Master's Thesis

Morphisms in Logic, Topology, and Formal Concept Analysis

by

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Abstract

The general topic of this thesis is the investigation of various notions of *morphisms* between logical deductive systems, motivated by the intuition that additional (categorical) structure is needed to model the interrelations of formal specifications. This general task necessarily involves considerations in various mathematical disciplines, some of which might be interesting in their own right and which can be read independently.

To find suitable morphisms, we review the relationships of formal logic, algebra, topology, domain theory, and formal concept analysis (FCA). This leads to a rather complete exposition of the representation theory of algebraic lattices, including some novel interpretations in terms of FCA and an explicit proof of the cartestian closedness of the emerging category. It also introduces the main concepts of “domain theory in logical form” for a particularly simple example.

In order to incorporate morphisms from FCA, we embark on the study of various context morphisms and their relationships. The discovered connections are summarized in a hierarchy of context morphisms, which includes *dual bonds*, *scale measures*, and *infomorphisms*.

Finally, we employ the well-known means of Stone duality to unify the topological and the FCA-based approach. A notion of logical consequence relation with a suggestive proof theoretical reading is introduced as a morphism between deductive systems, and special instances of these relations are identified with morphisms from topology, FCA, and lattice theory. Especially, scale measures are recognized as topologically continuous mappings, and infomorphisms are identified both with coherent maps and with Lindenbaum algebra homomorphisms.

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Although many people contributed to my academic education, the knowledge that enabled me to write this thesis largely goes back to two people: Pascal Hitzler and Matthias Wendt. My thanks to Pascal cannot possibly account for his influence on my studies, which traces back to my first contact with formal logic in undergraduate courses. Over the years, he provided me with numerous opportunities, hints, discussions, and an inexhaustible optimism that was often a major source of my motivation.

The discussions with Matthias have been extremely inspiring, though I was usually content to follow his ideas – at least in parts. My understanding of algebraic semantics, Stone duality, and also *Logic* in general, mainly goes back to this influence. I regret that, now that I come to comprehend some of these topics, he is already concerned with new subjects beyond my current mathematical horizon.

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Contents

Chapter 1

Introduction

In Computer Science, formal logics generally are perceived as a tool for *specification* and *reasoning*, where the latter – partly due to the efforts of Proof Theory – is often identified with a process of *computation*. This intuition turns out to be feasible for many logical formalisms, and today numerous concrete implementations of reasoning mechanisms are available. Classically, such implementations are the domain of *logic programming* [Llo87], but growing demands lead to developments in other areas as well. Most recently, *ontology research* opened up new applications for knowledge representation and reasoning, and gave rise to novel logic-based reasoning formalisms, such as *F-Logic* [KLW95] or *Description Logic* [BCM+03].

Many more approaches, both theoretical and practical in nature, engaged in similar efforts to provide means of specification and reasoning for some particular application area. However, in most cases, "specification and reasoning" restricts to the specification of and the reasoning on top of some particular deductive system (i.e. logic program, ontology, . . .). What is often neglected is the question of how to specify the relationships *between* such deductive systems and how to infer consequences for such interrelations. Nonetheless this question appears to be vital for the success of some – probably most – of the targeted applications of formal logics. On the one hand, use-cases of practical dimensions can hardly be based on a single huge specification, but will rather require *modularization* into numerous smaller ones. Reasoning in such a setting clearly requires the specification not only of the modules themselves, but also of the exact relationships between them. On the other hand, situations with even higher levels of heterogeneity naturally occur in ontology research, e.g. in the context of a *semantic web*. There, one faces a scenery of multiple distributed specifications which may not even use a common logical language, and which have not been conceived as modules of some overall deductive system. This situation represents a considerable challenge to current research, and neither theoretical nor practical approaches to this problem are developed to a satisfactory extent.

Given the amount and diversity of available logical formalisms, one obviously cannot expect this problem to have a simple solution. In fact, the first question that arises is how to specify the aforementioned "relationships" between deductive systems at all. Initially, one is faced with a mere collection of specifications, lacking additional structure that could be used for interrelating them. A priori, it is not clear how this additional structure should look like, and indeed there might be various reasonable choices, strongly depending on the particular kind of logical formalisms that are to be taken into account. However, the primitive concepts of such investigations most certainly are the relationships between a single pair of specifications. In ontology research, such relationships are sometimes called *ontology mappings* [KS03]. In this generality, this notion does not yield a lot of structural information, and we therefore make the additional assumption that relationships between specifications have a *direction*. This can be justified on practical grounds as well, since relationships between specifications often come with a preferred direction for the *flow of information*. Examples include modules which are to be included into some bigger specification, and ontologies that have been gathered from the semantic web to be processed in (the deductive system of) a local reasoner.

Of course this setting still appears to be very abstract. Yet directed relationships between objects are a familiar concept in mathematics, where they are generally referred to as "morphisms." Now such morphisms usually come with the additional property that they can be *composed* in a well behaved way.[1] This actually is reasonable from a practical viewpoint as well: if one is given a relationship between specifications A and B, and another relationship between specifications B and C, then it should also be possible to compose these relations to relate A to C. Nonetheless, it should be remarked that sufficiently well behaved compositions may not be available for all imaginable notions of morphisms.[2] Given a means of composing morphisms, one usually expects that there are *identity* morphisms from each specification to itself, acting as a neutral element to composition. Intuitively, such relationships correspond to the possibility of relating every part of a deductive system to itself. In another reading, identity morphisms represent translations of the content of a particular specification into itself, in a way that does neither add nor remove information.

Summing up, we wish to consider logical specifications together with a collection of mutual interrelations, called morphisms, which can be composed in well mannered way that allows for identity relations. In other words we are inter-

[1] "Well behaved" essentially means "associative" but we save formal details for later chapters.

[2] Especially, it is to be expected that ontology research, where a great amount of possible types of ontology-mappings has been proposed, came up with such unpleasant relationships. It is beyond our current ability and interest to provide a theoretical basis for these approaches as well.

ested in *categories of specifications*, which, quite naturally, are the topic of this work. Categories, which are indeed just collections of objects with the very simple structural constraints introduced above, have been studied intensively in the last decades (see e.g. [Bor94a, LR03, Mac71, McL92]), and a wealth of results is immediately available when dealing with such a structure. In particular, a surprisingly rich amount of concrete constructions can be defined only based on the structure of morphisms, and these constructions are also of interest when dealing with specifications. Typical examples include the construction of a specification from its parts or the merging of ontologies (see [KHES04] for a gentle introduction).

However, the focus of this work is not to give a general account of the possible applications of category theory in knowledge representation and reasoning. Instead, we consider very concrete categories of propositional logics and compare known logical morphisms in this context. Nevertheless our view on propositional logic is quite general. Especially, our investigations are simplified by not restricting logical languages in size, i.e. by allowing for uncountable sets of atomic formulae. A deductive system[3] of such propositional logics is not at all trivial: since infinite theories are taken into account, the *grounded* versions of logic programs are just special cases of this setting.

Although the central motif of this work is this logical view, the results obtained *en route* are interesting in their own right. Our findings are shortly summarized in the outline of the chapters which is given below.

Review: morphisms in logic

As explained above, the available supply of theoretically sound notions of morphisms between logics is rather small. A notable exception from the general disinterest for logical categories is *Institution Theory* [GB92], which goes back to the 80s and which encompasses a broad range of logical formalisms. The aims of the theory largely agree with the aforementioned general motives for the use of categories, though the aspect of modular logical specifications received particular interest in the first decades, leading even to the development of category theory based programming languages.

The basic principle of institution theory is the representation of logics in terms of their model theories. More precisely, the theory considers formalisms that can be described via a semantical consequence relation $\models$ between models and formulae. All further investigations are founded on binary relations in place of deductive systems. The predominant type of morphisms between these relations are so-called *infomorphisms*, each described by a mapping on formulae and a map-

[3] I.e. a logical calculus together with a background theory of presupposed assertions.

ping on models in the opposite direction, with the property that the image of a formula relates to a given model if and only if the image of the model relates to the formula.

These morphisms have several advantages: other than being motivated in logical terms, they can easily be described for arbitrary binary relations and they lead to some pleasing properties of the resulting categories. The latter reason also motivates the usage of this definition in other mathematical areas, for example in the theory of Chu spaces [Pra03]. On the other hand, the framework of institution theory is rater general, and it is not always clear how it relates to other possible morphisms that appear in concrete settings. Nonetheless, institution theory inspired a recent theory of Information Flow [BS97], which takes a similar categorical viewpoint based on the same notions of objects and morphisms.

Another ramification of institution theory has not been exploited yet. Binary relations as the basic objects of study are known as *formal contexts* in Formal Concept Analysis (FCA). In turn, FCA provides a number of possible morphisms, though the interrelation of these is not well understood either. However, this raises questions concerning the relevance of morphisms from FCA for logical investigations. Two such morphisms will turn out to be particularly interesting: *dual bonds*, a special type of binary relation between formal contexts, and *scale measures*, a class of functions that is characterized by certain continuity properties.

In contrast to these morphisms, part of which – to the best of our knowledge – have not yet been considered from a logical viewpoint at all, there is another collection of morphisms whose relationship to propositional logics is known for more than 70 years. It is based on Marshall Stone's celebrated representation theorems for *Boolean algebras* [Sto36, Sto37a] and *Brouwerian* (aka *intuitionistic*) *logics* [Sto37b]. From a logical perspective, these representation theorems can be explained as follows. First note that any logical formula – up to semantical equivalence – is described by the set of its models. Now one considers the collection of all sets of models that arise in this way. It turns out that this collection with the order of subset inclusion is a Boolean algebra, and that this algebra is isomorphic to the set of logical formulae, ordered by logical entailment and with semantically equivalent formulae identified. This is not surprising yet, since the relation of Boolean algebras and classical propositional logic was well known for a long time.

Now Stone's important step was to recognize that one can construct a topological space[4] from this Boolean algebra of sets of models by taking arbitrary unions of such sets, and that this process can be inverted to obtain Boolean algebras from certain topological spaces. Thus he obtained a correspondence of Boolean alge-

[4]This is just a system of sets (called "open sets") that is closed under arbitrary unions and finite intersections. For details see Section 2.4.

bras and a class of topologies that is now known as *Stone spaces*. This result extends to morphisms as well: homomorphisms of Boolean algebras correspond exactly to *coherent maps* between the associated topological spaces. But the topological perspective enables us to import further morphisms as well: the typical morphisms in topology are *continuous maps* and we will subsequently study their connection to the other candidates of logical morphisms mentioned above.

Connections to domain theory

Stone's duality theorems have been generalized to other types of order structures, finally leading to the creation of *locale theory* ("pointless topology") as an alternative to classical point-set topology. Surprisingly, this line of research exposed connections to *domain theory* [AJ94, GHK$^+$03], a branch of order theory that was originally established as a tool for constructing denotational semantics for certain *lambda calculi*. In consequence, domain theory encompasses various important aspects of automatic computation: most notably it formalizes *approximation* and it generally supports a wealth of *type constructions*.[5]

In its role as a framework for modelling computational processes, domain theory also includes notions of computational feasibility, describing circumstances under which approximating computations reach fixed points after only countably many iterations. The basic objects of study in domain theory are partially ordered sets which have specific properties to support such computations. The common term for these structures is "domains" though no particular definition is generally associated with this terminology.

The connection to topology and to Stone duality is made by defining topologies on a domain, where the order structure is employed to characterize open sets. For the converse, the points of a topology are ordered based on the collection of open sets within which they are contained.[6] The emerging connections to Stone duality lead to Abramsky's "Domain theory in logical form" [Abr91] and gave rise to numerous further studies [Bon98, DG90, CC00, CZ00, Zha91, Zha92].

The link between logics and domain theory is highly relevant for Computer Science since it establishes a correspondence of denotational and axiomatic semantics, i.e. of computation and specification. Motivated both by the possibility to connect logical considerations to aspects of computation and by the pleasing mathematical relationships, domain theoretical notions will also be discussed within this thesis.

[5] Here "type" has the usual meaning of Computer Science as a collection of input and output values. Typical examples of type constructions are cartesian products and function spaces.

[6] This yields the *order of specialization*, see Section 2.4.

Outline of this work

We shortly review the contents of the subsequent chapters and point out the main results from each of them. As a general rule, all theorems that occur elsewhere in the literature – to the extent of being known to the author – are properly attributed to their respective sources, such that the exact contributions of each chapter should become clear.

The following Chapter 2 gives a general introduction to the mathematical prerequisites needed to follow the rest of this work. Specifically, these include basic notions from order and lattice theory, and an extra section on the according morphisms with emphasis on Galois connections and closure operators. The presentation of Galois connections accounts for both the antitone and the monotone versions found in the literature, since the former is more common in domain theory while the latter is preferred in FCA. We then give an introduction to the mathematical basics of FCA and introduce some extended notation for images and preimages of binary relations. Our following exposition of notions from domain theory, general topology and category theory is again fairly standard. Since our exposition is restricted in space, we advice the reader to consult the cited textbooks for a more thorough introduction to the respective fields. Only our treatment of Galois connections and FCA shows some slight deviations from the literature. Likewise, the chosen notation should yield little surprises to the knowledgeable reader, who may thus prefer to skip familiar material and come back only when additional details are needed. The index at the end of this work is intended to aid this strategy.

Chapter 3 embarks on the representation theory of algeraic lattices, thus emphasizing the mentioned domain theoretical perspective on logics. The main contribution of this chapter lies in relating the aforementioned areas – domain theory, logic, FCA, lattice theory, and topology – for the concrete case of algebraic lattices. This setting is particularly simple and, in consequence, is well suited for a detailed introduction to Stone duality. Although much of the material covers standard results of the involved areas, no similarly extensive treatement is known to us. This chapter will also prepare the consideration of topological morphisms for deductive systems in Chapter 5.

Further major results of Chapter 3 are found in Section 3.3, where a novel representation of algebraic lattices via formal concepts is studied. This representation was originally proposed in [ZS0x] and first lifted to a categorical setting in [HZ04]. In the present work, we enhance our understanding of this approach by relating it to various other means of representing algebraic lattices. In addition, *cartesian closedness* of the emerging category of formal contexts is shown explicitly by giving new descriptions and proofs for the required categorical constructions.

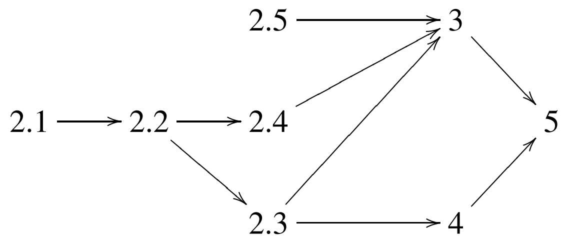
In Chapter 4 we prepare our later consideration of morphisms from FCA within the logical context of Chapter 5. The aim of Chapter 4 thus is to understand the relationships between certain well-known morphisms from FCA, most notably between *dual bonds*, *scale measures*, and *infomorphisms*. Our investigations will finally lead us to the insight that these apparently diverse morphisms can be accurately characterized as special cases of dual bonds. Using *attribute exploration*, the resulting hierarchy of context morphisms is cast into the formal context of Figure 4.2 that summarizes the essential relationships established in Chapter 4. To the best of our knowledge, most of the results of Chapter 4 are new, with the exception of some theorems taken from one of [GW99, Gan04, Xia93].

Finally, Chapter 5 unifies the considerations of Stone duality, topology and FCA in a common logical setting. We formaly explain the representation of various propositional logics in terms of formal contexts and review some known relations to Stone duality. The decisive step then is to recognize dual bonds between *logical contexts* as a multi lingual version of the common syntactical consequence relations known from proof theory. In consequence, our work yields a general framework for the interpretation of such consequence relations in terms of topology and FCA. This connects up with [JKM99], where similar relations in their classical proof theoretical formulation were studied as *multi lingual sequent calculi* between non-reflexive positive logics.

It is then shown that deductive systems and consequence relations constitute categories. The exact relationship to topology and FCA become apparent in Theorem 5.3.1, where a sub-class of consequence relations is shown to correspond to (i) continuous functions between the associated topological spaces, (ii) to scale measures between the associated formal contexts, and (iii) to frame homomorphisms between the associated concept lattices. Thus we arrive at the well-known connection between continuous functions and frame homomorphisms by the route of formal concept analysis. A second main result is Theorem 5.4.2, where we further specialize the considered class of consequence relations to obtain a correspondence between (i) coherent functions of topology, (ii) infomorphisms of FCA, and (iii) homomophisms of the Lindenbaum algebras of the deductive systems. These are the main results of Chapter 5, establishing the desired relationships between morphisms from logic, topology, institution theory and FCA. Most of these results are new, the only exception being some standard facts from Stone duality.

Interdependence of the chapters

As expounded above, the general theme of this work is the investigation of morphisms in logical settings. Yet, some parts of this work can be read rather independently. Especially, this applies to Chapters 3 and 4, the contents of which largely corresponds to the papers [HKZ04] and [KHZ05], respectively. The following graph describes the interdependence of the various parts of this thesis:



Chapter 2

Preliminaries

In order to make this work as self-contained as possible, the current chapter will present most of the mathematical preliminaries that are required to understand what follows. We shall assume the reader to be familiar with naive set theory, while everything else is expounded below. However, for readers without prior knowledge of a given area, it will usually be preferable to consult some of the more easy-paced treatments which we highlight at the beginning of each section. In particular, our introduction of logics in mainly algebraic terms, without any reference to their actual purpose of knowledge representation and reasoning, presumes that the reader already has some intuitions about the practical use of formal logics.

