*}} [F \Rightarrow G_2]] \in \mathcal{K}^+$ by the construction of $\text{Con}(F, \pi^*)$ and Lemma 9
- (6) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow C_1]] \in \mathcal{K}^+$ by (2), (4) and rule C.2
- (7) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow C_2]] \in \mathcal{K}^+$ by (3), (5) and rule C.2
- (8) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow D]] \in \mathcal{K}^+$ by (6), (7) and rule E.4, thus $D \in \Lambda_{\pi}(\delta')$ as desired

$\phi = \Box_{s'} [C \sqsubseteq \Box_s D]$ Assume $\Box_{s'} [C \sqsubseteq \Box_s D] \in \mathcal{K}$ (i.e. $\Box_{s'} [C \sqsubseteq \Box_s [\top \Rightarrow D]] \in \mathcal{K}^+$).

Base Case : First, let us show that $\mathfrak{D}_0 \models \Box_{s'} [C \sqsubseteq \Box_s [\top \Rightarrow D]]$. We let $C \in \Lambda_{\pi'}(\delta)$ for $\pi' \in \sigma_0(s')$, and we show that $D \in \Lambda_{\pi}(\delta)$ for $\pi \in \sigma_0(s)$.

- (1) $\Box_{S_{\pi'}} [C \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow D]] \in \mathcal{K}^+$, by assumption and Lemma 9
- (2) $\Box_u [\{a\} \sqsubseteq \Box_{S_{\pi'}} [\top \Rightarrow C]] \in \mathcal{K}^+$ for some s' such that $\pi \in \sigma_0(s')$, by construction and the assumption that $C \in \Lambda_{\pi'}(\delta)$
- (3) $\Box_u [\{a\} \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow D]] \in \mathcal{K}^+$ by rule F.1, (1) and (2), thus $D \in \Lambda_{\pi}(\delta')$ as desired

From this we obtain that if $\pi \in \Pi$ then $\mathfrak{D}_0 \models \Box_{s'} [C \sqsubseteq \Box_s [\top \Rightarrow D]]$ by the semantics.

Inductive Step : Now, we assume that $\mathfrak{D}_i \models \Box_{s'} [C \sqsubseteq \Box_s [\top \Rightarrow D]]$ and show that $\mathfrak{D}_{i+1} \models \Box_{s'} [C \sqsubseteq \Box_s [\top \Rightarrow D]]$. Consider $\delta \in \Delta_{i+1} \setminus \Delta_i$ (the case of $\delta \in \Delta_i$ is trivial). Thus, δ has been introduced to satisfy some axiom $\Box_t [E \sqsubseteq \exists R.F] \in \mathcal{K}^+$ with $t \in \sigma_{i+1}(\pi^*)$. We must show that if $C \in \Lambda_{\pi'}(\delta)$ then $D \in \Lambda_{\pi}(\delta)$ for all $\pi \in \sigma_{i+1}(s')$ and $\pi' \in \sigma_{i+1}(s)$. We consider the three cases.

Case 1 : Assume $\pi' = \pi^*$, hence $s', t \in \sigma_{i+1}^{-1}(\pi')$ and $C \in \Lambda_{\pi'}(\delta)$.

- (1) $\Box_* [\top \sqsubseteq \Box_{S_{\pi}} [F \Rightarrow C]] \in \mathcal{K}^+$ from $C \in \Lambda_{\pi'}(\delta)$, rule XXX and Lemma 9
- (2) $\Box_{S_{\pi'}} [C \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow D]] \in \mathcal{K}^+$, by assumption and Lemma 9, hence $D \in \Lambda_{\pi}(\delta)$ if $\pi = \Pi \setminus \pi^*$ as desired
- (3) $\Box_* [\top \sqsubseteq \Box_* [C \Rightarrow \top]] \in \mathcal{K}^+$ by axiom T.4
- (4) $\Box_* [\top \sqsubseteq \Box_* [E \Rightarrow \top]] \in \mathcal{K}^+$ by axiom T.4
- (5) $\Box_* [C \sqsubseteq \Box_* [E \Rightarrow \top]] \in \mathcal{K}^+$ by (3), (4) and rule C.2
- (6) $\Box_{S_{\pi'}} [C \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow \top]] \in \mathcal{K}^+$ by (5) Lemma 9
- (7) $\Box_{S_{\pi'}} [C \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow D]] \in \mathcal{K}^+$, from (2), (6) and rule C.1, hence $D \in \Lambda_{\pi}(\delta)$ if $\pi = \pi_{\delta', \diamond_{s_1} E}$ as desired
- (8) $\Box_* [\top \sqsubseteq \Box_{S_{\pi}} [C \Rightarrow D]] \in \mathcal{K}^+$, by (2) and rules I.1
- (9) $\Box_{S_{\pi'}} [\top \sqsubseteq \Box_{S_{\pi}} [C \Rightarrow D]] \in \mathcal{K}^+$, by (8) and Lemma 9
- (10) $\Box_{S_{\pi'}} [\top \sqsubseteq \Box_{S_{\pi}} [F \Rightarrow D]] \in \mathcal{K}^+$, by (1), (9) and rule C.1, hence $D \in \Lambda_{\pi}(\delta)$ if $\pi = \pi^*$ as desired

Case 2 : Assume $\pi' \in \Pi_i \setminus \{\pi^*\}$, and $C \in \Lambda_{\pi'}(\delta)$