Not all of the preliminaries are required to follow specific parts of this thesis, so the reader may prefer to skip most of what follows and come back when additional background or notation is needed. We will try to give appropriate reference to the according parts of this chapter when using concepts and results later on. Also note that there is a list of symbols and an index at the end of this work.

The following sections collect material in a way that is motivated by our later usage. Section 2.1 treats partially ordered sets and lattices, before Section 2.2 introduces the appropriate morphisms, including Galois connections and closure operators. Section 2.3 makes use of these basics to introduce formal concept analysis whereas Section 2.4 develops order theory in another direction to present domains and the related parts of topology. Finally, Section 2.5 introduces necessary facts from category theory.

2.1 Orders and lattices

This section introduces the basics of order theory and the related field of lattice theory. Together with additional introductory remarks and numerous illustrating examples, the following can also be found in [DP02] or online at [WP, Article "Order Theory"]. More in depth treatments of order theory are to be found in [Bir73, GHK+03].

Definition 2.1.1 A *partial order* is a relation $\leq$ on some set P which is reflexive ($x \leq x$), antisymmetric ($x \leq x'$ and $x' \leq x$ implies $x = x'$), and transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$). A *partially ordered set* (*poset*) is a tuple $(P, \leq)$, where $\leq$ is a partial order on the set P. If no confusion is likely, a poset $(P, \leq)$ will be denoted by its carrier P. Given elements $x, y \in P$, x is *smaller than* (or *below*) y if $x \leq y$.

For a poset $(P, \leq)$, its *order dual* P^{op} is defined to be the poset $(P, \geq)$, with $\geq$ being the inverse relation of $\leq$ as usual.

Given a poset $(P, \leq)$ any subset $S \subseteq P$ induces a subposet $(S, \leq|_S)$ obtained by restricting the order of P. Another way for obtaining new posets is to multiply two partially ordered sets.

Definition 2.1.2 Given posets P and Q, the *product* $P \times Q$ is defined to be the cartesian product of the carrier sets together with the order defined by

$$(p, q) \leq (p', q') \text{ if and only if } p \leq_P p' \text{ and } q \leq_Q q'.$$

We are often interested in the following constructions within posets.

Definition 2.1.3 Consider a poset P and a subset $X \subseteq P$. An *upper bound* of X in P is an element which is greater than all elements of X. An element of P is the *least upper bound* (*supremum, join*) of X in P, denoted $\bigvee X$, if it is smaller than all upper bounds of X. For two-element sets we denote $\bigvee\{x, y\}$ by $x \vee y$. *(Greatest) lower bounds* are defined dually (with dual notation $\bigwedge X$ and $x \wedge y$).

When dealing with more than one poset at a time, we will sometimes annotate $\leq$, $\bigvee$, $\wedge$, etc. with the name of the poset that they refer to, thus writing $\leq_P$, $\bigvee_Q$, $\wedge_L$, etc.

The supremum of the empty set (or, equivalently, the infimum of the whole poset) is the *least element* $\bot$ of the poset. Dual remarks apply to the *greatest element* $\top$. The observation that suprema and infima need not exist for all sets gives rise to the next definition.

Definition 2.1.4 A poset P is a *join-semilattice* if any two elements of P have a join (supremum). *Meet-semilattices* are defined dually. A *lattice* is a poset which is both a meet- and a join-semilattice. It is *bounded* if it has a least and a greatest element. A lattice L is *distributive* if one finds that $x \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z)$

holds for all elements $x, y, z \in L$ (which is equivalent to the dual condition with $\vee$ and $\wedge$ exchanged).

A poset is a *complete lattice* if all of its subsets have both a supremum and an infimum.

We recall the standard result that a poset which has all infima also has all suprema, and vice versa, so that one of these conditions is in fact sufficient to define complete lattices.

We give some easy examples, starting with a complete lattice that we will deal with throughout this document.

Example 2.1.5 Given some set G, the *powerset* of G is the set $2^G := \{O \mid O \subseteq G\}$. The poset $(2^G, \subseteq)$ is a complete lattice, the infima and suprema of which are computed as intersections and unions of sets, respectively. In the following, the notation 2^G will always refer to this complete lattice.

Similarly, by $\mathsf{Fin}(G)$ we denote the set of all finite subsets of G. Unless G itself is finite, this is not a lattice since it misses a greatest element. However, it is a meet-semilattice with least element $\emptyset$.

If numerous infima or suprema exist within a poset, then it makes sense to consider subsets of elements which are *dense* with respect to these constructions, i.e. which yield all other elements as suprema or infima.

Definition 2.1.6 Given a poset P, a subset $X \subseteq P$ is *meet-dense* (or *infimum-dense*) in P if we find that

$$y = \bigwedge\{x \in X \mid y \leq x\}, \text{ for all } y \in P.$$

Especially, the above infimum exists for all subsets of X of the given form. A subset of P is *join-dense* (or *supremum-dense*) in P if it is meet-dense in P^{op}.

Clearly, P is always meet-dense and join-dense in itself. More useful cases of density are those where the dense subset is substantially smaller than the poset itself. For example, in a powerset lattice 2^S, the strictly smaller set of all finite subsets of S is join-dense.

Finally, we define various types of subsets of a partially ordered set that are of special interest to us.

Definition 2.1.7 Let P be a poset and let $X \subseteq P$. The *lower closure* of X is the set $\downarrow X := \{y \in P \mid y \leq x \text{ for some } x \in X\}$. The *upper closure* $\uparrow X$ is defined dually. X is an *upper* (*lower*) *set* in P if X is upward (downward) closed, i.e. if $X = \uparrow X$ ($X = \downarrow X$).

X is *directed* if it is nonempty and, for any two elements $x, y \in X$, there is some element $z \in X$ such that $x \leq z$ and $y \leq z$. An *ideal* is directed lower set.[1] A *principal ideal* of P is an ideal which has a greatest element when considered as a subposet of P, i.e. which is of the form $\downarrow\{x\}$ for some $x \in P$.

An ideal I is *prime* if it is *inaccessible by binary infima*, i.e. if for any $x, y \in P$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$. An ideal is *completely prime* if it is inaccessible even by arbitrary infima.

A *filter* of P is an ideal of P^{op}, i.e. an upper subset of P which is *filtered* (directed with respect to P^{op}). Principal and (completely) prime filters are defined accordingly.

As usual, we will write $\downarrow x$ ($\uparrow x$) instead of $\downarrow\{x\}$ ($\uparrow\{x\}$). Note that a set I is a prime ideal if and only if its set complement if a prime filter. If, as in the cases we consider below, the underlying order is a lattice, the notion of a prime ideal is but a special case of the following more general concept of a prime element.

Definition 2.1.8 Given a lattice L, an element $x \in L$ is called

- *meet-irreducible* if $y \wedge z = x$ implies $y = x$ or $z = x$,
- *meet-prime* if $y \wedge z \leq x$ implies $y \leq x$ or $z \leq x$.

Join-irreducible and join-prime elements are defined dually.

In a distributive lattice, the meet-irreducibles are exactly the meet-primes, and this will be the only case considered in this paper. The prime ideals of a lattice are known to be the meet-prime elements in the complete lattice of all ideals (within which meets are computed as set intersections).

Our investigations will often rely on the existence of sufficiently many prime filters and ideals. Unfortunately, the supply of prime ideals that can be deduced in classical Zermelo-Fraenkel set theory is not sufficient for our purposes. We overcome this problem by postulating the required property.

Axiom 2.1.9 (Prime Ideal Theorem) Let I be an ideal of a distributive lattice and let F be a filter disjoint from I. Then there exists a prime ideal J which contains I and is disjoint from F.

The name for this axiom stems from the fact that it can also be obtained as a consequence of the strictly stronger *Axiom of Choice* (typically using the equivalent condition of *Zorn's Lemma*, see [Joh82, Lemma 2.3]). The above prime ideal theorem for distributive lattices (DPI) is equivalent to the *Boolean Prime Ideal Theorem* (BPI) – for details see [DP02, Joh82]. We will try to point out whenever a result in our subsequent investigations directly depends on DPI, which is typically the case for the investigations of Stone duality in Chapters 3 and 5.

[1] Note that this definition implies that $\emptyset$ (which is not directed) is not an ideal.

2.2 Morphisms of partially ordered sets

Now we shall turn to the most important types of *morphisms* (here: *functions*) between posets and lattices. Suggested references are the same as in Section 2.1, though [GHK+03] is our primary reference for our rather general treatment of Galois connections. Another good source on this topic is [EKMS93].

Before looking at particular types of functions, we remark that any collection of functions between two posets can itself be equipped with a partial order.

Definition 2.2.1 Given a set F of functions $f : P \to Q$ between posets P and Q, the *pointwise order* on F is defined by setting

$$f \leq g \quad \text{iff} \quad f(p) \leq g(p) \text{ for all } p \in P.$$

Note that this definition does not depend on the order of P, such that one could as well take a simple set here. However, as the following definition shows, the order on P plays an important role for describing appropriate collections of mappings between the posets.

Definition 2.2.2 Consider posets P and Q, and a function $f : P \to Q$. Then f is *monotone* (*antitone*) if it is *order-preserving* (*order-reversing*), i.e. if $x \leq y$ implies $f(x) \leq f(y)$ ($f(x) \geq f(y)$) for all $x, y \in P$. f is *order-reflecting* if $f(x) \leq f(y)$ implies $x \leq y$. An *order-isomorphism* is a bijective function which preserves and reflects the order.

Given a subset $X \subseteq P$ with supremum $\bigvee X$, f *preserves* the supremum of X if $\bigvee\{f(x) \mid x \in X\}$ exists and is equal to $f(\bigvee X)$. f *preserves all suprema* if it preserves the supremum of all subsets of P that have a supremum. Preservation of binary, directed, and (non-)empty suprema is defined analogously. The dual statements give rise to preservation properties for infima. A function that preserves directed suprema is also called *Scott continuous*.

Note that monotony can also be described as the preservation of infima (or, equivalently, suprema) of all sets of the form $\{x, y\}$, $x \leq y$. Especially, any function that preserves binary, directed, or non-empty suprema is necessarily monotone.

2.2.1 Galois connections

A homomorphism between posets of a specific type is usually assumed to be a mapping that preserves all of the required structural data. For example, a homomorphism of join-semilattices with least elements is a function that preserves binary joins and least elements (empty joins), and a homomorphism of bounded distributive lattices preserves all finite (including empty) meets and joins. As complete lattices can be defined using only suprema (or infima), mappings that preserve either all suprema or all infima are equally interesting in this case. It turns out that the corresponding type of morphism is conveniently characterized with the help of the following notion.

Definition 2.2.3 Consider posets P and Q, and a pair of monotone functions $\overrightarrow{g} : P \to Q$ and $\overleftarrow{g} : Q \to P$. Then $(\overrightarrow{g}, \overleftarrow{g})$ is a *monotone Galois connection* if, for all elements $x \in P$, $y \in Q$, one finds that

$$y \leq_Q \overrightarrow{g}(x) \quad \text{if and only if} \quad \overleftarrow{g}(y) \leq_P x.$$

In this case, $\overrightarrow{g}$ is called the *upper adjoint* and $\overleftarrow{g}$ the *lower adjoint* of the Galois connection.

As remarked in [GHK$^+$03], speaking of "adjoints" is motivated by close connections to category theory, while the use of "upper" and "lower" (instead of "left" and "right" as in category theory) is intended to avoid possible confusion arising from different categorical interpretations of posets that were considered in the literature. This terminology is easy to memorize by observing that the upper adjoint appears on the greater side of the order-relation in the above condition.

An *antitone Galois connection* from P to Q is a monotone Galois connection from P^{op} to Q.[2] Stated explicitly, one obtains pairs of maps as above such that $y \leq_Q \overrightarrow{g}(x)$ if and only if $x \leq_P \overleftarrow{g}(y)$. This is the historic definition of Galois connections which is still preferred in some areas (especially in Formal Concept Analysis, see Section 2.3). In many other cases, Galois connections are considered to be monotone by default. We avoid associated terminological confusion by making the desired meaning explicit. Introducing both notions allows us to concentrate on the formulation which is most convenient for a given purpose (or most common in a given subject area).

Note that, if $(\overrightarrow{g}, \overleftarrow{g})$ is a monotone Galois connection from P to Q then $(\overleftarrow{g}, \overrightarrow{g})$ is a monotone Galois connection from Q^{op} to P^{op}. Likewise, if $(\overrightarrow{g}, \overleftarrow{g})$ is an antitone Galois connection from P to Q then $(\overleftarrow{g}, \overrightarrow{g})$ is an antitone Galois connection from Q to P. Care must be taken not to confuse these statements to draw wrong conclusions. For example, an antitone Galois connection from P to Q is certainly not the same as an antitone Galois connection from P^{op} to Q^{op}.

Furthermore, as the next result shows, each part of a Galois connection determines the other uniquely.

[2] Since this yields a symmetrical definition, the distinction of lower and upper adjoints is not adequate in this context. However, we will still speak of *adjoints* when referring to the respective mappings.

Theorem 2.2.4 For functions $f : P \to Q$ and $g : Q \to P$ between posets P and Q, the following are equivalent:

(i) (f, g) is a monotone Galois connection,

(ii) f is monotone and $g(y) = \min f^{-1}(\uparrow y)$, for all $y \in Q$,

(iii) g is monotone and $f(x) = \max g^{-1}(\downarrow x)$, for all $x \in P$.

Proof. See [GHK⁺03, Theorem O-3.2]. □

The adjoints of any monotone Galois connection preserve infima and suprema, respectively, while the converse is only true under additional assumptions.

Theorem 2.2.5 The upper adjoint of a monotone Galois connection preserves all infima, the lower adjoint preserves all suprema.

Conversely, consider a function $f : L \to P$ with L a complete lattice and P a poset. If f preserves all infima, then f is the upper adjoint of a monotone Galois connection. The corresponding lower adjoint maps an element $x \in P$ to $\bigwedge f^{-1}(\uparrow x)$.

Proof. See [GHK⁺03, Theorems O-3.3 and O-3.4]. □

From the previous result we conclude that both adjoints of an antitone Galois connection transform suprema into infima, i.e. $\vec{g}\,(\bigvee X) = \bigwedge\{\vec{g}(x) \mid x \in X\}$ and likewise for $\overleftarrow{g}$. We emphasize that the dual statement is not true in general.

Given a lattice with element a, the mapping $\cdot \wedge a : x \mapsto x \wedge a$ clearly preserves all meets, and indeed is upper adjoint to the identity mapping. The converse is not true in general, such that the property that $\cdot \wedge a$ is a lower adjoint actually defines a further type of lattices. However, the emerging definition, compact as it may be, provides very little intuition about the (logical) nature of the defined structures. Interested readers are therefore referred to [DP02, Joh82, Bor94b] for further context.

Definition 2.2.6 A *Heyting algebra* is a bounded lattice L within which the mappings $\cdot \wedge a$ for arbitrary $a \in L$ are lower adjoints of a monotone Galois connection. The (necessarily unique) upper adjoints are denoted $a \to \cdot$.

A *Boolean algebra* is a Heyting algebra L for which one has $a \vee (a \to \bot) = \top$ for every $a \in L$, where $\bot$ and $\top$ denote the least and greatest element of L, respectively.

Note that any Heyting algebra is necessarily distributive: by Theorem 2.2.5, the lower adjoints $\cdot \wedge a$ preserve joins, and preservation of binary joins by these maps is just what we called distributivity. Heyting algebras and Boolean algebras will first appear at the end of Chapter 3, but our main use for these concepts is in the discussion of intuitionistic and classical propositional logics in Chapter 5.

2.2.2 Closure operators

Galois connections are closely connected to a class of order-theoretic functions known as closure operators.

Definition 2.2.7 Given a poset P, a *closure operator* on P is a function $f : P \to P$ which is

(i) monotone, i.e. $x \leq y$ implies $f(x) \leq f(y)$,
(ii) idempotent, i.e. $f(x) = f(f(x))$,
(iii) inflationary, i.e. $x \leq f(x)$,

for all $x, y \in P$. An element $x \in P$ is closed (under f) if $x = f(x)$.

The exact relationship to Galois connections is as follows.

Theorem 2.2.8 Consider posets P, Q, and a monotone or antitone Galois connection $(\vec{g}, \overleftarrow{g}) : P \to Q$. Then the composition $\vec{g} \circ \overleftarrow{g} : y \mapsto \vec{g}(\overleftarrow{g}(y))$ is a closure operator on Q.

Conversely, if $f : Q \to Q$ is a closure operator on Q, then there is the obvious factorization

$$Q \xrightarrow{\ f^{\circ}\ } f(Q) \xrightarrow{\ f_{\circ}\ } Q$$

into the corestriction f° and the inclusion $f_{\circ}$, and $(f_{\circ}, f^{\circ})$ is a monotone Galois connection from $f(Q) = \{f(y) \mid y \in Q\}$ to Q.