- (1) $\Box_{S_{\pi^*}} [G \sqsubseteq \Box_{S_{\pi'}} [\top \Rightarrow C]] \in \mathcal{K}^+$ from $C \in \Lambda_{\pi'}(\delta)$ for some $G \in \text{Con}(F, \pi^*)$ by construction
- (2) $\Box_{S_{\pi'}} [C \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow D]] \in \mathcal{K}^+$, by assumption and Lemma 9
- (3) $\Box_{S_{\pi^*}} [G \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow D]] \in \mathcal{K}^+$, by (1), (2) and rule F.1, hence $D \in \Lambda_{\pi}(\delta)$ if $\pi = \Pi \setminus \pi^*$ as desired
- (4) $\Box_* [\top \sqsubseteq \Box_* [G \Rightarrow \top]] \in \mathcal{K}^+$ by axiom T.4
- (5) $\Box_* [\top \sqsubseteq \Box_* [E \Rightarrow \top]] \in \mathcal{K}^+$ by axiom T.4
- (6) $\Box_* [G \sqsubseteq \Box_* [E \Rightarrow \top]] \in \mathcal{K}^+$ by (4), (5) and rule C.2
- (7) $\Box_{S_{\pi'}} [G \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow \top]] \in \mathcal{K}^+$ by (6) and Lemma 9
- (8) $\Box_{S_{\pi'}} [G \sqsubseteq \Box_{S_{\pi}} [E \Rightarrow D]] \in \mathcal{K}^+$, from (3), (7) and rule C.1, hence $D \in \Lambda_{\pi}(\delta)$ if $\pi = \pi_{\delta', \diamond_{s_1} E}$ as desired
- (9) $\Box_u [\top \sqsubseteq \Box_{S_{\pi^*}} [F \Rightarrow G]] \in \mathcal{K}^+$ from 1 by construction and Lemma 9
- (10) $\Box_{S_{\pi^*}} [F \sqsubseteq \Box_{S_{\pi}} [\top \Rightarrow D]] \in \mathcal{K}^+$ from (3), (9) and rule C.2

(11) $\Box_*[\top \sqsubseteq \Box_{S_\pi}[F \Rightarrow D]] \in \mathcal{K}^+$, by (10) and rules I.1, hence $D \in \Lambda_\pi(\delta)$ if $\pi = \pi^*$ as desired

Case 3 : Assume $\pi' = \pi_{\delta', \diamond_{s_1} E} \in \Pi_{i+1} \setminus \Pi_i$ and $C \in \Lambda_{\pi'}(\delta)$

- (1) $\Box_*[C \sqsubseteq \Box_{S_\pi}[\top \Rightarrow D]] \in \mathcal{K}^+$ by assumption
- (2) $\Box_{S_{\pi^*}}[J \sqsubseteq \exists R.F] \in \mathcal{K}^+$ by construction from the existential that triggered the iteration \mathfrak{D}_{i+1} and Lemma 9
- (3) $\Box_{u_1}[\top \sqsubseteq \Box_{S_{\pi^*}}[F \Rightarrow G]] \in \mathcal{K}^+$ by construction for $G \in \text{Con}(F, \pi^*)$ and Lemma 9
- (4) $\Box_{u_2}[\top \sqsubseteq \Box_{S_{\pi^*}}[F \Rightarrow H]] \in \mathcal{K}^+$ by construction for $H \in \text{Con}(F, \pi^*)$ and Lemma 9
- (5) $\Box_{S_{\pi^*}}[H \sqsubseteq \diamond_{S_{\pi'}} E] \in \mathcal{K}^+$ by construction and Lemma 9
- (6) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_{\pi'}}[E \Rightarrow C]] \in \mathcal{K}^+$ by construction and Lemma 9
- (7) $\Box_*[\top \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$, by (1), Lemma 9 and rule I.1
- (8) $\Box_*[\top \sqsubseteq \Box_*[G \Rightarrow \top]] \in \mathcal{K}^+$, by axiom T.4
- (9) $\Box_*[G \sqsubseteq \Box_{S_\pi}[C \Rightarrow D]] \in \mathcal{K}^+$, by (7), (8) and rule C.2
- (10) if $\pi = \pi'$, $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[E \Rightarrow D]] \in \mathcal{K}^+$ from (6), (9), Lemma 9 and rule C.1, hence $D \in \Lambda_\pi(\delta)$ if $\pi = \pi_{\delta', \diamond_{s_1} E}$ as desired
- (11) $\Box_*[C \sqsubseteq \Box_{S_\pi}[\top \Rightarrow D]] \in \mathcal{K}^+$, from (1) and rule S.4.
- (12) $\Box_{S_{\pi^*}}[F \sqsubseteq \diamond_{S_{\pi'}} E] \in \mathcal{K}^+$ from (4), (5) and rule C.3
- (13) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_{\pi'}}[E \Rightarrow C]] \in \mathcal{K}^+$ from (6), (12) and rule S.3
- (14) $\Box_{S_{\pi^*}}[F \sqsubseteq \Box_{S_{\pi'}}[E \Rightarrow C]] \in \mathcal{K}^+$ from (3), (6) and rule C.2
- (15) $\Box_{S_{\pi^*}}[F \sqsubseteq \diamond_{S_{\pi'}} C] \in \mathcal{K}^+$ from (12), (14) and rule C.4
- (16) $\Box_{S_{\pi^*}}[F \sqsubseteq \Box_{S_\pi}[\top \Rightarrow D]] \in \mathcal{K}^+$, from (11), (15) and rule F.3, hence $D \in \Lambda_\pi(\delta)$ if $\pi = \Pi \setminus \pi^*$ as desired
- (17) $\Box_{S_{\pi'}}[C \sqsubseteq \Box_{S_{\pi^*}}[\top \Rightarrow D]] \in \mathcal{K}^+$, from (11) and Lemma 9
- (18) if $\pi = \pi^*$, $\Box_*[\top \sqsubseteq \Box_{S_{\pi^*}}[F \Rightarrow D]]$, by (17) and rule I.1, hence $D \in \Lambda_\pi(\delta)$ if $\pi = \pi^*$ as desired

$\phi = \Box_{s'}[C \sqsubseteq \diamond_s D]$ Assume $\Box_{s'}[C \sqsubseteq \diamond_s D] \in \mathcal{K}$

Base Case : First, let us show that $\mathfrak{D}_0 \models \Box_{s'}[C \sqsubseteq \diamond_s D]$. Let $\pi' \in \sigma_0(s')$ and $C \in \Lambda_{\pi'}(\delta)$.