Proof. A full proof of these basic facts is given in [GHK⁺03, Proposition O-3.10]. The important first part is also to be found in [DP02, GW99]. □

The above formulation is correct, but might invite to the wrong conclusion that the composition of the adjoints of either a monotone or antitone Galois connection will always yield closure operators. This is true only for antitone Galois connections where both adjoints are interchangeable. In contrast, for monotone Galois connections, the composition $\overleftarrow{g} \circ \vec{g} : P \to P$ is a closure operator with respect to P^{op}.[3]

As a corollary of Theorem 2.2.8, we obtain additional properties of closure operators, especially when considered in conjunction with complete lattices.

Corollary 2.2.9 ([GHK⁺03] Proposition O-3.13) The image of any closure operator $f : L \to L$ is closed under arbitrary infima, i.e. – provided that it exists – the infimum of a collection of closed elements is closed.

[3] Such an operator – called *kernel operator* in [GHK⁺03] – is still monotone and idempotent on P, but it is "deflationary", i.e. the image of any element is below the element.

Furthermore, if L is a complete lattice, then the poset $f(L)$ is closed under arbitrary infima in L and thus is a complete lattice as well. Conversely, any subset C of L that is closed under arbitrary infima in L induces a unique closure operator c with image C, given by $c : L \to L : x \mapsto \bigwedge\{y \in C \mid x \leq y\}$.

Proof. Consider a collection $X \subset f(L)$ of closed elements with infimum $\bigwedge_L X$ in L. By monotonicity, $f(\bigwedge_L X)$ is the greatest lower bound of X in $f(L)$. Since $\bigwedge_L X \leq f(\bigwedge_L X)$, both infima are in fact equal, and thus the infimum of X in L is indeed closed.

If L is a complete lattice, the above entails that $f(L)$ has all infima, and, consequently, is a complete lattice as well. Now if $C \subset L$ is closed under arbitrary infima in L, then C is a complete lattice and the inclusion $f_\circ : f(L) \to L$ preserves infima. Thus by Theorem 2.2.5, $f_\circ$ is the upper adjoint of a Galois connection, the lower adjoint of which is the map $f^\circ : L \to f(L) : x \mapsto \bigwedge\{y \in C \mid x \leq y\}$. By Theorem 2.2.8, $f_\circ \circ f^\circ : L \to L$ is the claimed closure operator. □

Motivated by the previous result, subsets C of a complete lattice that are closed under infima are called *closure systems* in algebra, especially in the case where infima are computed as intersections of sets. In some areas (e.g. in topology, Section 2.4), more specific types of closure systems are considered, but we will always use the term in this most general sense.

The next proposition collects some additional facts to improve our understanding of Galois connections and to prepare our introduction of formal concept analysis in Section 2.3.

Proposition 2.2.10 For every Galois connection $(\vec{g}, \overleftarrow{g})$ between posets P and Q, one finds that

$$\vec{g}(x) = \vec{g}\overleftarrow{g}\vec{g}(x) \quad \text{and} \quad \overleftarrow{g}(y) = \overleftarrow{g}\vec{g}\overleftarrow{g}(y)$$

Especially, every element $\vec{g}(x)$ is closed under the closure operator $\vec{g} \circ \overleftarrow{g}$.

If $(\vec{g}, \overleftarrow{g})$ is an antitone Galois connection, then the subposets of P and Q that consist of the elements which are closed under $\overleftarrow{g} \circ \vec{g}$ and $\vec{g} \circ \overleftarrow{g}$, respectively, are dually isomorphic[4] and $(\vec{g}, \overleftarrow{g})$ provides the required isomorphism.

Proof. Proofs are for example given in [GW99, Propositions 5 and 8]. □

The second part of the previous proposition refers to antitone Galois connections, since this is the case for which we will use this result below. Of course it could as well have been stated for the monotone setting.

[4] I.e. each of the posets is order-isomorphic (Definition 2.2.2) to the dual of the other.

2.3 Formal concept analysis

Our notation for formal concept analysis mostly follows [GW99], with a few exceptions which enhance readability for our purposes. Especially, we avoid the use of the operator $'$ to denote the operations that are induced by a formal context. This will both clarify the exposition and allow us to use $'$ to enrich our pool of possible entity names (like in "$g, g' \in G$").

Furthermore, we introduce some additional notation for the (pre-)image of binary relations, as given in the next definition.

Definition 2.3.1 Let $R \subseteq G \times M$ be a binary relation between sets G and M. For subsets $O \subseteq G$ and $A \subseteq M$, we define

- $R(O) := \{m \in M \mid g\ R\ m \text{ for some } g \in O\}$, the *image* of O under R,
- $R^{-1}(A) := \{g \in G \mid g\ R\ m \text{ for some } m \in A\}$, the *preimage* of A under R,
- $O^R := \{m \in M \mid g\ R\ m \text{ for all } g \in O\}$, and
- $A^R := \{g \in G \mid g\ R\ m \text{ for all } m \in A\}$.

Note that, though we generally use R^{-1} to denote the inverse relation of R, we prefer the notation A^R over $A^{R^{-1}}$. We will be careful to avoid possible confusion that could arise from this notation if it is not clear whether A is a subset of G or of M. Furthermore, we adopt the usual abbreviations $g^R := \{g\}^R$, $R(g) := R(\{g\})$, etc.

The sets O^R and A^R turn out to be closed under certain closure operators (see Definition 2.2.7).

Proposition 2.3.2 Consider a binary relation $R \subseteq G \times M$. The mappings

$$\cdot^R : 2^G \to 2^M \quad \text{and} \quad \cdot^R : M \to G$$

constitute an antitone Galois connection between the powersets 2^G and 2^M, ordered by subset-inclusion.

Especially, $\cdot^{RR} : 2^G \to 2^G$ and $\cdot^{RR} : 2^M \to 2^M$ are closure operators and all sets of the form O^R, $O \subseteq G$, and A^R, $A \subseteq M$, are closed with respect to the respective operator.

Proof. Using Definition 2.3.1 it is straightforward to derive the condition of Definition 2.2.3 to show that the above mappings are indeed adjoints of an antitone Galois connection. The other results follow from Theorem 2.2.8 and Proposition 2.2.10. A more direct proof is to be found in [GW99]. □

At this stage we already have most of the background knowledge on FCA which will be required within this work. It remains to introduce some terms that are typically used in this area. For example, binary relations are called *contexts* in FCA:

Definition 2.3.3 A (*formal*) *context* $\mathbb{K}$ is a tuple (G, M, I), where $I \subseteq G \times M$ is a binary relation between G and M. G and M are referred to as the sets of *objects* and *attributes*, respectively, and I is called the *incidence relation* of $\mathbb{K}$.

A subset $O \subseteq G$ is an *extent* of $\mathbb{K}$ whenever $O = O^{II}$. Likewise, an *intent* of $\mathbb{K}$ is a closed subset $A = A^{II} \subseteq M$. An *attribute extent* (*object extent*) is a set of the form m^I, $m \in M$ (g^{II}, $g \in G$). *Object intents* and *attribute intents* are defined dually.

The intuitive reading in terms of knowledge representation is that $\cdot^I : 2^G \to 2^M$ yields all attributes common to a set of objects, while $\cdot^I : 2^M \to 2^G$ maps a set of attributes to all objects that fall under all of these attributes.

Note that, by Proposition 2.2.10, attribute extents and object intents are indeed closed and the extents of a context are exactly the sets of the form A^I for some $A \subseteq M$. Moreover, according to Corollary 2.2.9, the above closure operators induce complete lattices as their closure systems. These are called *concept lattices* in FCA.

Theorem 2.3.4 ([GW99] Theorem 3) Consider a context $\mathbb{K} = (G, M, I)$. The set of extents $\mathsf{B}_o(\mathbb{K}) := \{O \subseteq G \mid O = O^{II}\}$, ordered by subset-inclusion, is a complete lattice. Given a set of extents $\mathcal{X} \subseteq \mathsf{B}_o(\mathbb{K})$, we have

$$\bigwedge \mathcal{X} = \bigcap \mathcal{X} \quad \text{and} \quad \bigvee \mathcal{X} = \left(\bigcup \mathcal{X}\right)^{II}.$$

Given a set $\mathcal{Y} \subseteq 2^M$ of attribute sets of $\mathbb{K}$, we have

$$\left(\bigcup \mathcal{Y}\right)^I = \bigcap \{A^I \mid A \in \mathcal{Y}\}.$$

Especially, the set of all attribute extents is meet-dense in $\mathsf{B}_o(\mathbb{K})$.

Dual statements hold for the complete lattice $\mathsf{B}_a(\mathbb{K})$ of all intents of $\mathbb{K}$. Furthermore, $\mathsf{B}_o(\mathbb{K})$ and $\mathsf{B}_a(\mathbb{K})$ are dually isomorphic with isomorphisms given by $\cdot^I : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_a(\mathbb{K})$ and $\cdot^I : \mathsf{B}_a(\mathbb{K}) \to \mathsf{B}_o(\mathbb{K})$.

Proof. The first part of the statement is immediate from Corollary 2.2.9 and the fact that $\cdot^{II}$ is a closure operator on 2^G (Proposition 2.3.2), where set-theoretic operations yield infimum and supremum.

The second part follows since $\cdot^I : 2^M \to 2^G$ is an antitone Galois connection (Proposition 2.3.2), which thus transforms suprema into infima (Theorem 2.2.5). Meet-density of the attribute extents follows since any extent $O \subseteq G$ is equal to O^{II}, which can be expressed as $\left(\bigcup \{\{a\} \mid a \in O^I\}\right)^I$. The claimed dual isomorphism has been established in Proposition 2.2.10. □

The closure systems $\mathsf{B}_o(\mathbb{K})$ and $\mathsf{B}_a(\mathbb{K})$ of the above theorem are called *object* and *attribute concept-lattice*, respectively.

An important aspect of FCA – at least from a mathematical perspective – is that contexts can be dualized and complemented to obtain new structures. These operations will turn out to be vital for our subsequent studies. Considering some context $\mathbb{K} = (G, M, I)$, the context *complementary to* $\mathbb{K}$ is $\mathbb{K}^{\mathsf{c}} = (G, M, \not{I})$ where $\not{I} = (G \times M) \setminus I$. The context *dual* to $\mathbb{K}$ is $\mathbb{K}^{\mathsf{d}} = (M, G, I^{-1})$. We already employed the latter construction implicitly when using the term "dually" in the above studies.

It is easy to see that dualizing a context does merely change the roles of extents and intents and thus the order of the concept lattices: $\mathsf{B}_o(\mathbb{K}^{\mathsf{d}}) = \mathsf{B}_a(\mathbb{K})$ and $\mathsf{B}_a(\mathbb{K}^{\mathsf{d}}) = \mathsf{B}_o(\mathbb{K})$. The situation for complementation is more involved since the concept lattices of $\mathbb{K}$ and $\mathbb{K}^{\mathsf{c}}$ are in general not (dually) isomorphic to each other. What we can observe immediately is that dualization and complementation commute: $\mathbb{K}^{\mathsf{cd}} = \mathbb{K}^{\mathsf{dc}}$. Furthermore, we will find the following lemma quite helpful.

Lemma 2.3.5 Given a context $\mathbb{K} = (G, M, I)$ with objects $g, h \in G$, we find that $g \in h^{II}$ if and only if $h \in g^{\not{I}\not{I}}$.

Proof. If $g \in h^{II}$ then $g\ I\ m$ for all $m \in h^{I}$. Thus $h\ I\ m$ implies $g\ I\ m$. Contrapositively, $g \not{I}\ m$ entails $h \not{I}\ m$, which shows $h \in g^{\not{I}\not{I}}$. □

Further specific notions, especially those that are related to morphisms between formal contexts, will be introduced in Chapter 4.

2.4 Topology and domain theory

Domain theory is a branch of order theory that, roughly speaking, is concerned with structures that model iterative computation and approximation in computer science. The original motive for such a formalism was finding an appropriate semantical description of certain *lambda calculi*.

In contrast, topology originally was introduced in order to study spacial relationships of geometric objects in an abstract way. However, further abstraction lead to the field of *general topology* and gave rise to structures of high relevance to theoretical computer science. These developments allow us to study domain theory and topology as too sides of the same coin.[5]

The viewpoint on topology that we adopt here is detailed in [Smy92], and we will not need topological background knowledge that goes beyond this treatment. Our main reference for domain theory is [GHK⁺03], though the lighter exposition in [DP02] might be more suitable for beginning the studies in this field. Another good source of domain theory is [AJ94], an additional advantage of which is that

[5] As will be explained in Chapter 3, logic is found on the side of topology.

it is freely available online. Introductory remarks on the motivation underlying domain theory as well as some of the relevant definitions are also to be found in [WP, Article "Domain Theory"].

2.4.1 Domain theory

Domain theory is concerned with various specific kinds posets, the most basic of which are the *directed complete partial orders*. Directed subsets have been introduced in Definition 2.1.7.

Definition 2.4.1 A poset P is a *directed complete partial order* (or *dcpo* for short), if it is directed complete, i.e. if all directed subsets of P have a supremum.

P is a *complete partial order* (or *cpo*), if it is a dcpo with a least element, i.e. within which the empty supremum exists.

Given that the defining property of dcpos is the existence of directed suprema, Scott continuous functions (Definition 2.2.2) suggest themselves as the natural type of morphism between dcpos.

The intuition underlying domain theory is to view elements of posets as the possible inputs or outcomes of a computation. The order then provides a *qualitative* measure for the information content of some particular piece of data in the respective input or output domains. In spite of the fact that ordering relations can not provide for a notion of distance to judge how "close" a particular output is to a desired result, it is still possible to formulate a notion of approximation on domains.

Definition 2.4.2 Consider elements x, y of some dcpo[6] P. Then x *approximates* y (or x is *way-below* y), written $x \ll y$, if we find that, for all directed sets $D \subseteq P$ with $y \leq \bigvee D$, there is some element $d \in D$ with $x \leq d$.

An element $x \in P$ is *compact* if it is way below itself, i.e. if $x \ll x$. The set of all compact elements of a poset P is denoted $\mathsf{K}(P)$.

The *order of approximation* $\ll$ intuitively states that one element is much simpler than another, and provides a useful alternative to the strict order $<$ (or $\subset$), which is not very meaningful in the case of infinite posets.

Example 2.4.3 Consider the set $\mathbb{N}$ of all natural numbers and its powerset $2^{\mathbb{N}}$. Given the set $U \subseteq \mathbb{N}$ of all odd numbers, we find that $U \setminus \{2147483647\}$ is strictly smaller than U, though it does rather not appear to be considerably simpler. In

[6] One can also discuss the given notions for posets that are not directed complete, but we have no need to take this additional effort.

contrast, the sets that are way below U are only the finite sets of odd numbers. Furthermore, any finite set of natural numbers is compact.[7]

Some easy facts about the approximation relation need to be recorded before we proceed.

Proposition 2.4.4 Let P be a dcpo and let $\ll$ be the order of approximation on P.

(i) $\ll \subseteq \leq$, i.e. $x \ll y$ implies $x \leq y$,

(ii) if $x' \leq x$, $x \ll y$, and $y \leq y'$, then one finds $x' \ll y'$,

(iii) $\ll$ is antisymmetric and transitive,

(iv) $x \ll y$ and $x' \ll y$ imply $x \vee x' \ll y$ (provided this supremum exists),

(v) if a least element $\perp$ exists, then $\perp \ll x$,

hold for all $x, y, x', y' \in P$.

Proof. Statement (i) is immediate when observing that $\{y\}$ is a directed subset which has y as its supremum. For (ii) one just has to note that any directed supremum above y' is also above y and that any element of this directed set which is above x is also above x'. Item (iii) follows from (i) and (ii).

For (iv), consider a directed subset $D \subseteq P$ with $q \leq \bigvee D$. Then there are elements d and $d' \in D$ with $x \leq d$ and $x' \leq d'$. By directedness, there is some $e \in D$ with $d \leq e$ and $d \leq e'$, and, in consequence, $x \vee x' \leq e$ as required.

Statement (v) is again immediate from the definition of $\ll$, where one has to note that directed sets can not be empty. □

In general, it is possible that some elements of a dcpo are not approximated by any element. We, however, are interested in cases where every element is the supremum of the set of elements that are way-below it and where this set is directed. Directed complete partial orders where this is the case are called *continuous*. Our treatment focuses on an even more specific case, where we can restrict to the set of compact elements to achieve these approximations. In addition, we will only have reason to consider dcpos of this type which are complete lattices (Definition 2.1.4).

Definition 2.4.5 A poset L is an *algebraic lattice* if

(i) L is a complete lattice,

(ii) L is algebraic, i.e. any element $x \in L$ is the supremum of the compact elements below it: $x = (\downarrow x \cap \mathsf{K}(L))$.

[7] Which is why "finite element" is used in place of "compact element" in parts of the literature. The term "compact" stems from a similar example found in topology (see Definition 2.4.11).

2.4.2 General topology

The central notion of topology is that of a topological space.

Definition 2.4.6 A *topological space* is a tuple (X, τ) where X is a set and $\tau \subseteq 2^X$, provided that the following hold:

(i) $X \in \tau$ and $\emptyset \in \tau$,

(ii) τ is closed under binary intersections, i.e. O, $O' \in \tau$ implies $O \cap O' \in \tau$,

(iii) τ is closed under arbitrary unions.

The elements of τ then are called *open sets* and the complete lattice $(\tau, \subseteq)$ is the *open set lattice*. The set-complements of open sets are the *closed sets*.

A subset B of τ is a *base* of τ if every open set is equal to the union of all members of B it contains.