- (1) $\Box_{S_{\pi'}}[C \sqsubseteq \diamond_s D] \in \mathcal{K}$ by assumption and Lemma 9
- (2) $\Box_u[\{a\} \sqsubseteq \Box_{S_{\pi'}}[\top \Rightarrow C]] \in \mathcal{K}^+$, by construction and Lemma 9
- (3) $s[a, D] \prec s$ by construction
- (4) $\Box_*[\{a\} \sqsubseteq \Box_s[D \Rightarrow P_{s,a,D}]]$ by construction
- (5) $\Box_*[P_{s,a,D} \sqsubseteq \Box_{s[a,D]}[\top \Rightarrow D]]$ by construction
- (6) $\Box_*[\{a\} \sqsubseteq \Box_{S_{\pi'}}[\top \Rightarrow C]] \in \mathcal{K}^+$, by (2) and rule A.3.
- (7) $\Box_{S_{\pi'}}[\{a\} \sqsubseteq \Box_{S_{\pi'}}[\top \Rightarrow C]] \in \mathcal{K}^+$, by (6) and rule S.3
- (8) $\Box_{S_{\pi'}}[\top \sqsubseteq \Box_{S_{\pi'}}[\{a\} \Rightarrow C]] \in \mathcal{K}^+$, by (7) and rule I.1
- (9) $\Box_{S_{\pi'}}[\{a\} \sqsubseteq \diamond_s D] \in \mathcal{K}$ by (8) and rule C.3
- (10) $\Box_{S_{\pi'}}[\{a\} \sqsubseteq \Box_s[D \Rightarrow P_{s,a,D}]]$ by (4) and rule S.3
- (11) $\Box_{S_{\pi'}}[\{a\} \sqsubseteq \diamond_s P_{s,a,D}] \in \mathcal{K}$ by (10) and rule C.4
- (12) $\Box_s[P_{s,a,D} \sqsubseteq \Box_{s[a,D]}[\top \Rightarrow D]]$ by (5) and rule S.3
- (13) $\Box_{S_{\pi'}}[\{a\} \sqsubseteq \Box_{s[a,D]}[\top \Rightarrow D]]$ by (11), (12) and rule F.3, as desired

From this we obtain that if $\pi' \in \sigma_0(s')$ and $C \in \Lambda_{\pi'}(\delta)$, then $D \in \Lambda_\pi(\delta)$ for $\pi \in \sigma_0(s[a, D])$ and hence $\pi \in \sigma_0(s)$, thus $\mathfrak{D}_0, \pi \models C \sqsubseteq \diamond_s D$ for all $\pi \in \sigma_0(s')$ and hence $\mathfrak{D}_0 \models \Box_{s'}[C \sqsubseteq \diamond_s D]$ by the semantics.

Inductive Step : Now, we assume that $\mathfrak{D}_i \models \Box_{s'}[C \sqsubseteq \diamond_s D]$ and show that $\mathfrak{D}_{i+1} \models \Box_{s'}[C \sqsubseteq \diamond_s D]$. Consider $\delta \in \Delta_{i+1} \setminus \Delta_i$ (the case of $\delta \in \Delta_i$ is trivial). Thus, δ has been introduced to satisfy some axiom $\Box_t[E \sqsubseteq \exists R.F] \in \mathcal{K}^+$. We must show that if $C \in \Lambda_{\pi'}(\delta)$ then $D \in \Lambda_\pi(\delta)$ for some $\pi \in \sigma_{i+1}(s)$, specifically for $\pi = \pi_{\delta', \diamond_{s_1} D}$. We consider the three cases.

Case 1 : Assume $\pi' = \pi^*$, $\pi \in \sigma_{i+1}(s')$ and $C \in \Lambda_{\pi'}(\delta)$.

- (1) $\Box_u[\top \sqsubseteq \Box_{S_{\pi'}}[F \Rightarrow C]] \in \mathcal{K}^+$ from $C \in \Lambda_\pi(\delta)$ and Lemma 9
- (2) $\Box_{S_{\pi'}}[C \sqsubseteq \diamond_s D]$, by assumption and Lemma 9
- (3) $\Box_*[\top \sqsubseteq \Box_*[D \Rightarrow D]]$ by axiom T.3
- (4) $\Box_*[\top \sqsubseteq \Box_*[C \Rightarrow \top]]$ by axiom T.4
- (5) $\Box_*[C \sqsubseteq \Box_*[D \Rightarrow D]]$ by (3), (4) and rule C.2
- (6) By (1), (2) and by construction, there is a precisification $\pi_{\delta', \diamond_{s_1} D} \in \pi \in \sigma_{i+1}(s)$ and by (5) we have $D \in \Lambda_{\pi_{\delta', \diamond_{s_1} D}}(\delta)$ as

desired.

Case 2 : Assume $\pi' \in \Pi_i \setminus \{\pi^*\}$, $\pi \in \sigma_{i+1}(s')$ and $C \in \Lambda_{\pi'}(\delta)$

- (1) $\Box_{\mathcal{S}_{\pi^*}}[G \sqsubseteq \Box_{\mathcal{S}_{\pi'}}[\top \Rightarrow C]] \in \mathcal{K}^+$ for some $G \in \text{Con}(F, \pi^*)$ by construction and Lemma 9.
- (2) $\Box_{\mathcal{S}_{\pi'}}[C \sqsubseteq \Diamond_{\mathcal{S}}D]$, by assumption
- (3) $\Box_{\mathcal{S}_{\pi^*}}[G \sqsubseteq \Diamond_{\mathcal{S}}D]$, by (1), (2) and rule F.2
- (4) From (1) and (3), by construction there is a precisification $\pi_{\delta', \Diamond_{\mathcal{S}}D} \in \pi \in \sigma_{i+1}(s)$, and by **Case 1** we have $D \in \Lambda_{\pi_{\delta', \Diamond_{\mathcal{S}}D}}(\delta)$ as desired

Case 3 : Assume $\pi' = \pi_{\delta', \Diamond_{\mathcal{S}_1}E} \in \Pi_{i+1} \setminus \Pi_i$, $\pi' \in \sigma_{i+1}(s')$ and $C \in \Lambda_{\pi'}(\delta)$