If confusion is unlikely, we will denote topological spaces by their sets of points. For a topological space X, we also use $\omega(X)$ to denote its open set lattice. Note that set complementation transforms unions into intersections and vice versa, such that the collection of all closed sets of a topological space is indeed a closure system (see remarks after Corollary 2.2.9), inducing a corresponding closure operator. However, not every closure operator on a powerset is topological, since the additional requirements (i) and (ii) might be violated.

Basic examples of a topology on some set X are the discrete topology $(X, 2^X)$ and the indiscrete topology $(X, \{X, \emptyset\})$. Another simple example is the *Sierpiński space*, which is the topological space defined on the two-element set $\{0, 1\}$ with open subsets $\emptyset$, $\{1\}$ and $\{0, 1\}$.

The appropriate morphisms between topological spaces are *continuous functions*.

Definition 2.4.7 Consider topological spaces X and Y, and a function $f : X \to Y$. Then f is *continuous* if its inverse image preserves open sets, i.e. for every open set $O \subseteq Y$, the set $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$ is open in X.

If f is bijective and both f and f^{-1} are continuous then f is a *homeomorphism*. The topological spaces X and Y are said to be *homeomorphic* if a homeomorphism between them exists.

One can now connect topology and order theory by defining a topology τ on a poset P, based on the structure given by the partial order. A simple example is the *Alexandrov topology* where one takes the collection of all upper sets (Definition 2.1.7) as opens. It is easy to check that this is indeed a topology. In fact, the Alexandrov topology is even closed under arbitrary intersections. The following definition introduces another topology that restricts to upper sets, and which is most important for our subsequent studies.

Definition 2.4.8 Consider a dcpo P. A subset $O \subseteq P$ is *Scott open* if the following hold:

(i) $x \in O$ and $x \leq y$ imply $y \in O$ (O is an upper set),

(ii) for any directed set $D \subseteq P$, $\bigvee D \in O$ implies $D \cap O \neq \emptyset$ (O is inaccessible by directed suprema).

The *Scott topology* on P is the collection of Scott open sets. We use $\sigma(P)$ to denote this collection and $\Sigma(P) = (P, \sigma(P))$ for the resulting topological space.

Note that according to this definition, a Scott closed set is a lower set which contains the suprema of all of its directed subsets.

One evidence that the Scott topology is a good choice from the viewpoint of domain theory is provided by the following result.

Proposition 2.4.9 A function between dcpos P and Q is Scott continuous if and only if it is topologically continuous when considered as a function between the spaces $\Sigma(P)$ and $\Sigma(Q)$.

The proof of this result is entirely standard (see, e.g., [AJ94, Proposition 2.3.4]), but we include it as an illustration of some typical domain-theoretic reasoning.

Proof. Consider a Scott continuous function $f : P \to Q$ and a Scott open set $O \subseteq Q$. Let $D \subseteq P$ be directed such that $\bigvee D \in f^{-1}(O)$. Then $f(\bigvee D) = \bigvee f(D) \in O$. Hence there is some element $f(d) \in O$ and thus $d \in f^{-1}(O)$. Since $f^{-1}(O)$ is clearly an upper set by monotonicity of f this shows that it is Scott open as required.

For the converse consider a continuous function $g : \Sigma(P) \to \Sigma(Q)$ and two elements $x \leq y \in P$. It is easy to see that principal ideals (Definition 2.1.7) in Q are Scott closed. Supposing that $f(x) \nleq f(y)$ one finds that $f(x) \notin {\downarrow} f(y)$, where the latter is Scott closed. Thus, $f^{-1}({\downarrow} f(y))$ is a closed set not containing x. But this contradicts the fact that closed sets are lower sets. This shows that f is monotone. Thus for any directed $D \subseteq L_1$, $f(D)$ is directed. Surely $\bigvee f(D) \leq f(\bigvee D)$ and $C = {\downarrow} \bigvee f(D)$ is closed. But then $f^{-1}(C)$ is a closed set containing D, thus also $\bigvee D \in f^{-1}(C)$. But this implies $f(\bigvee D) \in C$ and $f(\bigvee D) \leq \bigvee(f(D))$ as required. □

On the other hand, one can also reverse the above process to obtain orders from topologies.

Definition 2.4.10 Consider a topological space (X, τ). Then τ defines a *specialization (pre-)order* $\leq$ on X by setting $x \leq y$ whenever $x \in O$ implies $y \in O$ for any $O \in \tau$. A topology on a partially ordered set is called *order consistent* if its specialization order coincides with the order of the poset.

The utility value of the previous definitions lies in the fact that we can hope to obtain a mutual correspondence between topological spaces and partial orders, thus substantiating the claimed relationship between both areas. It is easy to see that the Alexandrov topology is indeed order consistent.[8]

For the Scott topology, the situation is slightly more complicated. For example, consider the unit interval $[0, 1]$ of real numbers in their natural order. A little reflection reveals that most subsets of this set are accessible by (trivially directed) suprema: the Scott topology is indiscrete, i.e. only $\emptyset$ and $[0, 1]$ are open. As this example shows, we need to impose additional restrictions in order to obtain order consistent Scott topologies. We shall provide the details in Chapter 3 below.

Finally, we come back to the notion of compactness of Definition 2.4.2. The terminology chosen in this general case derives from the following topological notions.

Definition 2.4.11 Given a topological space X, an open set $O \in \omega(X)$ is *compact* if it is a compact element in the open set lattice $\omega(X)$. The space X is compact if X itself is a compact open set.

A space is *coherent* if the intersection of any two compact opens is again compact.

Since $\omega(X)$ is a (complete) lattice, the explicit definition of compact open sets need not include the notion of a directed set. The reason is that an arbitrary subset X of a lattice can be transformed into a directed set that has the same supremum as X (provided it exists). This is achieved by adding to X the suprema of all finite subsets of X. Accordingly, the traditional formulation of compactness in topology includes finite unions of open sets.

2.5 Category theory

Category theory provides us with some valuable tools to improve our understanding of the relationships between the various types of objects studied within this work. Yet, we will use only very little of the huge amount of knowledge available in this area.

[8] In fact, one can even define Alexandrov spaces without explicitly mentioning the poset, by just requiring that arbitrary intersections of opens be open. For details see [WP, Article "Alexandrov topology"].

For a first introduction, we strongly recommend [LR03]. A more advanced but still very accessible treatment is [McL92]. Finally, [Mac71] and [Bor94a] offer increasingly comprehensive treatments, which go far beyond what will be needed to follow this text.

A category consists of a (usually large) collection of objects which are connected by "arrows" (called *morphisms*) to form a directed graph. As a simple example, one can consider the collection of all sets[9] with arrows given by the functions between each pair of sets. Functions provide a good first intuition for understanding the concept of morphisms in category theory, but we remark that the nature of the chosen arrows is rather arbitrary. For example, the collection of all sets with (directed) binary relations is also a category. However, one requires the availability of a *composition operation* between morphisms that essentially behaves like the composition of functions or the product of relations.

Definition 2.5.1 A *category* $\mathbf{C}$ consists of

(i) a class $|\mathbf{C}|$ of *objects* of the category,
(ii) for all $A, B \in |\mathbf{C}|$, a set $\mathbf{C}(A, B)$ of *morphisms* from A to B,
(iii) for all $A, B, C \in |\mathbf{C}|$, a composition operation
$\circ : \mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$,
(iv) for all $A \in |\mathbf{C}|$, an *identity morphism* $\mathrm{id}_A \in \mathbf{C}(A, A)$,

such that for all $f \in \mathbf{C}(A, B)$, $g \in \mathbf{C}(B, C)$, $h \in \mathbf{C}(C, D)$, the associativity axiom $h \circ (g \circ f) = (h \circ g) \circ f$, and the identity axioms $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$ are satisfied.

As usual, we write $f : A \to B$ for morphisms $f \in \mathbf{C}(A, B)$ and refer to $\mathbf{C}(A, B)$ as the *homset*[10] of A and B. Note that we already considered a number of different categories in the previous sections. For example, we can combine arbitrary classes of posets with any type of order-theoretic functions, since it is clear that the usual composition of functions and identity functions satisfy the conditions of Definition 2.5.1. In Chapter 3, we will especially be interested in the category **Alg** of algebraic lattices and Scott continuous functions. In Section 2.4, we encountered the category **Top** of topological spaces and continuous functions.

[9] According to *Russel's Paradox*, this cannot be a set. The reader may now choose among various possible solutions that have been proposed to handle this problem: one can restrict all considerations to a certain *universe*, a super-set which provides an overall setting for all set-theoretical operations [Mac71], or one can resort to some appropriate theory of *classes* (i.e. "large sets"), e.g. to the *von Neumann-Bernays-Gödel axioms* of set theory [Bor94a]. In this document, we shall ignore these issues completely, generally speaking of "(proper) classes" when we encounter collections of objects that cannot be sets.

[10] This is a rather historical expression, deriving from the first types of morphisms being homo.

An example which is much smaller than the categories above is obtained by considering a single poset as a category. To this end, one considers the elements of a poset P as objects of a category, and defines a morphism $p \to q$ if and only if $p \leq q$. Thus there is at most one morphism between two objects. Composition and identities are obtained from transitivity and reflexivity of the order. Though this example yields an overly complicated view on posets, it is well-suited for demonstrating that morphisms need not be functions in general.

The process of dualizing a partial order can easily be generalized to arbitrary categories. Accordingly, the *opposite* $\mathbf{C}^{\mathsf{op}}$ of a category $\mathbf{C}$ is defined by setting $|\mathbf{C}^{\mathsf{op}}| = |\mathbf{C}|$ and $\mathbf{C}^{\mathsf{op}}(A, B) = \mathbf{C}(B, A)$.

Another means of constructing new categories from old ones is the *product of categories*.

Definition 2.5.2 Given categories $\mathbf{A}$ and $\mathbf{B}$, the *product category* $\mathbf{A} \times \mathbf{B}$ is defined as follows:

- the objects of $\mathbf{A} \times \mathbf{B}$ are the pairs (A, B) of objects from $A \in |\mathbf{A}|$ and $B \in |\mathbf{B}|$,
- a morphism $(A, B) \to (A', B')$ in $\mathbf{A} \times \mathbf{B}$ is a pair (f, g) of morphisms $f : A \to A'$ and $g : B \to B'$.

In Section 2.2, we defined order isomorphisms as a means to establish the equivalence of two ordered sets. Likewise, Section 2.4 introduced the notion of a homeomorphism to achieve a similar comparison between topological spaces. In order to lift these ideas to a categorical level, we define an *isomorphism* in a category to be a morphism $f : A \to A'$ that has an inverse, i.e. for which there is a morphism $g : A' \to A$ with $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_{A'}$. It is not too hard to show that only one morphism g can be inverse to f in this sense. One can now check that this categorical formulation indeed yields the usual types isomorphisms when applied to specific categories.

In order to compare categories among each other, we need the following notion of "morphisms between categories."

Definition 2.5.3 A *functor* F from a category $\mathbf{A}$ to a category $\mathbf{B}$ consists of

(i) a mapping $|\mathbf{A}| \to |\mathbf{B}|$ of objects, where the image of an object $A \in |\mathbf{A}|$ is denoted by $\mathsf{F}A$,

(ii) for all $A, A' \in |\mathbf{A}|$, a mapping $\mathbf{A}(A, A') \to \mathbf{B}(\mathsf{F}A, \mathsf{F}A')$, where the image of a morphism $f \in \mathbf{A}(A, A')$ is denoted by $\mathsf{F}f$,

such that for all $A, B, C \in |\mathbf{A}|$ and all $f \in \mathbf{A}(A, B)$ and $g \in \mathbf{A}(B, C)$ we have $\mathsf{F}(f \circ g) = \mathsf{F}f \circ \mathsf{F}g$ and $\mathsf{F}\,\mathrm{id}_A = \mathrm{id}_{\mathsf{F}A}$.

Note that it is common in category theory to omit parentheses to simplify notation. This also applies to composition, where, for example, one could write GF instead of $(\mathsf{G} \circ \mathsf{F})$.

Simple examples of functors are obtained by "forgetting" special structures that the objects of some particular category have. For instance, there is an obvious functor from the category of complete lattices and meet-preserving functions (upper adjoints) to the category of posets and monotone functions, which maps each lattice and function to itself. Also note that a functor between posets (considered as categories) is just a monotone function. Another particularly important functor is the *identity functor* $\mathrm{id}_{\mathbf{A}}$ on a category $\mathbf{A}$, which maps all objects and morphisms to themselves.

It is obvious that functors can be composed by composing the associated mappings on objects and morphisms, so that the collection of all categories and all functors is again a category. However, this would again be a kind of "set of all sets" which is why one usually considers only the category of all small categories (those having only a set of objects).

Yet the concept of a category of categories hints at a suitable notion for an *isomorphism of categories*: it is a pair of inverse functors between two categories, such that the compositions of both functors each yield the identity functor on the respective category. However, such a situation rarely occurs in reality and one often must be content when finding pairs of functors that are "inverse up to isomorphism," i.e. whose composition maps objects to isomorphic ones, but not necessarily to themselves.

To formalize such generalized isomorphisms, we first have to state the following definition, which yields a notion of morphisms between functors.

Definition 2.5.4 Given two functors $\mathsf{F}, \mathsf{G} : \mathbf{A} \to \mathbf{B}$, a family of morphisms $\eta = (\eta_A : \mathsf{F}A \to \mathsf{G}A)_{A \in |\mathbf{A}|}$ is a *natural transformation* from F to G, if, for all morphisms $f : A \to A'$ of $\mathbf{A}$, one has that $\eta_{A'} \circ \mathsf{F}f = \mathsf{G}f \circ \eta_A$. This situation is denoted $\eta : \mathsf{F} \Rightarrow \mathsf{G}$. A natural transformation $(\eta_A)_{A \in |\mathbf{A}|}$ is a *natural isomorphism* if all of its members are isomorphisms.

Natural transformations, functors, and of course categories, are the three basic ingredients of category theory and already enable us to derive many important notions. Especially, we now have all the machinery needed to define a generalized version of isomorphisms for the comparison of categories.

Definition 2.5.5 Consider categories $\mathbf{A}$ and $\mathbf{B}$. An *equivalence of categories* $\mathbf{A}$ and $\mathbf{B}$ is constituted by a pair of functors $\mathsf{F} : \mathbf{A} \to \mathbf{B}$ and $\mathsf{G} : \mathbf{B} \to \mathbf{A}$, together with a pair of natural isomorphisms $\eta : \mathsf{GF} \Rightarrow \mathrm{id}_{\mathbf{A}}$ and $\epsilon : \mathsf{FG} \Rightarrow \mathrm{id}_{\mathbf{B}}$, where $\mathrm{id}_{\mathbf{A}}$ and $\mathrm{id}_{\mathbf{B}}$ denote the identity functors on the respective categories.

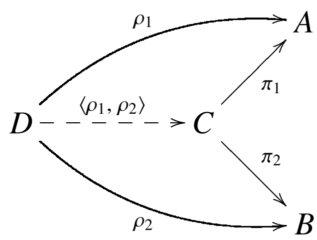


Figure 2.1: The categorical product.

Two categories are *equivalent* whenever there is an equivalence of categories between them. There are many alternative characterizations for this situation, but we shall only make use of the one given here.

Next we consider constructions which can be defined on the objects of some particularly well behaved categories. This process can be compared to the supremum (and infimum) operations on posets, which indeed provide a special case for this situation.

Definition 2.5.6 The *terminal object* of a category $\mathbf{C}$ is an object $\mathbf{1}$ of $\mathbf{C}$ such that for every object $C \in |\mathbf{C}|$, there is exactly one morphism $C \to \mathbf{1}$.

This is the simplest case of the general notion of a *limit* of a graph within a category. Considering a poset as a category, the terminal object coincides with the greatest element. This also illustrates that terminal objects need not exist in all cases.

Another such *limit* that we will encounter in this work is the binary product of two objects.

Definition 2.5.7 Consider objects A and B of a category $\mathbf{C}$. An object C with morphisms $\pi_1 : C \to A$ and $\pi_2 : C \to B$ is the *product* of A and B if, given any object D and morphisms ρ_1 and ρ_2 as in Figure 2.1, there is a unique morphism $\langle \rho_1, \rho_2 \rangle$ that makes this diagram commute, i.e. for which we have $\pi_1 \circ \langle \rho_1, \rho_2 \rangle = \rho_1$ and $\pi_2 \circ \langle \rho_1, \rho_2 \rangle = \rho_2$.

Although the *projections* π_1 and π_2 belong to the constructed product, it is common to denote a product (C, π_1, π_2) by the object C. As in most situations in category theory, a product of two objects may (i) fail to exist at all and (ii) need not be unique if it exists. However, the objects of all existing products are always isomorphic.

For example, in the category of posets and monotone maps, the product is constructed as the product order (Definition 2.1.2), together with the obvious projections to the first and second component. If a single poset is considered as a category, the product of two elements is just its infimum. Many other products considered in mathematics can be described similarly as the product in a suitable category. Especially, the product category yields a product in a category of categories.

Also note that the definition of a binary product can easily be extended to arbitrary collections of factors. In the case of zero factors, the categorical product is just the terminal object introduced above. Clearly, a category has all finite products whenever it has binary products and a terminal object.

Given a category with binary products, one can go further and consider the construction of *function spaces*.