- (1) $\Box_{\mathcal{S}_{\pi^*}}[G \sqsubseteq \Box_{\mathcal{S}_{\pi'}}[E \Rightarrow C]] \in \mathcal{K}^+$ for some $G \in \text{Con}(F, \pi^*)$ by construction and Lemma 9
- (2) $\Box_{\mathcal{S}_{\pi'}}[C \sqsubseteq \Diamond_{\mathcal{S}}D]$, by assumption and Lemma 9
- (3) $\Box_{\mathcal{S}_{\pi^*}}[H \sqsubseteq \Diamond_{\mathcal{S}_{\pi'}}E]$ for some $H \in \text{Con}(F, \pi^*)$ by construction and Lemma 9
- (4) $\Box_{u_1}[\top \sqsubseteq \Box_{\mathcal{S}_{\pi^*}}[F \Rightarrow G]] \in \mathcal{K}^+$ by construction since $G \in \text{Con}(F, \pi^*)$ and Lemma 9
- (5) $\Box_{u_2}[\top \sqsubseteq \Box_{\mathcal{S}_{\pi^*}}[F \Rightarrow H]] \in \mathcal{K}^+$ by construction since $H \in \text{Con}(F, \pi^*)$ and Lemma 9
- (6) $\Box_{\mathcal{S}_{\pi^*}}[F \sqsubseteq \Diamond_{\mathcal{S}_{\pi'}}E] \in \mathcal{K}^+$ from (5), (3) and rule C.3
- (7) $\Box_{\mathcal{S}_{\pi^*}}[F \sqsubseteq \Box_{\mathcal{S}_{\pi'}}[E \Rightarrow C]] \in \mathcal{K}^+$ from (4), (1) and rule C.2
- (8) $\Box_{\mathcal{S}_{\pi^*}}[F \sqsubseteq \Diamond_{\mathcal{S}_{\pi'}}C] \in \mathcal{K}^+$ from (6), (7) and rule (C.4)
- (9) $\Box_{\mathcal{S}_{\pi^*}}[F \sqsubseteq \Diamond_{\mathcal{S}}D] \in \mathcal{K}^+$ from (2), (8) and rule (F.4)
- (10) From (9) by construction there is a precisification $\pi_{\delta', \Diamond_{\mathcal{S}}D} \in \sigma_{i+1}(s)$, and by **Case 1** we have $D \in \Lambda_{\pi_{\delta', \Diamond_{\mathcal{S}}D}}(\delta)$ as desired.

Consequently, from **Cases 1-3**, for all $\pi \in \sigma_{i+1}(s')$, if $C \in \Lambda_{\pi}(\delta)$ then there is some $\pi_{\delta', \Diamond_{\mathcal{S}}D} \in \sigma_{i+1}(s)$ with $D \in \Lambda_{\pi_{\delta', \Diamond_{\mathcal{S}}D}}(\delta)$, hence $\mathfrak{D}_0 \models \Box_{s'}[C \sqsubseteq \Diamond_{\mathcal{S}}D]$ and by induction also $\mathfrak{D} \models \Box_{s'}[C \sqsubseteq \Diamond_{\mathcal{S}}D]$.

$\phi = \Box_{\mathcal{S}}[C \sqsubseteq \exists R.D]$ (notice that $D \neq \text{Self}$) Let us now show that if $\Box_{\mathcal{S}}[C \sqsubseteq \exists R.D] \in \mathcal{K}^+$ then $\mathfrak{D} \models \Box_{\mathcal{S}}[C \sqsubseteq \exists R.D]$. That is, we must show that if $C \in \Lambda_{\pi}(\delta)$ for some $\pi \in \sigma(s)$ then there exists δ' such that $(\delta, \delta') \in R^{\gamma(\pi)}$ and $D \in \Lambda_{\pi}(\delta')$. Assume that for some \mathfrak{D}_i we have $C \in \Lambda_{\pi}(\delta)$ for $\pi \in \sigma_i(s)$ but there is no δ' such that $(\delta, \delta') \in R^{\gamma_i(\pi)}$ and $D \in \Lambda_{\pi}(\delta')$. Then for some iteration $k > i$ we will pick $\pi = \pi^*$ and $\delta = \delta^*$ to produce \mathfrak{D}_{i+1} and set $\Delta_{i+1} = \Delta_i \cup \{\delta'\}$, where δ' is a fresh domain element. By the construction, $\Lambda_{\pi}(\delta') = \{A \mid \Box_{u}[\top \sqsubseteq \Box_{\mathcal{S}}[D \Rightarrow A]] \in \mathcal{K}^+, s \in \sigma^{-1}(\pi^*)\}$. By axiom T.3 we have $\Box_{*}[\top \sqsubseteq \Box_{*}[D \Rightarrow D]]$, hence $D \in \Lambda_{\pi}(\delta')$ as desired. We must now show that also $(\delta, \delta') \in R^{\gamma_{i+1}(\pi)}$. This easily follows since $[R^{\gamma_{i+1}(\pi)}]_0 = \text{Self} \cup \text{Other}$ and $(\delta, \delta') \in \text{Other}$. Thus $(\delta, \delta') \in R^{\gamma_{i+1}(\pi)}$ as required.

$\phi = \Box_{\mathcal{S}}[\exists R.C \sqsubseteq D]$ Let us show that if $\Box_{\mathcal{S}}[\exists R.C \sqsubseteq D] \in \mathcal{K}^+$ then $\mathfrak{D}_i \models \Box_{\mathcal{S}}[\exists R.C \sqsubseteq D]$. That is, we must show that if $C \in \Lambda_{\pi}(\delta_2)$ and $(\delta_1, \delta_2) \in R^{\gamma_i(\pi)}$ for $\pi \in \sigma_i(s)$, then $D \in \Lambda_{\pi}(\delta_1)$.

Remark 1 : If there is a sequence $\Box_{t_1}[R_1 \sqsubseteq R_2], \dots, \Box_{t_j}[R_j \sqsubseteq R_{j+1}] \in \mathcal{K}^+$ for $j \geq 0$, such that $\Box_{t_{j+1}}[R_{j+1} \sqsubseteq R] \in \mathcal{K}^+$ and $t_1, \dots, t_{j+1} \in \sigma_i^{-1}(\pi)$, then by Lemma 9 and the successive application of rule R.1 we have $\Box_{\mathcal{S}_{\pi}}[R_1 \sqsubseteq R] \in \mathcal{K}^+$

Base Case : First, let us show that $\mathfrak{D}_0 \models \Box_{\mathcal{S}}[\exists R.C \sqsubseteq D]$. We let $\pi \in \sigma_0(s)$, $(\delta_1, \delta_2) \in R^{\gamma_0(\pi)}$ and $C \in \Lambda_{\pi}(\delta_2)$.

- (1) $\Box_{\mathcal{S}_{\pi}}[\exists R.C \sqsubseteq D] \in \mathcal{K}$ by assumption and Lemma 9
- (2) $\Box_{*}[\{b\} \sqsubseteq \Box_{\mathcal{S}_{\pi}}[\top \Rightarrow C]] \in \mathcal{K}$ by construction (since $\delta_2 = b^{\gamma_0}$ for some atom b) and Lemma 9
- (3) $\Box_{\mathcal{S}_{\pi}}[R(a, b)] \in \mathcal{K}$ for some atom a with $\delta_1 = a^{\gamma_0}$ by construction and Lemma 9
- (4) $\Box_{\mathcal{S}_{\pi}}[\{a\} \sqsubseteq \exists R.C] \in \mathcal{K}$ by (2), (3) and rule A.6
- (5) $\Box_{*}[\top \sqsubseteq \Box_{\mathcal{S}_{\pi}}[\{a\} \Rightarrow D]] \in \mathcal{K}$ by (1), (4) and rule E.3
- (6) $\Box_{\mathcal{S}_{\pi}}[\{a\} \sqsubseteq \Box_{\mathcal{S}_{\pi}}[\top \Rightarrow D]] \in \mathcal{K}$ by (5), Lemma 9 and rule A.2, hence $D \in \Lambda_{\pi}(\delta_1)$ as desired

Inductive Step : Now, we assume that $\mathfrak{D}_{i-1} \models \Box_{\mathcal{S}}[\exists R.C \sqsubseteq D]$ and show that $\mathfrak{D}_i \models \Box_{\mathcal{S}}[\exists R.C \sqsubseteq D]$. So, we need to consider the case where δ_2 is fresh, $\delta_2 \in \Delta_i \setminus \Delta_{i-1}$. Thus, δ_2 has been introduced to satisfy some axiom $\Box_t[E \sqsubseteq \exists T.F] \in \mathcal{K}^+$. We must show that if $(\delta_1, \delta_2) \in R^{\gamma_i(\pi)}$ and $C \in \Lambda_{\pi}(\delta_2)$ then $D \in \Lambda_{\pi}(\delta_1)$ for $\pi \in \sigma_i(s)$. Hence $\delta_2 = \delta'$. First, (**Case Loop**) we consider the -uniquely- self-loop case (where $\delta_1 = \delta'$) for all $\pi \in \Pi$. Then, (**Case Forward**) we assume that $\delta_1 \neq \delta'$, in which case we focus on $\pi = \pi^*$ since this is the only case that introduces a fresh relation into the saturation process, hence the other cases are trivial. Notice that (**Case Forward**) may contain loops but they do not need to be treated specially.

(**Case Loop**) Assume $\delta_1 = \delta_2 = \delta'$, $C \in \Lambda_{\pi}(\delta')$ and $(\delta', \delta') \in R^{\gamma_i(\pi)}$.

Case 1 : Assume $\pi = \pi^*$. Since $C \in \Lambda_{\pi}(\delta)$ and by Lemma 9 then $\Box_u[\top \sqsubseteq \Box_{\mathcal{S}_{\pi}}[F \Rightarrow C]] \in \mathcal{K}^+$. And since $(\delta', \delta') \in R^{\gamma_i(\pi)} \setminus R^{\gamma_{i-1}(\pi)}$, then there must be $\Box_{u'}[\top \sqsubseteq \Box_{\mathcal{S}_{\pi}}[F \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$. By application of rules L.2 and E.1,

we obtain $\Box_{S_\pi}[F \sqsubseteq \exists R.C] \in \mathcal{K}^+$. By the premise and Lemma 9, we obtain $\Box_{S_\pi}[\exists R.C \sqsubseteq D] \in \mathcal{K}^+$. Now by rule E.3 we obtain $\Box_*[\top \sqsubseteq \Box_{S_\pi}[F \Rightarrow D]] \in \mathcal{K}^+$ and hence $D \in \Lambda_\pi(\delta)$ as desired

Case 2 : Assume $\pi \in \Pi_i \setminus \{\pi^*\}$ and $\pi \in \sigma(s)$. Since $C \in \Lambda_\pi(\delta')$ then for some $G, H \in \text{Con}(F, \pi^*)$ there is $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C]] \in \mathcal{K}^+$ by Lemma 9. And since $(\delta', \delta') \in R^{\gamma_i(\pi)} \setminus R^{\gamma_{i-1}(\pi)}$, then there must be $\Box_{S_{\pi^*}}[H \sqsubseteq \Box_{S_\pi}[\top \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$.

- (1) $\Box_{S_\pi}[\exists R.C \sqsubseteq D] \in \mathcal{K}^+$ by the assumption
- (2) $\Box_{S_{\pi^*}}[E \sqsubseteq \exists R.F] \in \mathcal{K}^+$, by construction and Lemma 9
- (3) $\Box_{u_1}[\top \sqsubseteq \Box_{S_{\pi^*}}[F \Rightarrow G]] \in \mathcal{K}^+$
- (4) $\Box_{u_2}[\top \sqsubseteq \Box_{S_{\pi^*}}[F \Rightarrow H]] \in \mathcal{K}^+$
- (5) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C]] \in \mathcal{K}^+$, by construction and Lemma 9
- (6) $\Box_{S_{\pi^*}}[H \sqsubseteq \Box_{S_\pi}[\top \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by construction and Lemma 9
- (7) $\Box_{S_{\pi^*}}[F \sqsubseteq \Box_{S_\pi}[\top \Rightarrow C]] \in \mathcal{K}^+$, by (3), (5) and rule C.2
- (8) $\Box_{S_{\pi^*}}[F \sqsubseteq \Box_{S_\pi}[\top \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (4), (6) and rule C.2
- (9) $\Box_{S_\pi}[\exists R.\text{Self} \sqcap C \sqsubseteq D] \in \mathcal{K}^+$ by (1) and rule L.3
- (10) $\Box_{S_{\pi^*}}[F \sqsubseteq \Box_{S_\pi}[\top \Rightarrow D]] \in \mathcal{K}^+$, by (8), (9), (10) and rule E.4, and hence $D \in \Lambda_\pi(\delta)$ as desired.