Definition 2.5.8 A category $\mathbf{C}$ is *cartesian closed* if it has all finite products, and there is a functor $\mathbf{C}^{\mathsf{op}} \times \mathbf{C} \to \mathbf{C} : (A, B) \mapsto B^A$ and a natural bijection between the homsets $\mathbf{C}(A \times B, C)$ and $\mathbf{C}(A, C^B)$. Objects of the form B^A are called *exponentials* or *function spaces*.

This definition is applied in Section 3.3, where cartesian closedness is shown for a particular category. This application also might provide some further clarification of this property.

Chapter 3

Algebraic Lattices

This chapter, which largely agrees with [HKZ04], details major parts of the representation theory of algebraic lattices. Above all, we study a new FCA-based representation of algebraic lattices which has been proposed in [ZS0x] and was subsequently extended in [HZ04]. In the process of relating this approach to known results, we give a comprehensive account of the well-known relationships between algebraic lattices, join- and meet-semilattices, Scott information systems, and the according types of topological spaces and propositional logics.

Part of this task essentially boils down to a specialization of [Abr91] to the case of algebraic lattices, but we are not aware of a similarly comprehensive treatment of the according relationships.

The structure of this chapter is as follows. Section 3.1 starts the discussion of algebraic lattices from a domain theoretic perspective, with special emphasis on the role of the semilattice of compact elements. Thereafter, Section 3.2 introduces appropriate notions of morphisms for such semilattices, which are shown to be equivalent to Scott continuous functions between the corresponding algebraic lattices. Section 3.3 then introduces a category of formal contexts equivalent to the category of algebraic lattices and Scott continuous functions, and gives an explicit proof of the cartesian closure of this new category. Building on the prototypical categorical equivalences established earlier, Section 3.4 introduces further representation theorems from logic and topology, which are then connected using Stone duality. Finally, Section 3.5 gives pointers to further literature and hints at possible extensions of the given results.

3.1 Algebraic lattices

In this section, we continue the recapitulation of some fundamental results on algebraic lattices, started in Section 2.4, where we introduced algebraic lattices as complete lattices that have a *base* of compact elements (Definition 2.4.5). Now we explain how algebraic lattices can be reconstructed from the poset of their compacts, which thus turns out to provide an alternative means of representation. Thereafter, we specialize the relationship between complete lattices and closure operators to the case of algebraic lattices and Scott continuous closures.

Definition 2.4.2 introduced the set of compact elements $\mathsf{K}(P)$ of a dcpo P. Using the close relationship of compactness to the order of approximation $\ll$, the following is an immediate consequence of Proposition 2.4.4.

Corollary 3.1.1 Let L be a complete lattice with compact elements $a, b \in \mathsf{K}(L)$ and least element $\bot$. Then $a \vee b$ and $\bot$ are compact.

We conclude that the poset $\mathsf{K}(L)$ of compact elements of a complete lattice is a join-semilattice with least element under the order of L (see Definition 2.1.4). However, for a full characterization one would also be interested in the opposite direction, i.e. given a join-semilattice, one would like to construct a complete lattice. The right tool for this endeavor is that of *ideal completion*, introduced next.

Definition 3.1.2 Given a partially ordered set P, the *ideal completion* $\mathsf{Idl}(P)$ is the collection of all ideals of P (Definition 2.1.7), partially ordered via subset inclusion.

The significance of this construction is based upon the fact that all sets of the form $\downarrow x \cap \mathsf{K}(L)$, $x \in L$, are ideals in $\mathsf{K}(L)$, which is the content of Corollary 3.1.1. The following representation theorem states that this correspondence of ideals and elements is bijective in the case of algebraic lattices.

Theorem 3.1.3 ([GHK+03] Proposition I-4.10) Let L be an algebraic lattice and let S be a join-semilattice with least element.

(i) $\mathsf{K}(L)$ is a join-semilattice with least element.

(ii) $\mathsf{Idl}(S)$ is a an algebraic lattice.

(iii) S is order-isomorphic to $\mathsf{K}(\mathsf{Idl}(S))$.

(iv) L is order-isomorphic to $\mathsf{Idl}(\mathsf{K}(L))$.

Proof. Claim (i) follows from Corollary 3.1.1. For (ii), we show first that $\mathsf{Idl}(S)$ is a complete lattice. Consider any subset $A \subseteq \mathsf{Idl}(S)$ of ideals of S. If A is empty, then its infimum is just the greatest element of $\mathsf{Idl}(S)$. It is easy to see that this element exists and is equal to S. This is an ideal since any join-semilattice is directed, where the existence of a least element guarantees non-emptiness.

On the other hand, if A is non-empty, then its infimum in $\mathsf{Idl}(S)$ is given by $\bigcap A$. To see that this intersection is an ideal, observe that any intersection of lower sets is necessarily a lower set. For directedness consider elements a and $b \in \bigcap A$, i.e. $a, b \in I$ for all $I \in A$. Now, by directedness, any such I contains some upper bound c of a and b. Since $a \vee b \leq c$ we find $a \vee b \in I$ for any $I \in A$. But then $a \vee b \in \bigcap A$ as required. Now clearly the ideal $\bigcap A$ is a lower bound of A and no lower bound can be greater with respect to the subset ordering. This proves that $\mathsf{Idl}(S)$ is a complete lattice, since arbitrary suprema can be expressed as the infima of the sets of upper bounds.

Next we identify the compact elements of $\mathsf{Idl}(S)$ as the principal ideals of S. First consider any directed set $D \subseteq \mathsf{Idl}(S)$. We show that $\bigcup D$ is an ideal and hence the supremum of D. Indeed, any union of lower sets is obviously a lower set. Now let $a, b \in \bigcup D$. Then there are ideals $I, J \in D$ such that $a \in I$ and $b \in J$. By directedness, D contains an upper bound K of I and J. But then $a, b \in K$ and hence $a \vee b \in K \subseteq \bigcup D$ as required. Now consider any $a \in S$ and let D be a directed set as before. If $\downarrow a \subseteq \bigcup D$ then clearly $a \in I$ for some $I \in D$. But then $\downarrow a \subseteq I$ such that $\downarrow a$ is indeed compact. Likewise, all compacts in $\mathsf{Idl}(S)$ are of this form: any ideal $I \in \mathsf{Idl}(S)$ is of the form $I = \bigcup\{\downarrow a \mid a \in I\}$, where the latter is directed since S is a join semilattice. But this also shows algebraicity of $\mathsf{Idl}(S)$ and settles Claim (ii).

For (iii) we note that the mapping $f : S \to \mathsf{K}(\mathsf{Idl}(S)) : a \mapsto \downarrow a$ is the required order-isomorphism. Indeed, by the above characterization of $K(\mathsf{Idl}(S))$, f is bijective and monotone. Its inverse $f^{-1} : \mathsf{K}(\mathsf{Idl}(S)) \to S$ clearly enjoys the same properties.

Finally, (iv) is established by defining the function $g : L \to \mathsf{Idl}(\mathsf{K}(L)) : x \mapsto \downarrow x \cap \mathsf{K}(L)$, which is clearly monotone. We claim that $h : \mathsf{Idl}(\mathsf{K}(L)) \to L : I \mapsto \bigvee I$ yields its inverse. By algebraicity, we know that $h(f(x)) = x$ for any $x \in L$. For the converse consider any $I \in \mathsf{Idl}(\mathsf{K}(L))$. It is easy to see that $I \subseteq f(h(I))$. Conversely, for any $a \in f(h(I))$, we have $a \in \mathsf{K}(L)$ and $a \leq \bigvee I$ by definition. Compactness of a requires the existence of an element $b \in I$ with $a \leq b$. Consequently $a \in I$ and $f(h(I)) \subseteq I$. This finishes the proof, since monotonicity of h is also obvious. □

This result demonstrates that we can represent any algebraic lattice – up to order-isomorphism – by an appropriate semilattice and vice versa. We subsequently obtain a number of alternative characterizations from this statement and its proof. A first observation is that Theorem 3.1.3 assures that every algebraic lattice is isomorphic to a lattice of sets. More precisely, for an algebraic lattice L, we established an isomorphism to a subset of the powerset of its compact elements $2^{\mathsf{K}(L)}$.

In order to characterize those substructures of powersets which yield algebraic lattices, one can make use of *closure operators* as discussed in Section 2.2.2. Indeed, the proof of Theorem 3.1.3(ii) shows that $\mathsf{Idl}(S)$ is closed under arbitrary intersections. Hence it is a *closure system* on 2^S which, according to Corollary 2.2.9, can be uniquely characterized by a closure operator. However, not every closure system is an algebraic lattice, such that a further restriction on the class of closure operators is required. It turns out that Scott continuity (Definition 2.2.2) is what is needed to further extend the representation of algebraic lattices.

Theorem 3.1.4 ([GHK⁺03] Corollary I-4.14) Any algebraic lattice L is order-isomorphic to the image of a Scott continuous closure operator on the powerset $2^{\mathsf{K}(L)}$.

Conversely, the image of any continuous closure operator is an algebraic lattice, where the compacts are exactly the images of finite sets of compacts.

Proof. The first statement is easily derived from what was already said. For an algebraic lattice L, we noted that $\mathsf{Idl}(\mathsf{K}(L))$ is closed under arbitrary infima in $2^{\mathsf{K}(L)}$. The induced closure operator $c : 2^{\mathsf{K}(L)} \to \mathsf{Idl}(\mathsf{K}(L))$ is given by assigning to any set of compacts the least ideal which contains this set.

For Scott continuity, consider any directed collection $D \subseteq 2^{\mathsf{K}(L)}$ of subsets of $\mathsf{K}(L)$. By monotonicity, the image $c(D)$ of D under c is directed. In the proof of Theorem 3.1.3, we showed that $\mathsf{Idl}(\mathsf{K}(L))$ is closed under directed unions, hence $\bigvee c(D) = \bigcup c(D)$. But this is clearly the least set that contains $c(d)$ for any set $d \in D$, thus it is the closure of $\bigcup D$, such that $c(\bigvee D) = \bigvee c(D)$ as required.

For the other direction, we recall that the compact elements of a powerset 2^S are just its finite subset. It is now easy to see that a Scott continuous closure operator c preserves compact elements: Consider a finite set $A \in 2^S$ and a directed set $D \subseteq c(2^S)$. If $c(A) \subseteq \bigcup D$ then $A \subseteq \bigcup D$ and thus $A \subseteq B$ for some $B \in D$. Monotonicity shows that $c(A) \subseteq c(B)$. But D is assumed to be a set of closed sets such that $c(B) = B$ and $c(A) \subseteq B$ as required. Note that preservation of directed unions now implies that every closed set is in fact the union of a set of finitely generated closures and $c(2^S)$ is indeed algebraic. Furthermore this shows that any compact element of the closure system is indeed finitely generated. □

This gives us a third characterization of algebraic lattices. One is tempted to develop a similar statement for join-semilattices with least element. Indeed, any closure operator on the semilattice of finite elements of a powerset can uniquely be extended to a Scott continuous closure on the powerset. However, it is not true that all join-semilattices are images of closure operators on the semilattice of finite subsets of some set. This is easy to see by noting that any collection of finite sets can only have finite descending chains, i.e. it satisfies the *descending chain condition* (see [DP02]). Yet there are join-semilattices with least element that do not have this property, like for example the non-negative rational numbers in their natural order. What we can say is the following.

Corollary 3.1.5 For any join-semilattice S with least element, there is a closure operator $c : 2^S \to 2^S$, such that S is isomorphic to the image of the finite elements of 2^S under c. Conversely, the finite-set image of any closure operator on a powerset is a join-semilattice with least element.

Proof. Note that any closure operator c on a powerset induces a unique Scott continuous closure c' by setting $c'(X) = \bigcup\{c(A) \mid A \subseteq X,\ A \text{ finite}\}$, where c' agrees with c on all finite sets. Then combine Theorems 3.1.3 and 3.1.4, especially the characterization of compact closed subsets. □

The significance of this statement will become apparent in Section 3.3.

3.2 Approximable mappings

So far we have only provided object-level correspondences between algebraic lattices and join-semilattices. We supplement this with suitable morphisms which turn these relations into an equivalence of the respective categories (see Definition 2.5.5). On the side of algebraic lattices, one typically employs Scott continuous functions to form the category **Alg**. This definition leads to a rather advantageous property, namely *cartesian closedness*, which will be discussed in the next section. The aim of this section is to identify a notion of morphism for join-semilattices that produces a category which is *equivalent* to **Alg**.

Now it is well-known that a functor $\mathsf{F} : \mathbf{A} \to \mathbf{B}$ that is part of an equivalence of categories must be *full and faithful*, i.e. there must be a bijection between the homsets $\mathbf{A}(A, A')$ (the set of all morphisms from A to A') and $\mathbf{B}(\mathsf{F}A, \mathsf{F}A')$. Thus our next goal is to define a set of morphisms between each pair of join-semilattices which corresponds bijectively to the set of Scott continuous mappings between the associated algebraic lattices. It is easy to see that we cannot expect to use functions for this purpose for mere cardinality reasons: the set of compacts can be significantly smaller than its algebraic lattice. This problem was already solved by Scott in the closely related case of his *information systems* [Sco82a], which we shall also encounter later on. The idea is to shift to a special set of relations, called *approximable mappings*. To our knowledge, the notion of approximable mappings has not yet been introduced to the study of join-semilattices, so we spell out the details.

Definition 3.2.1 Consider join-semilattices S and T with least elements $\bot_S$ and $\bot_T$, respectively. A relation $\rightsquigarrow \subseteq S \times T$ is an *approximable mapping* if the following hold:

(am1) $a \rightsquigarrow \bot_T$ (non-emptiness) (am2) $a \rightsquigarrow b$ and $a \rightsquigarrow b'$ implies $a \rightsquigarrow b \vee b'$ (directedness)

(am3) $a \leq a'$, $a \rightsquigarrow b$, and $b' \leq b$ imply $a' \rightsquigarrow b'$ (monotonicity and downward closure)

for all elements $a, a' \in S$ and $b, b' \in T$. This situation is denoted by writing $S \rightsquigarrow T$.

The labels for the above properties already indicate their purpose: for every element $a \in S$ the set $\{b \in T \mid a \rightsquigarrow b\}$ is an ideal of T and the resulting assignment $S \to \mathsf{Idl}(T)$ is monotone. It is now rather obvious how this encodes Scott continuous functions: The image of a compact element is given explicitly via the ideal of compacts which approximates it. The image of a non-compact element is obtained by representing it as directed supremum of compacts and applying Scott continuity.

Some easy checks show that join-semilattices with least element together with approximable mappings indeed constitute a category $\mathbf{Sem}_\vee$, where composition of morphisms is defined as the usual composition of relations. Thus for two approximable mappings $S \rightsquigarrow_1 R$ and $R \rightsquigarrow_2 T$, one defines

$$\rightsquigarrow_2 \circ \rightsquigarrow_1 = \{(s,t) \mid \text{there is } r \in R \text{ such that } (s,r) \in \rightsquigarrow_1 \text{ and } (r,t) \in \rightsquigarrow_2\}.$$

Clearly, $\rightsquigarrow_2 \circ \rightsquigarrow_1$ satisfies (am1) since $a \rightsquigarrow_1 \bot_R$ and $\bot_R \rightsquigarrow_2 \bot_T$. Likewise, under the assumptions of (am2), one finds intermediate values $r, r' \in R$ with $a \rightsquigarrow_1 r \rightsquigarrow_2 b$ and $a \rightsquigarrow_1 r' \rightsquigarrow_2 b'$. By (am2) $a \rightsquigarrow_1 r \vee r'$, and by (am3) $r \vee r' \rightsquigarrow_2 b$ and $r \vee r' \rightsquigarrow_2 b'$. Hence $a \rightsquigarrow_1 r \vee r' \rightsquigarrow_2 b \vee b'$ by another application of (am2). Finally, suppose the assumptions for (am3) hold for $\rightsquigarrow_2 \circ \rightsquigarrow_1$. Then there is $r \in R$ such that $a \rightsquigarrow_1 r \rightsquigarrow_2 b$ and hence $a' \rightsquigarrow_1 r \rightsquigarrow_2 b'$ as required. The identity morphism on a semilattice $S \in |\mathbf{Sem}_\vee|$ is just its greater-or-equal relation $\geq_S$. The fact that this yields an identity under relational composition is just statement (am3). Associativity is inherited from relational composition.

Lemma 3.2.2 The object mappings Idl and K from Section 3.1 can be extended to morphisms as follows.

- For any approximable mapping $\rightsquigarrow \subseteq S \times T$, define $\mathsf{Idl}(\rightsquigarrow) : \mathsf{Idl}(S) \to \mathsf{Idl}(T)$ as $\mathsf{Idl}(\rightsquigarrow)(I) = \{b \mid \text{there is } a \in I \text{ with } a \rightsquigarrow b\}$.[1]
- For any Scott continuous mapping $f : L \to M$, define $\mathsf{K}f \subseteq \mathsf{K}L \times \mathsf{K}M$ by setting $\mathsf{K}f = \{(a,b) \mid b \leq f(a)\}$.

These definitions produce functors $\mathsf{Idl} : \mathbf{Sem}_\vee \to \mathbf{Alg}$ and $\mathsf{K} : \mathbf{Alg} \to \mathbf{Sem}_\vee$.

[1]Note that in the notation of Definition 2.3.1 this mapping could also be written as $I \mapsto \rightsquigarrow(I)$.