Case 3 : Assume $\pi = \pi_{\delta', \diamond_s E}$ and $\pi \in \sigma(s)$. Since $C \in \Lambda_\pi(\delta')$ then for some $G, H \in \text{Con}(F, \pi^*)$ by Lemma 9 there is $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[E \Rightarrow C]] \in \mathcal{K}^+$ and also $\Box_{S_{\pi^*}}[H \sqsubseteq \Box_{S_\pi}[E \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$

- (1) $\Box_{S_\pi}[\exists R.C \sqsubseteq D] \in \mathcal{K}^+$ by the assumption
- (2) $\Box_{S_{\pi^*}}[E \sqsubseteq \exists R.F] \in \mathcal{K}^+$, by construction and Lemma 9
- (3) $\Box_{u_1}[\top \sqsubseteq \Box_{S_{\pi^*}}[F \Rightarrow G]] \in \mathcal{K}^+$
- (4) $\Box_{u_2}[\top \sqsubseteq \Box_{S_{\pi^*}}[F \Rightarrow H]] \in \mathcal{K}^+$
- (5) $\Box_{S_{\pi^*}}[E \sqsubseteq \exists R.F] \in \mathcal{K}^+$, by construction and Lemma 9
- (6) $\Box_{S_{\pi^*}}[G \sqsubseteq \Box_{S_\pi}[E \Rightarrow C]] \in \mathcal{K}^+$, by construction and Lemma 9
- (7) $\Box_{S_{\pi^*}}[H \sqsubseteq \Box_{S_\pi}[E \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by construction and Lemma 9
- (8) $\Box_{S_{\pi^*}}[F \sqsubseteq \Box_{S_\pi}[E \Rightarrow C]] \in \mathcal{K}^+$, by (3), (6) and rule C.2
- (9) $\Box_{S_{\pi^*}}[F \sqsubseteq \Box_{S_\pi}[E \Rightarrow \exists R.\text{Self}]] \in \mathcal{K}^+$, by (4), (7) and rule C.2
- (10) $\Box_{S_\pi}[\exists R.\text{Self} \sqcap C \sqsubseteq D] \in \mathcal{K}^+$ by (1) and rule L.3
- (11) $\Box_{S_{\pi^*}}[F \sqsubseteq \Box_{S_\pi}[E \Rightarrow D]] \in \mathcal{K}^+$, by (8), (9), (10) and rule E.4, and hence $D \in \Lambda_\pi(\delta')$ as desired

(Case Forward) Now, since we have $(\delta_1, \delta') \in R^{\gamma_{i+1}(\pi)}$ then by the saturation process that constructs $[R^{\gamma_{i+1}(\pi)}]_k$ there must be a sequence of domain elements $\delta'_1, \dots, \delta'_m, \delta'_{m+1}$ with $\delta'_1 = \delta_1$ and $\delta'_{m+1} = \delta$. We notice that by construction we have the following formulas:

- (P1) We have $C = C'_{m+1}$, $R = R'_{1|m}$ and $T = R_{m|m}$
- (P2) $C_j, C'_j \in \Lambda_\pi(\delta'_j)$ for all $j \in \{k, \dots, m\}$
- (P3) There is some $k \in \{1, \dots, m-1\}$ such that
 - (P3.a) $\Box_{S_\pi}[C'_j \sqsubseteq \exists R_{j|j}.C_{j+1}] \in \mathcal{K}^+$ for all $j \in \{k, \dots, m\}$, and
 - (P3.b) $\Box_{S_\pi}[R_{j|j}(a_j, a_{j+1})] \in \mathcal{K}^+$ for all $j \in \{1, \dots, k-1\}$
- (P4) $\Box_{S_\pi}[R_{j|j} \sqsubseteq R'_{j|j}], \Box_{S_\pi}[R_{j|m} \sqsubseteq R'_{j|m}] \in \mathcal{K}^+$ (Notice that by axiom T.5 and Lemma 9 we can obtain formulas $\Box_{S_\pi}[R' \sqsubseteq R] \in \mathcal{K}^+$)
- (P5) If $1 \leq j < m$, $\Box_{S_\pi}[R'_{j|j} \circ R'_{j+1|m} \sqsubseteq R_{j|m}] \in \mathcal{K}^+$

We first show that we can obtain $\Box_{S_\pi}[C'_m \sqsubseteq \exists R'_{m|m}.C'_{m+1}] \in \mathcal{K}^+$:

- (1) $\Box_u[\top \sqsubseteq \Box_{S_\pi}[C_{m+1} \Rightarrow C'_{m+1}]] \in \mathcal{K}^+$ from $C \in \Lambda_\pi(\delta'_{m+1})$ by construction with $C = C'_{m+1}$
- (2) $\Box_{S_\pi}[C'_m \sqsubseteq \exists R_{m|m}.C_{m+1}] \in \mathcal{K}^+$ by the assumption of **Case 1**
- (3) $\Box_{S_\pi}[R_{m|m} \sqsubseteq R'_{m|m}] \in \mathcal{K}^+$ by construction (P4)
- (4) $\Box_{S_\pi}[C'_m \sqsubseteq \exists R'_{m|m}.C'_{m+1}] \in \mathcal{K}^+$ by (1), (2), (3) and rule E.1

We show that for $j > k$, if we have $\Box_{S_\pi}[C'_j \sqsubseteq \exists R'_{j|m}.C'_{m+1}] \in \mathcal{K}^+$ then we can obtain $\Box_{S_\pi}[C'_{j-1} \sqsubseteq \exists R'_{j-1|m}.C'_{m+1}] \in \mathcal{K}^+$.

- (1) $\Box_u[\top \sqsubseteq \Box_{s_\pi}[C_j \Rightarrow C'_j]] \in \mathcal{K}^+$ from $C'_j \in \Lambda_\pi(\delta'_j)$ by construction
- (2) $\Box_{s_\pi}[C'_{j-1} \sqsubseteq \exists R_{j-1|j-1}.C_j] \in \mathcal{K}^+$ by construction (P3.a)
- (3) $\Box_{s_\pi}[R_{j-1|j-1} \sqsubseteq R'_{j-1|j-1}] \in \mathcal{K}^+$ by construction (P4)
- (4) $\Box_{s_\pi}[R'_{j-1|j-1} \circ R'_{j|m} \sqsubseteq R_{j-1|m}] \in \mathcal{K}^+$ by construction (P5)
- (5) $\Box_{s_\pi}[C'_j \sqsubseteq \exists R'_{j|m}.C'_{m+1}] \in \mathcal{K}^+$ by inductive hypothesis
- (6) $\Box_{s_\pi}[C'_{j-1} \sqsubseteq \exists R'_{j-1|j-1}.C'_j] \in \mathcal{K}^+$ by (1), (2), (3) and rule E.1
- (7) $\Box_{s_\pi}[C'_{j-1} \sqsubseteq \exists R'_{j-1|m}.C'_{m+1}] \in \mathcal{K}^+$ by (4), (5), (6) and rule E.2

Now, assume $k = 1$; we show the base case where $j = 1$.

- (1) $\Box_{s_\pi}[\exists R'_{1|m}.C'_{m+1} \sqsubseteq D] \in \mathcal{K}^+$ by the assumption
- (2) $\Box_{s_\pi}[C'_1 \sqsubseteq \exists R'_{1|m}.C'_{m+1}] \in \mathcal{K}^+$ by the inductive hypothesis
- (3) $\Box_u[\top \sqsubseteq \Box_{s_\pi}[F \Rightarrow C'_1]] \in \mathcal{K}^+$ from $C'_1 \in \Lambda_\pi(\delta')$ by construction
- (4) $\Box_*[\top \sqsubseteq \Box_{s_\pi}[C'_1 \Rightarrow D]] \in \mathcal{K}^+$ by (1), (2) and rule E.3
- (5) $\Box_u[\top \sqsubseteq \Box_{s_\pi}[C'_1 \Rightarrow D]] \in \mathcal{K}^+$ by (4), and rule S.3
- (6) $\Box_u[\top \sqsubseteq \Box_{s_\pi}[F \Rightarrow D]] \in \mathcal{K}^+$ by (3), (5) and rule C.1 and hence $D \in \Lambda_\pi(\delta_1)$ as desired

Else, in case $j = k \neq 1$, we first we obtain

- (1) $\Box_{s_\pi}[R'_{j-1|j-1}(a_{j-1}, a_j) \sqsubseteq D] \in \mathcal{K}^+$ by assumption (P3.b)
- (2) $\Box_{s_\pi}[C'_j \sqsubseteq \exists R'_{j|m}.C'_{m+1}] \in \mathcal{K}^+$ by assumption
- (3) $\Box_*[\{a_j\} \sqsubseteq \Box_{s_\pi}[\top \Rightarrow C'_j]] \in \mathcal{K}^+$
- (4) $\Box_{s_\pi}[R'_{j-1|j-1} \circ R'_{j|m} \sqsubseteq R_{j|m}] \in \mathcal{K}^+$ by assumption (P5)
- (5) $\Box_{s_\pi}[R_{j-1|m} \sqsubseteq R'_{j-1|m}] \in \mathcal{K}^+$ by construction (P4)
- (6) $\Box_*[\{a_{j-1}\} \sqsubseteq \exists R_{j-1|m}.C'_{m+1}] \in \mathcal{K}^+$ by (1), (2), (3), (4) and rule A.8
- (7) $\Box_*[\{a_{j-1}\} \sqsubseteq \exists R'_{j-1|m}.C'_{m+1}] \in \mathcal{K}^+$ by (5), (6) and rule A.4

Else, in case $j \in \{2, \dots, k-1\}$ if we have $\Box_*[\{a_j\} \sqsubseteq \exists R'_{j|m}.C'_{m+1}] \in \mathcal{K}^+$ then we can obtain $\Box_*[\{a_{j-1}\} \sqsubseteq \exists R'_{j-1|m}.C'_{m+1}] \in \mathcal{K}^+$.

- (1) $\Box_{s_\pi}[R_{j-1|j-1}(a_{j-1}, a_j) \sqsubseteq D] \in \mathcal{K}^+$ by construction (P3.b)
- (2) $\Box_*[\{a_j\} \sqsubseteq \exists R'_{j|m}.C'_{m+1}] \in \mathcal{K}^+$ by assumption
- (3) $\Box_{s_\pi}[R'_{j-1|j-1} \circ R'_{j|m} \sqsubseteq R_{j|m}] \in \mathcal{K}^+$ by assumption (P5)
- (4) $\Box_{s_\pi}[R_{j-1|m} \sqsubseteq R'_{j-1|m}] \in \mathcal{K}^+$ by construction (P4)
- (5) $\Box_*[\{a_{j-1}\} \sqsubseteq \exists R_{j-1|m}.C'_{m+1}] \in \mathcal{K}^+$ by (1), (2), (3) and rule A.7
- (6) $\Box_*[\{a_{j-1}\} \sqsubseteq \exists R'_{j-1|m}.C'_{m+1}] \in \mathcal{K}^+$ by (4), (5) and rule A.4

Finally, assume the base case where $j = 1$ and $k \neq 1$

- (1) $\Box_{s_\pi}[\exists R'_{1|m}.C'_{m+1} \sqsubseteq D] \in \mathcal{K}^+$ by the assumption
- (2) $\Box_*[\{a_1\} \sqsubseteq \exists R'_{1|m}.C'_{m+1}] \in \mathcal{K}^+$ by induction
- (3) $\Box_u[\top \sqsubseteq \Box_{s_\pi}[\{a_1\} \Rightarrow D]] \in \mathcal{K}^+$ by (1), (2) and rule E.3
- (4) $\Box_u[\{a_1\} \sqsubseteq \Box_{s_\pi}[\top \Rightarrow D]] \in \mathcal{K}^+$ by (3) and rule I.1, $D \in \Lambda_\pi(\delta_1)$ as desired

$\phi = \Box_s(C(a))$ Assume $\Box_s(C(a)) \in \mathcal{K}$ (i.e. $\Box_*[\{a\} \sqsubseteq \Box_s[\top \Rightarrow C]] \in \mathcal{K}^+$). Then, by construction $\delta_a = a^{\gamma_0}$ and $C \in \Lambda_\pi(\delta_a)$.