Proof. To see that Idl is indeed well-defined, observe that for any $a \in S$, $\mathsf{Idl}(\rightsquigarrow)(\downarrow a) = \{b \mid a \rightsquigarrow b\}$, by (am3). This set has already been recognized as an ideal, and hence $\mathsf{Idl}(\rightsquigarrow)$ is well-defined for the compact elements of $\mathsf{Idl}(S)$. By algebraicity, any ideal I is equal to the directed union $\bigcup_{a \in I} \downarrow a$, and hence, observing that $\mathsf{Idl}(\rightsquigarrow)$ preserves all unions, $\mathsf{Idl}(\rightsquigarrow)(I) = \bigcup_{a \in I} \mathsf{Idl}(\rightsquigarrow)(\downarrow a)$. This observation shows that, as a directed union of ideals, $\mathsf{Idl}(\rightsquigarrow)(I)$ is an ideal, and that $\mathsf{Idl}(\rightsquigarrow)$ is Scott continuous.

It is immediate that $\mathsf{Idl}(\rightsquigarrow)$ maps the identity approximable mapping $\geq$ to the identity function. To see that it also preserves composition, note that Scott continuity allows us to restrict to the case of principal ideals. Thus consider two approximable mappings $S \rightsquigarrow_1 R$ and $R \rightsquigarrow_2 T$ and some principal ideal $\downarrow a$, $a \in S$. Preservation of composition is established by the following computation

$$\begin{aligned}(\mathsf{Idl}(\rightsquigarrow_2) \circ \mathsf{Idl}(\rightsquigarrow_1))(\downarrow a) &= \mathsf{Idl}(\rightsquigarrow_2)\{r \mid a \rightsquigarrow_1 r\} \\ &= \{b \mid \text{there is } r \text{ with } a \rightsquigarrow_1 r \text{ and } r \rightsquigarrow_2 b\} \\ &= \{b \mid a(\rightsquigarrow_2 \circ \rightsquigarrow_1)b\} \\ &= \mathsf{Idl}(\rightsquigarrow_2 \circ \rightsquigarrow_1)(\downarrow a).\end{aligned}$$

In the case of K, first observe that $\mathsf{K}f$ clearly has properties (am1) to (am3). For functoriality consider Scott continuous functions $f_1 : L \to M$ and $f_2 : M \to N$. It is easy to see that for $a \in \mathsf{K}L$ and $c \in \mathsf{K}N$, whenever there is $b \in \mathsf{K}M$ with $b \leq f_1(a)$ and $c \leq f_2(b)$, one has $c \leq f_2(f_1(a))$. Since the converse also holds, we find that

$$\begin{aligned}\mathsf{K}(f_2 \circ f_1) &= \{(a, c) \mid c \leq f_2(f_1(a))\} \\ &= \{(a, c) \mid \text{there is } b \in \mathsf{K}M \text{ with } b \leq f_1(a) \text{ and } c \leq f_2(b)\} \\ &= \mathsf{K}f_2 \circ \mathsf{K}f_1.\end{aligned}$$

Finally, applying K to the identity function clearly yields the identity approximable mapping. □

We finish this section by showing the expected categorical equivalence:

Theorem 3.2.3 The functors Idl and K of Section 3.1 yield an equivalence of the categories **Alg** and $\mathbf{Sem}_\vee$.

Proof. For an algebraic lattice L let $\eta_L : L \to \mathsf{Idl}(\mathsf{K}(L)) : x \mapsto \downarrow x \cap \mathsf{K}(L)$ be the isomorphism as established in Theorem 3.1.3. Now consider an algebraic lattice M and a Scott continuous function $f : L \to M$. For any element $x \in L$, $\mathsf{Idl}(\mathsf{K}(f))$ maps the ideal $\eta_L(x)$ to the ideal $\{b \mid \text{there is } a \in \mathsf{K}(L) \text{ with } a \leq x \text{ and } b \leq f(a)\}$. Since Scott continuity guarantees that the supremum of $\{f(a) \mid a \in \mathsf{K}(L), a \leq x\}$ is $f(x)$, the above is just the set $\eta_M(f(x))$ of all compacts below $f(x)$. Consequently, $\mathsf{Idl}(\mathsf{K}(f))(\eta_L(x)) = \eta_M(f(x))$, i.e. η is a natural transformation (Definition 2.5.4).

For a join-semilattice S with least element, we define $\epsilon_S \subseteq S \times \mathsf{K}(\mathsf{Idl}(S))$ by setting $\epsilon_S = \{(a, I) \mid I \subseteq \downarrow a\}$. From Theorem 3.1.3 we derive that every compact ideal I is of the form $\downarrow b$, hence $\epsilon_S = \{(a, \downarrow b) \mid b \leq a\}$. It is now obvious that ϵ_S is an isomorphism whose inverse is given by $\{(\downarrow b, a) \mid a \leq b\}$. For naturality of ϵ, consider some approximable mapping $S \rightsquigarrow T$. We compute $\mathsf{K}(\mathsf{Idl}(\rightsquigarrow)) \circ \epsilon_S = \{(a, \downarrow b) \mid \text{there is } a' \in S \text{ with } a' \leq a \text{ and } (\downarrow a', \downarrow b) \in \mathsf{K}(\mathsf{Idl}(\rightsquigarrow))\}$. Expanding the condition $(\downarrow a', \downarrow b) \in \mathsf{K}(\mathsf{Idl}(\rightsquigarrow))$, we find it equivalent to $\downarrow b \subseteq \mathsf{Idl}(\rightsquigarrow)(\downarrow a')$, which in turn is true iff $\downarrow b \subseteq \{t \mid a' \rightsquigarrow t\}$, exploiting the fact that $\downarrow a'$ is compact. Finally, by (am3) this is equivalent to $a' \rightsquigarrow b$, and we obtain $\mathsf{K}(\mathsf{Idl}(\rightsquigarrow)) \circ \epsilon_S = \{(a, \downarrow b) \mid a \rightsquigarrow b\}$, again by (am3). On the other hand, $\epsilon_T \circ \rightsquigarrow = \{(a, \downarrow b) \mid \text{there is } b' \in T \text{ with } a \rightsquigarrow b' \text{ and } b \leq b'\}$. Using (am3) once more, this evaluates to $\{(a, \downarrow b) \mid a \rightsquigarrow b\}$, which finishes the proof of naturality of ϵ. □

3.3 A cartesian closed category of formal contexts

In Section 2.3, we introduced formal concept analysis as an alternative representation of complete lattices as *concept lattices* of certain *formal contexts*. In the present section, we further investigate an alternative means of deriving lattices from formal contexts, which produces complete lattices that are algebraic and that was originally proposed in [ZS0x]. We review the result of [HZ04] that the according construction extends to an equivalence of categories between **Alg** and an appropriate category of formal contexts, though we will take a more direct route for showing this. Furthermore, we take the opportunity to establish cartesian closedness of **Alg** (Definition 2.5.8) by developing the necessary constructions for formal contexts.

As explained in Section 2.3, the complete lattice usually assigned to a formal context (G, M, I) is the closure system that is induced by the extent closure operator $\cdot^{II} : 2^G \to 2^G$. We now take a slightly different approach and focus our attention on the operation of $\cdot^{II} : 2^M \to 2^M$ on $\mathsf{K}(2^M)$, the join-semilattice with least element given by the finite subsets of M. The decision for working with intents instead of extents is quite arbitrary (compare Theorem 2.3.4), but it turns out to be more in line with existing literature when we come to the treatment of logics in Section 3.4.1, since formulae are usually expected to be attributes rather than objects.

Given that we already found algebraic lattices to be equivalent with certain semilattices, the desired representation reduces to constructing arbitrary join-semilattices with least element from formal contexts. To this end, Corollary 3.1.5 suggests the following solution.

Corollary 3.3.1 For every formal context $\mathbb{K} = (G, M, I)$, the set $\mathsf{Sem}(\mathbb{K}) = \{X^{II} \mid X \in \mathsf{K}(2^M)\}$ is a join-semilattice with least element. Conversely, every such semilattice can (up to isomorphism) be represented in this way.

Proof. In spite of our earlier considerations, we give the easy direct proof. For two finite sets X and Y, $(X \cup Y)^{II}$ is the least closed set that contains X and Y (Corollary 2.2.9), and thus also X^{II} and Y^{II}. Hence $X^{II} \vee Y^{II} = (X \cup Y)^{II}$ (which is just a specialization of Theorem 2.3.4). The first part of the proof is finished by noting that $\emptyset^{II}$ is the least closed set.

Conversely, for a join-semilattice with least element S, consider the context (S, S, I), with $I = \geq_S$. Then for any finite $X \subseteq S$, X^{II} is the set of all lower bounds of all upper bounds of X. But this is easily recognized as $\downarrow \bigvee X$. Note that the least upper bound of the empty set is just the least element. The obvious isomorphism between S and the semilattice $(\{\downarrow s \mid s \in S\}, \subseteq)$ suffices to complete the proof. □

By Theorem 3.1.3 the above shows that every algebraic lattice can be represented by some formal context and vice versa. To make this explicit, we can extend the closure operator of Corollary 3.3.1 to a Scott continuous closure operator on 2^A, as done before in the proof of Corollary 3.1.5. In this way we can recover the following result from [ZS0x].

Corollary 3.3.2 Consider a formal context $\mathbb{K} = (G, M, I)$ and the mapping $c : 2^M \to 2^M : x \mapsto \bigcup\{X^{II} \mid X \subseteq x,\ X \text{ finite}\}$. Then $\mathsf{Alg}(\mathbb{K}) = c(2^M)$ is an algebraic lattice and every algebraic lattice is of this form (up to isomorphism).

Proof. Clearly, c is just the unique Scott continuous closure operator induced by $\cdot^{II}$ as in Corollary 3.1.5. By Theorem 3.1.4 its closure system is indeed an algebraic lattice. For the other direction combine Theorem 3.1.4 and Theorem 3.1.3 to see that $c(2^M)$ is isomorphic to the ideal completion of $\mathsf{Sem}(\mathbb{K})$. Since every algebraic lattice is of this form for some join-semilattice with least element, the claim follows from Corollary 3.3.1. □

Closed sets with respect to the operator c from the above proposition have been termed *approximable concepts* in [ZS0x]. Naturally, it is also possible to extend this result to a categorical equivalence. For this purpose we define a category **Cxt** of formal contexts. The morphisms between two contexts $\mathbb{K}$ and $\mathbb{L}$ are defined by setting $\mathbf{Cxt}(\mathbb{K}, \mathbb{L}) = \mathbf{Sem}_\vee(\mathsf{Sem}(\mathbb{K}), \mathsf{Sem}(\mathbb{L}))$.[2] The following is readily seen.

Theorem 3.3.3 The categories $\mathbf{Sem}_\vee$ and **Cxt** are equivalent.

The functors needed for this result are obvious: on the object level, we obtain suitable mapping from Corollary 3.3.1, and the situation for morphisms is trivial.

[2]In [HZ04] a slightly different definition of morphisms is given. In the formulation given there, the corresponding approximable mapping is not defined on the closed sets $\mathsf{Sem}(\mathbb{K})$ but on all finite attribute sets. We get a context morphism in this sense by extending our approximable mappings, relating two finite sets iff their closures are related.

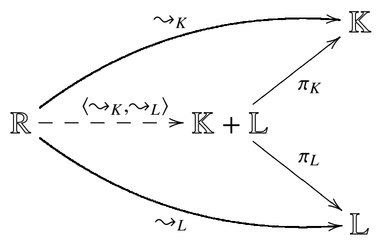


Figure 3.1: The product construction in **Cxt**.

The construction of the natural isomorphisms is similar to the one of ϵ in Theorem 3.2.3, where the identity approximable mapping was modified using the given order-isomorphism of the semilattices.

In the remainder of this section we investigate the categorical constructions that are possible within the categories **Alg**, $\mathbf{Sem}_\vee$, and **Cxt**, where the latter will be the explicit object of study. Because **Cxt** is equivalent to **Alg**, we know that it is *cartesian closed* (see [GHK$^+$03]). We make the required constructions explicit in the sequel, and thus give a mostly self-contained proof of cartesian closedness of **Cxt**.

Exact requirements for showing this were given in Definition 2.5.8. We first consider the empty product of **Cxt**, i.e. the terminal object, which turns out to be given by the formal context $\mathbf{1} = (\emptyset, \emptyset, \emptyset)$. Indeed, for every formal context $\mathbb{K} = (G, M, I)$, there is a unique approximable mapping $\mathbb{K} \leadsto \mathbf{1}$ that relates every finite subset of M to the empty set. The situation for binary products is not much more difficult.

Proposition 3.3.4 Consider two formal contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, and define a formal context $\mathbb{K} + \mathbb{L} = (G \uplus H, M \uplus N, \oplus)$, where $\oplus = I \uplus J \uplus (G \times N) \uplus (H \times M)$, $\uplus$ denoting disjoint union.

Then $\mathbb{K} + \mathbb{L}$ is the categorical product of $\mathbb{K}$ and $\mathbb{L}$, i.e. there are approximable mappings $\pi_K : \mathbb{K} + \mathbb{L} \to \mathbb{K}$ and $\pi_L : \mathbb{K} + \mathbb{L} \to \mathbb{L}$ such that, given approximable mappings $\leadsto_K$ and $\leadsto_L$ as in Figure 3.1, there is a unique approximable mapping $\langle \leadsto_K, \leadsto_L \rangle$ that makes this diagram commute.

Proof. Since context morphisms were defined with reference to the induced semilattices, we first look at $\mathsf{Sem}(\mathbb{K}+\mathbb{L})$. It is easy to see that concept closure in $\mathbb{K}+\mathbb{L}$ is computed by taking disjoint unions of closures in $\mathbb{K}$ and $\mathbb{L}$, i.e. for sets $X \subseteq M$ and $Y \subseteq N$, one finds that $(X \uplus Y)^{\oplus\oplus} = X^{II} \uplus Y^{JJ}$. Hence every element of $\mathsf{Sem}(\mathbb{K} + \mathbb{L})$ corresponds to a unique disjoint union of elements of $\mathsf{Sem}(\mathbb{K})$ and $\mathsf{Sem}(\mathbb{L})$.

We can now define the projections by setting $(X \uplus Y, X') \in \pi_K$ iff $X' \subseteq X$ and $(X \uplus Y, Y') \in \pi_L$ iff $Y' \subseteq Y$, for all X, $X' \in \mathsf{Sem}(\mathbb{K})$ and Y, $Y' \in \mathsf{Sem}(\mathbb{L})$. It is readily seen that these morphisms satisfy the properties of Definition 3.2.1.

Now consider $\rightsquigarrow_K$ and $\rightsquigarrow_L$ as in Figure 3.1. We define the relation $\langle\rightsquigarrow_K,\rightsquigarrow_L\rangle$ by setting $(Z, X \uplus Y) \in \langle\rightsquigarrow_K,\rightsquigarrow_L\rangle$ iff $Z \rightsquigarrow_K X$ and $Z \rightsquigarrow_L Y$, for all concepts X, Y, Z from the corresponding semilattices. Again it is easy to check the conditions of Definition 3.2.1, since they follow immediately from the corresponding properties of $\rightsquigarrow_K$ and $\rightsquigarrow_L$. Furthermore, if there is $X \uplus Y \in \mathsf{Sem}(\mathbb{K}+\mathbb{L})$ with $(Z, X \uplus Y) \in \langle\rightsquigarrow_K,\rightsquigarrow_L\rangle$ and $(X \uplus Y, X') \in \pi_K$ then $Z \rightsquigarrow_K X'$ by the definition of π_K and (am3). Conversely, if $Z \rightsquigarrow_K X'$ then one finds that $X' \uplus \emptyset^{\oplus\oplus} \in \mathsf{Sem}(\mathbb{K}+\mathbb{L})$ yields the required intermediate element to show that $(Z, X') \in \pi_K \circ \langle\rightsquigarrow_K,\rightsquigarrow_L\rangle$. Since a similar reasoning applies to $\rightsquigarrow_L$, Figure 3.1 commutes as required.

Finally, for uniqueness of $\langle\rightsquigarrow_K,\rightsquigarrow_L\rangle$ consider $\mathbb{R} \rightsquigarrow \mathbb{K}+\mathbb{L}$ with $\pi_K \circ \rightsquigarrow\; = \;\rightsquigarrow_K$ and $\pi_L \circ \rightsquigarrow\; = \;\rightsquigarrow_L$. If $Z \rightsquigarrow X \uplus Y$, then $(Z, X) \in \pi_K \circ \rightsquigarrow$ and hence $Z \rightsquigarrow_K X$ and, by a similar reasoning, $Z \rightsquigarrow_L Y$. Conversely, if $Z \rightsquigarrow_K X$ then there must be some X' and Y' such that $X \subseteq X'$ and $Z \rightsquigarrow X' \uplus Y'$. By (am3) this implies $Z \rightsquigarrow X \uplus Y'$. The same argument can be applied to $\rightsquigarrow_L$. Thus whenever $Z \rightsquigarrow_K X$ and $Z \rightsquigarrow_L Y$, there are X' and Y' with $Z \rightsquigarrow X \uplus Y'$ and $Z \rightsquigarrow X' \uplus Y$. Invoking properties (am2) and (am3) for $\rightsquigarrow$, this shows that $Z \rightsquigarrow X \uplus Y$. We have just shown that $Z \rightsquigarrow X \uplus Y$ iff $Z \rightsquigarrow_K X$ and $Z \rightsquigarrow_L Y$, and hence that $\rightsquigarrow\; = \langle\rightsquigarrow_K,\rightsquigarrow_L\rangle$ as required. □

The above product construction is also known in formal concept analysis as the *direct sum* of two contexts [GW99] (which is the reason for our choice of notation). However, it is not the only possible specification of the products in **Alg**. For each formal context $\mathbb{K} = (G, M, I)$, we define a context $\mathbb{K}^+ = (G^+, M^+, I^+)$, where $G^+ = G \cup \{o\}$ and $M^+ = M \cup \{a\}$, with o and a being fresh elements: $o \notin G$ and $a \notin M$. For defining the incidence relation, we set $g\ I^+\ m$ whenever $g\ I\ m$ (requiring that $m \in M$ and $g \in G$) or $g = o$ or $m = a$. Thus $\mathbb{K}^+$ emerges from $\mathbb{K}$ by "adding a full row and a full column."