$\phi = \Box_s(R(a, b))$ Assume $\Box_s(R(a, b)) \in \mathcal{K}$. Then, by construction $\delta_a = a^{\gamma_0}$, $\delta_b = b^{\gamma_0}$ and $(\delta_a, \delta_b) \in R^{\gamma_0(\pi)}$ for all $\pi \in \sigma_0(s)$. Thus also by construction $\delta_a = a^\gamma$, $\delta_b = b^\gamma$ and $(\delta_a, \delta_b) \in R^{\gamma(\pi)}$ for all $\pi \in \sigma(s)$ and hence $\mathfrak{D} \models \Box_s(R(a, b))$.

With this, Claim 7 is proved.

Now, returning to the proof of the completeness theorem, we show that it follows from the claim above. It remains to show that \mathfrak{D} is indeed a model. First, we observe that clearly, by the construction of Π_0 and σ_0 , no standpoint is empty. Moreover, by axiom T.1 we obtain that $\sigma_0(*) = \Pi_0$ as required, and we remark that by construction this carries to $\sigma(*) = \Pi$. Finally, we

need to make sure that for all $\pi \in \Pi$ and $\delta \in \Delta$, $\perp \notin \Lambda_\pi(\delta)$, so all domain elements can be instantiated in all precisifications.

Assume for the sake of contradiction that $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]] \notin \mathcal{K}^+$ but in \mathfrak{D} there is $\perp \in \Lambda_\pi(\delta')$ for some $\delta' \in \Delta$. First, assume that $\delta' \in \Delta_0$ of \mathfrak{D}_0 , and thus $\perp \in \Lambda_\pi(\delta')$. Then by construction there is some $\Box_u[\{a\} \sqsubseteq \Box_{s_\pi}[\top \Rightarrow \perp]] \in \mathcal{K}^+$ for some named individual a . But then by rule B.4 we have $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]] \in \mathcal{K}^+$, thus reaching a contradiction. Thus, if $\delta' \notin \Delta_0$ of \mathfrak{D}_0 , there must be a \mathfrak{D}_{i+1} with $\delta' \in \Delta_{i+1} \setminus \Delta_i$ and $\perp \in \Lambda_\pi(\delta')$. Notice that the iteration has been triggered at π^* to satisfy an axiom of the form $\Box_t[E \sqsubseteq \exists R.F]$, and by Lemma 9 we have $\Box_{s_{\pi^*}}[E \sqsubseteq \exists R.F]$.

Case $\pi = \pi^*$ By construction we have $\Lambda_{\pi^*}(\delta') = \{A \mid \Box_u[\top \sqsubseteq \Box_s[F \Rightarrow A]] \in \mathcal{K}^+, s \in \sigma^{-1}(\pi^*)\}$ and thus by the assumption we have some $\Box_u[\top \sqsubseteq \Box_{s_{\pi^*}}[F \Rightarrow \perp]]$. By axiom T.5 and Lemma 9 we obtain also $\Box_{s_{\pi^*}}[R \sqsubseteq R]$ and by rule E.1 we get $\Box_{s_{\pi^*}}[E \sqsubseteq \exists R.\perp]$. Then, by rule B.1 we obtain $\Box_*[\top \sqsubseteq \Box_{s_{\pi^*}}[E \Rightarrow \perp]]$. This means that in \mathfrak{D}_i there is $\perp \in \Lambda_{\pi^*}(\delta^*)$ since $\delta^* \in \Delta_i$.

Case $\pi \in \Pi_i \setminus \pi^*$ By construction and Lemma 9 if $\perp \in \Lambda_\pi(\delta')$ then $\Box_{s_{\pi^*}}[G \sqsubseteq \Box_{s_\pi}[\top \Rightarrow \perp]] \in \mathcal{K}^+$ for $G \in \Lambda_{\pi^*}(\delta')$. By rule B.2 we obtain $\Box_*[\top \sqsubseteq \Box_{s_{\pi^*}}[G \Rightarrow \perp]] \in \mathcal{K}^+$, which leads to $\perp \in \Lambda_{\pi^*}(\delta')$ and, by the previous case (**Case $\pi = \pi^*$**), in \mathfrak{D}_i there is $\perp \in \Lambda_{\pi^*}(\delta^*)$ since $\delta^* \in \Delta_i$.

Case $\pi = \pi_{\delta', \diamond_s E}$ By construction and Lemma 9 if $\perp \in \Lambda_{\pi_{\delta', \diamond_s E}}(\delta')$ then $\Box_{s_{\pi^*}}[G \sqsubseteq \Box_{s_\pi}[E \Rightarrow \perp]] \in \mathcal{K}^+$ and $\Box_{s_{\pi^*}}[H \sqsubseteq \diamond_{s_\pi} E] \in \mathcal{K}^+$ for $G, H \in \Lambda_{\pi^*}(\delta')$. By rule C.2 we obtain that also $\Box_{s_{\pi^*}}[F \sqsubseteq \Box_{s_\pi}[E \Rightarrow \perp]] \in \mathcal{K}^+$ and by rule C.3 we obtain $\Box_{s_{\pi^*}}[F \sqsubseteq \diamond_{s_\pi} E] \in \mathcal{K}^+$. Then by rule C.4 we obtain $\Box_{s_{\pi^*}}[F \sqsubseteq \diamond_{s_\pi} \perp] \in \mathcal{K}^+$ and by rule B.3 we get $\Box_*[\top \sqsubseteq \Box_{s_{\pi^*}}[F \Rightarrow \perp]] \in \mathcal{K}^+$. Hence, by the first case (**Case $\pi = \pi^*$**) in \mathfrak{D}_i there is $\perp \in \Lambda_{\pi^*}(\delta^*)$ since $\delta^* \in \Delta_i$.

Thus, if there is some $\delta \in \Delta_i$ of \mathfrak{D}_i such that $\perp \in \Lambda_\pi(\delta)$, then there is $\delta' \in \Delta_0$ of \mathfrak{D}_0 such that $\perp \in \Lambda_\pi(\delta')$, but as we showed this implies $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]] \in \mathcal{K}^+$, thus reaching a contradiction. \square