Now let $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ be formal contexts. Define a new formal context $\mathbb{K} \bowtie \mathbb{L} = (G^+ \times H^+, M^+ \times N^+, \otimes)$ of $\mathbb{K}$ and $\mathbb{L}$ by setting $(g, h) \otimes (m, n)$ iff $g\ I^+\ m$ and $h\ J^+\ n$. This turns out to be an alternative description of the products in **Cxt**.

Proposition 3.3.5 Given formal contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, the contexts $\mathbb{K}+\mathbb{L}$ and $\mathbb{K} \bowtie \mathbb{L}$ are isomorphic in **Cxt**. Equivalently, $\mathbb{K} \bowtie \mathbb{L}$ is the object part of the categorical product of $\mathbb{K}$ and $\mathbb{L}$ in **Cxt**.

Proof. The required isomorphism corresponds to an iso approximable mapping between the semilattices $\mathsf{Sem}(\mathbb{K}+\mathbb{L})$ and $\mathsf{Sem}(\mathbb{K} \bowtie \mathbb{L})$. The elements of the former were already recognized as disjoint unions of concepts from $\mathbb{K}$ and $\mathbb{L}$. In the latter case, concepts are easily recognized as products of concepts from $\mathbb{K}^+$ and $\mathbb{L}^+$. Adding the additional elements a and o guarantees that neither of these extended formal contexts allows for the empty set as a concept, so that each element of $\mathsf{Sem}(\mathbb{K} \otimes \mathbb{L})$ is indeed of the form $X \times Y$ for two uniquely determined concepts $X = X^{II} \in \mathsf{Sem}(\mathbb{K}^+)$ and $Y = Y^{JJ} \in \mathsf{Sem}(\mathbb{L}^+)$.

We define a relation $\rightsquigarrow^+ \subseteq \mathsf{Sem}(\mathbb{K} + \mathbb{L}) \times \mathsf{Sem}(\mathbb{K} \bowtie \mathbb{L})$ by setting $X \rightsquigarrow^+ Y$ whenever $p_1(Y) \cap M \subseteq X$ and $p_2(Y) \cap N \subseteq X$, where p_i denotes the projection to the ith components in a set of pairs. Conversely, a relation $\rightsquigarrow^- \subseteq \mathsf{Sem}(\mathbb{K} \bowtie \mathbb{L}) \times \mathsf{Sem}(\mathbb{K} + \mathbb{L})$ is specified by setting $Y \rightsquigarrow^- X$ whenever $X \cap M \subseteq p_1(Y)$ and $X \cap N \subseteq p_2(Y)$.

We claim that $\rightsquigarrow^+$ and $\rightsquigarrow^-$ are mutually inverse approximable mappings between $\mathsf{Sem}(\mathbb{K} + \mathbb{L})$ and $\mathsf{Sem}(\mathbb{K} \bowtie \mathbb{L})$. The properties of Definition 3.2.1 follow immediately from our use of set-theoretic operations in the definitions. Furthermore it is easy to see that $X(\rightsquigarrow^- \circ \rightsquigarrow^+)X'$ implies $X' \subseteq X$ for any two elements $X, X' \in \mathsf{Sem}(\mathbb{K} + \mathbb{L})$. The converse implication also holds, which can be concluded from the obvious relationships $X \rightsquigarrow^+ (X \cap M)^{II} \times (X \cap N)^{JJ}$, $(X' \cap M)^{II} \times (X' \cap N)^{JJ} \rightsquigarrow^- X'$, and $(X' \cap M)^{II} \times (X' \cap N)^{JJ} \subseteq (X \cap M)^{II} \times (X \cap N)^{JJ}$. Hence $\rightsquigarrow^- \circ \rightsquigarrow^+$ is indeed the identity approximable mapping. A similar reasoning shows that the same is true for $\rightsquigarrow^+ \circ \rightsquigarrow^-$, thus finishing the proof.

Finally, the assertion that this makes $\bowtie$ an alternative product construction is a basic fact from category theory. The required projections are obtained by composing $\rightsquigarrow^-$ with the projections from the proof of Proposition 3.3.4. $\square$

The construction of exponentials in **Cxt** turns out to be slightly more intricate. To fully understand the following definition, it is helpful to look at the function spaces in **Alg**. These are just the sets of all Scott continuous maps between two algebraic lattices under the pointwise order of functions (Definition 2.2.1). The standard technique for describing the compact elements of this lattice are so-called *step functions*. Given two algebraic lattices L and M and two compacts $a \in \mathsf{K}(L)$ and $b \in \mathsf{K}(M)$, one defines a function $|a \Rightarrow b| : L \to M$, that maps an element x to b whenever $a \leq x$, and to $\bot_M$ otherwise. It is well-known that any such step function is Scott continuous and compact in the function space of L and M (see [GHK+03]). However, not all compacts are of this form, since finite joins of step functions are also compact maps that can usually take more than two different values.

Our goal is to construct a formal context that represents the join-semilattice of all compact Scott continuous functions in the sense of Corollary 3.3.1. Intuitively, the collection of all step functions suggests itself as the set of attributes. Finitely generated concepts should represent finite joins of step functions, which in turn correspond bijectively to lower sets with respect to the pointwise order of step functions. In order to obtain a formal context that yields this lower closure, one is tempted to take some subset of Scott continuous functions for objects, and to employ the inverted pointwise order as an entailment relation. This is indeed feasible, but our supply of step functions unfortunately is insufficient to serve as object set in this case. We end up with the following definition:

Definition 3.3.6 Consider two formal contexts $\mathbb{K}$ and $\mathbb{L}$, and the sets $M = \mathsf{Sem}(\mathbb{K}) \times \mathsf{Sem}(\mathbb{L})$ and $G = \mathsf{Fin}(M)$. A formal context $[\mathbb{K} \rightsquigarrow \mathbb{L}] = (G, M, \circleddash)$ is defined by setting $\{(a_i, b_i)\} \circleddash (a, b)$ iff $b \subseteq \bigvee\{b_i \mid a_i \subseteq a\}$, where $\bigvee$ is the join operation from the semilattice $\mathsf{Sem}(\mathbb{L})$.

This definition derives from the above discussion by representing step functions $|a \Rightarrow b|$ via pairs (a, b).[3] Hence, the approximable concepts of $[\mathbb{K} \rightsquigarrow \mathbb{L}]$ as obtained in Corollary 3.3.2 are sets of such pairs, i.e. relations between $\mathsf{Sem}(\mathbb{K})$ and $\mathsf{Sem}(\mathbb{L})$. The reader's suspicion about the true nature of these relations shall be confirmed:

Lemma 3.3.7 Given contexts $\mathbb{K}$ and $\mathbb{L}$, the algebraic lattice $L = \mathsf{Alg}[\mathbb{K} \rightsquigarrow \mathbb{L}]$ of approximable concepts of $[\mathbb{K} \rightsquigarrow \mathbb{L}]$ coincides with the lattice of all approximable mappings from $\mathbb{K}$ to $\mathbb{L}$, ordered by subset inclusion.

Proof. Consider any approximable concept $x \in L$. Definition 3.3.6 implies that the pairs of arbitrary elements $a \in \mathsf{Sem}(\mathbb{K})$ and the least element of $\mathsf{Sem}(\mathbb{L})$ are modelled by any object of $[\mathbb{K} \rightsquigarrow \mathbb{L}]$, i.e. (am1) of Definition 3.2.1 holds for x. For (am2), assume $(a, b_1) \in x$ and $(a, b_2) \in x$. Following the construction in Corollary 3.3.2, one finds that $\{(a, b_1), (a, b_2)\}^{\circleddash\circleddash} \subseteq x$. However, for any object o of $[\mathbb{K} \rightsquigarrow \mathbb{L}]$, $o \circleddash (a, b_1)$ and $o \circleddash (a, b_2)$ clearly implies $o \circleddash (a, b_1 \vee b_2)$, by expanding the definition of $\circleddash$, and thus $(a, b_1 \vee b_2) \in x$. Finally, for (am3) consider some $(a, b) \in x$, $a' \supseteq a$, and $b' \subseteq b$. Clearly, we have $\{(a, b)\}^{\circleddash\circleddash} \subseteq x$. The definition of $\circleddash$ shows immediately that every object that models (a, b) must also model (a', b'), and thus $(a', b') \in \{(a, b)\}^{\circleddash\circleddash}$ as required.

For the converse consider any approximable mapping $\mathbb{K} \rightsquigarrow \mathbb{L}$. We show that $\rightsquigarrow \in L$. Given any finite subset $X = \{(a_i, b_i)\} \subseteq \rightsquigarrow$, one finds that $X \circleddash (a_n, b_n)$ for all $(a_n, b_n) \in X$. Thus $X \in X^{\circleddash}$ and, whenever $(a, b) \in X^{\circleddash\circleddash}$, one also has $X \circleddash (a, b)$, i.e. $b \subseteq \bigvee\{b_j \mid a_j \subseteq a\}$. Defining $J = \{j \mid a_j \subseteq a\}$, one finds that for every $n \in J$, $a_n \subseteq \bigvee\{a_j \mid j \in J\}$ and hence $\bigvee\{a_j \mid j \in J\} \rightsquigarrow b_n$ by (am3). Since J is finite, one can employ an easy induction to show that $\bigvee\{a_j \mid j \in J\} \rightsquigarrow \bigvee\{b_j \mid j \in J\}$, where the case $J = \emptyset$ follows from (am1) and the induction step uses (am2). Obviously $\bigvee\{a_j \mid j \in J\} \subseteq a$ and $b \subseteq \bigvee\{b_j \mid j \in J\}$, and hence $a \rightsquigarrow b$ by (am3). This shows that $\rightsquigarrow$ is an approximable concept. □

The above considerations shed additional light on approximable mappings in general: they can in fact be viewed as lower sets of step functions, the joins of which uniquely determine an arbitrary Scott continuous map between the induced

[3] This correspondence is not injective. In fact, the context $[\mathbb{K} \rightsquigarrow \mathbb{L}]$ in general contains both duplicate rows and duplicate columns.

algebraic lattices. We remark that this also hints at an alternative formulation of the constructions in Lemma 3.2.2.

It remains to show that the above construction does indeed yield a function space in the sense of category theory:

Proposition 3.3.8 The construction $[\cdot \rightsquigarrow \cdot]$ yields the categorical function space of two contexts, i.e. for all contexts $\mathbb{K}$, $\mathbb{L}$, and $\mathbb{R}$, there is a bijection between the sets $\mathbf{Cxt}(\mathbb{K} \times \mathbb{L}, \mathbb{R})$ and $\mathbf{Cxt}(\mathbb{K}, [\mathbb{L} \rightsquigarrow \mathbb{R}])$, and this bijection is natural in all arguments.

Proof. Our earlier results can be employed to simplify this proof. The algebraic lattices associated with the above contexts is denoted by $L = \mathsf{Alg}(\mathbb{K})$, $M = \mathsf{Alg}(\mathbb{L})$, and $N = \mathsf{Alg}(\mathbb{R})$, and we write $[M \to N]$ for the lattice of all Scott continuous functions from M to N, ordered pointwise. The categorical equivalences between **Cxt**, $\mathbf{Sem}_\vee$, and **Alg** (Theorems 3.2.3 ald 3.3.3) and the categorical role of the product construction $\mathbb{L}\times\mathbb{R}$ (Proposition 3.3.4) establish natural bijections between the sets $\mathbf{Cxt}(\mathbb{K} \times \mathbb{L}, \mathbb{R})$ and $\mathbf{Alg}(L \times M, N)$, where $L \times N$ is the standard product order. Likewise, using the same equivalences and the bijection of function spaces from Lemma 3.3.7, one finds another natural bijection between $\mathbf{Cxt}(\mathbb{K}, [\mathbb{L} \rightsquigarrow \mathbb{R}])$ and $\mathbf{Alg}(L, [N \to M])$.

The proof is completed by providing the well-known natural bijection of the sets $\mathbf{Alg}(L\times M, N)$ and $\mathbf{Alg}(L, [N \to M])$. This standard proof can for example be found in [GHK+03].[4] □

Summing up these results, we obtain:

Theorem 3.3.9 The categories **Alg**, $\mathbf{Sem}_\vee$, and **Cxt** are cartesian closed.

Proof. Cxt was shown cartesian closed in Proposition 3.3.4 and Proposition 3.3.8. Closure of the other categories follows by their categorical equivalence (Theorem 3.2.3 and Theorem 3.3.3). □

We stress the fact that our novel interpretation of formal contexts agrees with the classical one, as long as finite contexts or lattices are considered, which covers most of the current FCA applications in Computer Science. On the other hand, the different treatment of infinite data structures displays a deviation from the classical philosophically motivated viewpoint towards one that respects the practical constraints of finiteness and computability. The drawback of this is of course, that duality between extents and intents as an essential feature of FCA is lost.

[4] Of course, this part of the proof is the essential one from the viewpoint of category theory.

3.4 Further representations

So far, we encountered three equivalent representations for algebraic lattices. In this respect, the hardest part was to establish the equivalence of the rather diverse categories **Alg** and $\mathbf{Sem}_\vee$. Many other equivalent categories can now be recognized by relating them to one of these two – an objective that will in general be accomplished rather easily. A typical example for this has already been given in form of the category **Cxt**, that was easily seen to be equivalent to $\mathbf{Sem}_\vee$.

The representations given below are grouped according to these observations: we start with "logical" descriptions that have their closest relationships to the categories **Cxt** and $\mathbf{Sem}_\vee$, and then proceed to formulations that can more naturally be connected to **Alg**. Classifying representations in this way is by no means arbitrary: as we will see the end of this section, our arrangement reflects the "localic" respectively "spacial" side of a very specific case of Stone duality.

3.4.1 Logic and information systems

The representation of join-semilattices via formal contexts did already incorporate some logical flavor: approximable concepts can be viewed as sets closed under a certain entailment relation. Scott continuity of this closure is reminiscent of the compactness property of a logic. However, we will see that a much closer connection to some very well-known logics can be made. The reader is referred to [DH01] for related considerations.

Definition 3.4.1 Given a set A of propositions, the set of well-formed *conjunctive propositional formulae* $\mathscr{S}(A)$ over A is given by the following expression:

$$\mathscr{S}(A) \ ::= \ \top \ \mid \ a \in A \ \mid \ (\mathscr{S}(A) \wedge \mathscr{S}(A))$$

A relation $\vdash \ \subseteq \mathscr{S}(A) \times \mathscr{S}(A)$ is a *consequence relation* of conjunctive propositional logic (CP logic) if it is closed under application of the following rules:

$$F \vdash \top \quad (\mathrm{T}) \qquad F \vdash F \quad (\mathrm{R}) \qquad \frac{F \vdash G,\ G \vdash H}{F \vdash H} \quad (\mathrm{Cut})$$

$$\frac{F \vdash (G \wedge H)}{F \vdash G} \quad (\mathrm{W1}) \qquad \frac{F \vdash (G \wedge H)}{F \vdash H} \quad (\mathrm{W2}) \qquad \frac{F \vdash G,\ F \vdash H}{F \vdash (G \wedge H)} \quad (\mathrm{And})$$

In this case $(\mathscr{S}(A), \vdash)$ is a *deductive system* (of CP logic). For any two formulae $F, G \in \mathscr{S}(A)$, the situation where $F \vdash G$ and $G \vdash F$ is denoted $F \approx G$.

Hence deductive systems are logical systems of the conjunctive fragment of propositional logic, together with a (not necessarily minimal) consequence relation. The following properties are easily verified.

Lemma 3.4.2 Consider a deductive system $(\mathscr{S}(A), \vdash)$. The following hold for all formulae F, G, and $H \in \mathscr{S}(A)$:

- $((F \wedge G) \wedge H) \approx (F \wedge (G \wedge H))$
- $(F \wedge G) \approx (G \wedge F)$
- $F \approx (F \wedge F)$
- $F \approx (F \wedge \top)$

Hence we see that the rules (W1), (W2), and (And) imply associativity, commutativity, and idempotency of $\wedge$. Furthermore, occurrences of $\top$ can be eliminated. Consequently, we henceforth write formulae of CP in the form $a_1 \wedge a_2 \wedge \ldots \wedge a_n$ ($a_i \in A$), knowing that this determines a set of "real" formulae up to proof-theoretic equivalence. Additionally, for the case $n = 0$ the above expression is interpreted as the singleton set $\{\top\}$. Any statement about formulae in this notation represents the corresponding set of statements about the original formulae. We can now consider the algebraic semantics (see [DH01]) of these logics. This is based largely on the following notion:

Definition 3.4.3 Consider a deductive system $(\mathscr{S}(A), \vdash)$. The *Lindenbaum algebra* of $(\mathscr{S}(A), \vdash)$ is the poset obtained from the preorder $(\mathscr{S}(A), \vdash)$ through factorization by the equivalence relation $\approx$, i.e. $[F]_\approx \leq [G]_\approx$ iff $F \vdash G$. The Lindenbaum algebra is denoted by $\mathsf{LA}(\mathscr{S}(A), \vdash)$.

Hence the Lindenbaum algebra is a partially ordered set of $\approx$-equivalence classes of formulae, ordered by syntactic entailment. Since it can cause hardly any confusion, we take the freedom to denote equivalence classes by one of their representatives or even by the simplified notation introduced above. Of course, this creates possible ambiguity between the conjunction symbol and the meet operation within the Lindenbaum algebra. The following lemma shows that this is not a problem.

Lemma 3.4.4 Consider a deductive system $(\mathscr{S}(A), \vdash)$ and formulae $F, G \in \mathscr{S}(A)$. Then $[F]_\approx \wedge [G]_\approx = [F \wedge G]_\approx$.

Proof. We have to show that $F \wedge G \vdash F$, $F \wedge G \vdash G$, and that for any formula H such that $H \vdash F$ and $H \vdash G$, we find $H \vdash F \wedge G$. These assertions are obvious consequences of the proof rules of CP. □

Since the meet operation yields a unique result, this shows that $F \approx F'$ and $G \approx G'$ imply $F \wedge G \approx F' \wedge G'$, which is just the *Replacement Theorem* [DH01] for CP logics. We now state the now obvious representation theorem.

Theorem 3.4.5 For any deductive system $(\mathscr{S}(A), \vdash)$, the Lindenbaum algebra $\mathsf{LA}(\mathscr{S}(A), \vdash)$ is a meet-semilattice with greatest element. Conversely, every such semilattice is isomorphic to the Lindenbaum algebra of some deductive system.

Proof. Lemma 3.4.4 already showed the existence of binary meets. We conclude the first part of the proof by noting that $[\top]_\approx$ is the required greatest element.

For the converse let S be a meet-semilattice with greatest element. We define a consequence relation $\vdash$ on $\mathscr{S}(S)$ by setting, for all $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in S$, $a_1 \wedge a_2 \wedge \ldots \wedge a_n \vdash b_1 \wedge b_2 \wedge \ldots \wedge b_m$ whenever $a_1 \wedge a_2 \wedge \ldots \wedge a_n \leq b_1 \wedge b_2 \wedge \ldots \wedge b_m$. One can easily check that this definition satisfies all of the required rules. Note that (T) follows by our convention to represent $\top$ by the empty conjunction. To reduce confusion, we denote meets in S by $\bigwedge$ and meets in $\mathsf{LA}(\mathscr{S}(S), \vdash)$ by $\bigwedge_\approx$.

We claim that S is isomorphic to $\mathsf{LA}(\mathscr{S}(S), \vdash)$. Indeed, one can define mappings $f : S \to \mathsf{LA}(\mathscr{S}(S), \vdash)$ and $g : \mathsf{LA}(\mathscr{S}(S), \vdash) \to S$ by setting $f(a) = [a]_\approx$ and, for propositions a_i, $1 \leq i \leq n$, $g[\bigwedge_\approx a_i]_\approx = \bigwedge a_i$. To see that g is well-defined, note that for any two formulae $\bigwedge_\approx a_i, \bigwedge_\approx b_j \in \mathscr{S}(S)$ we have that $\bigwedge_\approx a_i \approx \bigwedge_\approx b_j$ (in $\mathscr{S}(S)$) implies $\bigwedge a_i = \bigwedge b_j$ (in S) by the definition of $\vdash$.

Finally, we show that g and f are inverse to each other. By what was said above, $g(f(a)) = a$ is immediate. On the other hand, any formula $\bigwedge_\approx a_i$ is syntactically equivalent to $\bigwedge a_i$ by the definition of $\vdash$. This shows bijectivity of f and g. Monotonicity of both functions is obvious from their definition. □

This relationship closes the gap to our prior category $\mathbf{Sem}_\vee$, since the above meet-semilattices are just the order duals of the objects within this category. By an approximable mapping between two meet-semilattices with least element or two deductive systems of CP logic, we mean an approximable mapping between the induced join-semilattices. The following then is immediate.

Theorem 3.4.6 Consider the categories $\mathbf{Sem}_\wedge$ and $\mathbf{CP}$ of meet-semilattices with greatest element and deductive systems of CP logic, respectively, together with approximable mappings as morphisms. Then $\mathbf{Sem}_\vee$, $\mathbf{Sem}_\wedge$, and $\mathbf{CP}$ are equivalent.

The insights just obtained allow to relate our study with results obtained in [HW03, Hit04], where the conjunctive fragment of the logic RZ (introduced in [RZ01]), was found to be closely related to concept closure in FCA. We derive a very similar result, but some preparations are needed first.

An *algebraic cpo D* is a dcpo with least element $\perp$ such that every $e \in D$ is the directed supremum of all compact elements below it. Note that this definition is in fact very similar to that of an algebraic lattice, with the additional requirement that the mentioned sets of compact elements be directed, which is automatic in the case of lattices (Corollary 3.1.1). A *coherent algebraic cpo* is an algebraic cpo with a coherent Scott topology (see Definitions 2.4.8 and 2.4.11).

These notions can be found in [RZ01], along with a characterization of the *Smyth Powerdomain* of any given coherent algebraic cpo D by means of a logic defined on D, which we call the *logic RZ*. We will only be concerned with the conjunctive fragment of RZ, which can be given as follows. For compact elements $c_1, \ldots, c_n, d_1, \ldots, d_m$ we write $c_1 \wedge \ldots \wedge c_n \vdash d_1 \wedge \ldots \wedge d_m$ iff any minimal upper bound of $\{c_1, \ldots, c_n\}$ is above all d_i. This way, we obtain a deductive system $(\mathsf{K}(D), \vdash)$, and the following result, which is related to those in [HW03, Hit04], and such considerations were put to use in [Hit04] for developing a generic non-monotonic rule-based reasoning paradigm over hierarchical knowledge.

Theorem 3.4.7 Let $\mathbb{K} = (G, M, I)$ be a formal context. Then there is a coherent algebraic cpo D and a mapping $\iota : M \to D$ such that for every finite set $X = \{m_1, \ldots, m_n\} \subseteq M$ we have $X^{II} = \{m \mid \iota(m_1) \wedge \ldots \wedge \iota(m_n) \vdash \iota(m)\}$.

Proof. Define $D = \mathsf{Alg}(\mathbb{K})$ and set $\iota(m) = \{m\}^{II}$ for $m \in M$. Since D is a complete algebraic lattice, it is a coherent algebraic cpo.

Now consider the finite set X as above. Using the completeness of the lattice, we obtain that $\iota(X)$ has X^{II} as supremum, which suffices. □

The difference between Theorem 3.4.7 and the results in [HW03, Hit04] lies in the fact that the latter were proven by taking D to be a sublattice of the (classical) formal concept lattice, instead of $\mathsf{Alg}(\mathbb{K})$, which facilitates reasoning with formal contexts in a natural way.

Finally, we come to another popular description of algebraic lattices, that fits well into the above discussion, and will also shed additional light on morphisms of **CP**.

Definition 3.4.8 Consider a structure $(A, \Vdash)$, where A is a set, and $\Vdash \subseteq \mathsf{Fin}(A) \times A$ is a relation between finite subsets of A and elements of A. Then $(A, \Vdash)$ is a *Scott information system* (with trivial consistency predicate) if the following hold:

(IS1) $a \in X$ implies $X \Vdash a$,

(IS2) if $X \Vdash y$ for all $y \in Y$ and $Y \vdash a$, then $X \Vdash a$.

Scott information systems were introduced in [Sco82a] as a logical characterization of order structures arising in denotational semantics. The connection to CP logic is as follows.

Proposition 3.4.9 There is a bijective relationship between Scott information systems and deductive systems of CP logic.

Proof. Consider a Scott information system $(A, \Vdash)$. Using the set A as propositions, we obtain the set of CP formulae $\mathscr{S}(A)$. A consequence relation $\vdash$ for $\mathscr{S}(A)$ is defined by setting $a_1 \wedge a_2 \wedge \ldots \wedge a_n \vdash b_1 \wedge b_2 \wedge \ldots \wedge b_m$ whenever $\{a_1, a_2, \ldots, a_n\} \Vdash b_i$ for all $i = 1, \ldots m$. We have to verify that $\vdash$ is closed under the rules given in Definition 3.4.1. For the case $m = 0$ the condition is obviously true so that we obtain axiom (T). Likewise, the conditions for axiom (R) are satisfied due to condition (IS1) in Definition 3.4.8. Similarly, the (Cut) rule follows immediately from (IS2). For the rules (W1), (W2), and (And), we simply notice that these are direct consequences from our definition of $\vdash$.

Now for the opposite direction, consider a deductive system $(\mathscr{S}(A), \vdash)$. Using the set of propositions of $\mathscr{S}(A)$ as attributes, we construct a Scott information system $(A, \Vdash)$, where we define $\{a_1, a_2, \ldots, a_n\} \Vdash b$ whenever $a_1 \wedge a_2 \wedge \ldots \wedge a_n \vdash b$. Again it is straightforward to check that this is indeed an information system. (IS1) can be deduced from the rules (R) and iterated applications of (W1) and (W2). Under the assumption of (IS2), we see that the (And) rule allows us to construct a conjunction that corresponds to the premise Y of the second rule. By (Cut) this yields the required entailment.

To complete the proof, we note that these two constructions are in fact inverse to each other. The identity on Scott information systems is trivial. For CP logics, we note that any sequent $a_1 \wedge a_2 \wedge \ldots \wedge a_n \vdash b_1 \wedge b_2 \wedge \ldots \wedge b_m$ induces via (W1)/(W2) the existence of sequents $a_1 \wedge a_2 \wedge \ldots \wedge a_n \vdash b_i$, for all $i = 1, \ldots, m$. The original sequent can then be reconstructed from the entailment of the Scott information system induced from these relations. □

Note that this proposition yields a bijective correspondence, not just a relationship up to isomorphism. Indeed Scott information systems are essentially an efficient formulation of conjunctive propositional logic, where the properties of $\wedge$ are obtained implicitly by using sets in the first place. The category of Scott information systems and approximable mappings between the induced semilattices is denoted **SIS**.[5] From Proposition 3.4.9 one easily concludes that **SIS** is *isomorphic* to **CP**, and hence also equivalent to all categories mentioned earlier.

Furthermore, approximable mappings between CP logics need not be expressed on the level of their Lindenbaum algebras, but could be formulated directly on formulae. From this viewpoint, approximable mappings appear as consequence relations between different logical languages. Indeed, all the requirements of Definition 3.2.1 do still have a very intuitive reading under this interpretation: (am1) and (am2) correspond to (T) and (And) of Definition 3.4.1, respectively, while (am3) can be viewed as a modified (Cut) rule, that also subsumes (W1) and (W2). Hence we recognize approximable mappings as a simple form of *multilingual sequent calculi* as introduced in [JKM99] for the more complicated case of

[5] Historically, this is indeed the first context for which *approximable mappings* were defined [Sco82a].

positive logics (i.e., logics including conjunction and disjunction) without the rule (T). We will come back to the idea of such *consequence relations* in Chapter 5.

We remark that one could as well have connected CP logic or information systems directly to algebraic lattices, instead of presenting the ideal completion for semilattices of compacts. In the case of logics, algebraic lattices are obtained directly as sets of *models* of a deductive system, where models are considered as deductively closed sets of (true) formulae. These turn out to be exactly the filters (Definition 2.1.7) within the corresponding Lindenbaum algebras, and the duality to ideal completion is immediate. The reader may care to consult [DH01] for a general treatment of such matters. For Scott information systems, algebraic lattices are constructed similarly as sets of *elements*. As defined in [Sco82a], an element of an information system $(A, \Vdash)$ is a subset $x \subseteq A$ such that $a \in x$ whenever there is some finite set $X \subseteq x$ with $X \Vdash a$.

Our logical considerations can also be put to practical use by noting that every *definite logic program* (see, e.g., [Llo87]) can be expressed by a deductive system in the above sense. This has also been mentioned in [Zha03]. Considering the fact that the theory of definite logic programs is quite well-developed, these insights are merely providing some further explanation for the situation in this field. In the light of the connections to Stone duality outlined below and the immediate connection to algebraic semantics of logical systems, one could also further analyze the situation for more expressive logical languages from this perspective.

Note that only a small portion of Scott information systems and algebraic lattices can be obtained from definite logic programs. The reason is that there are only countably many different programs, but uncountably many Scott information systems (even for countable sets of generators). We also remark that, while algebraicity always makes fixed point computation possible in theory, the specific structure of the information systems of logic programs is employed to ensure that the semantic operator suitable for logic programs is indeed effectively computable.

We do not bother to give a category of logic programs, although this could be done by adjusting the formalism of approximable mappings. However, it is not clear to us how the subcategory of algebraic lattices that arises in this way can be characterized.

3.4.2 The Scott topology

Next we want to study the spacial side of Stone duality. It is here where we find the models and their semantic entailment, while the *localic* side is inhabited by syntactic representations and their proof theory. We already mentioned that models in our case take the specific form of algebraic lattices. From a domain theoretic perspective, the natural topology for these structures is the Scott topology as introduced in Definition 2.4.8. The practical justification for this choice is that this topology, when defined on an algebraic lattice, has some rather specific properties such as order consistency and coherency (Definitions 2.4.10 and 2.4.11). Proof for the following statements can also be found in [AJ94, GHK⁺03, Joh82].

Proposition 3.4.10 Consider an algebraic lattice L. We have the following:

(i) $\Sigma(L)$ is order consistent.

(ii) The set $B = \{\uparrow c \mid c \in \mathsf{K}(L)\}$ is a base for $\sigma(L)$.

(iii) The compact opens of $\Sigma(L)$ are exactly the finite unions of members of B.

(iv) $\sigma(L)$ is coherent.

(v) $\sigma(L)$ is sober.[6]

Proof. For (i) consider $x \le y$. Then clearly y is contained in any Scott open that contains x. We still have to show the converse: if $x \not\le y$ then there is a Scott open set O such that $x \in O$ and $y \notin O$. By algebraicity, every element is the supremum of the compact elements below it. Thus, there is a compact $c \le x$ such that $c \not\le y$, since otherwise monotonicity of join would imply $x \le y$. Now it is easy to see that $\uparrow c$ is Scott open: it is certainly an upper set and inaccessibility by directed suprema is just the compactness of c. Since $x \in \uparrow c$ and $y \notin \uparrow c$ this shows that $x \not\le y$ in the specialization order.

For (ii) we already noted that all members of B are Scott open. Now whenever a Scott open contains an element x then it also contains some compact element below x, since the directed supremum of all such compacts is x. Hence any open set is indeed the union of all members of B it contains.

To show Claim (iii), note that a finite union O of elements of B is compact, since any directed set of opens is closed under finite unions. The converse is obvious.

Item (iv) is an easy consequence of (iii). Consider two compact Scott opens O and O', where $O = \bigcup \uparrow c_i$ and $O' = \bigcup \uparrow c_j$ for two finite families $(c_i)_{i \in I}$ and $(c_j)_{j \in J}$ of compacts. Then $O \cap O' = \bigcup \{\uparrow (c_i \vee c_j) \mid i \in I,\ j \in J\}$, where each of the joins is compact by Proposition 3.1.1. By (iii) $O \cap O'$ is compact open.

For (v) see [GHK⁺03, Corollary II-1.12]. We also obtain this as a corollary of our topological presentation theorems in the next section. □

Order consistency insures that algebraic lattices and their Scott topologies uniquely characterize each other. A category $\mathbf{\Sigma_{Alg}}$ of Scott topologies on algebraic lattices is readily obtained by employing continuous maps between topologies as morphisms.

[6] We did not define sobriety in this document. Readers who are not familiar with this concept may safely ignore this statement.

Theorem 3.4.11 The categories **Alg** and $\mathbf{\Sigma_{Alg}}$ are isomorphic, hence equivalent.

Proof. The required functors are defined on objects by taking the Scott topology and the specialization order of the arguments, respectively. By order consistency of the topologies, this yields a bijection between the classes of objects. Since the carrier sets of lattices and topologies remain unchanged, one can consider every function between algebraic lattices directly as a function between spaces and vice versa. To finish the proof, we use that a function between algebraic lattices is Scott continuous iff it is continuous with respect to the Scott topologies (Proposition 2.4.9). □

In the next section, we see that the topological spaces of $\mathbf{\Sigma_{Alg}}$ are indeed very specific.

3.4.3 Stone duality

Since the very beginning of the theory, Stone duality has been recognized as a tool for relating proof theory, algebraic semantics, and model theory of logical systems (see [Sto37b]). One direction of this investigation has already been mentioned in Section 3.4.1: Lindenbaum algebras can be represented by corresponding model theories, where models are characterized as subsets (filters) of formulae. Dually, one could also have presented every formula by the set of its models. The conceptual step from such systems of specific subsets to *topological spaces* was the key to the strength and utility of Stone's original representation theorems.

However, it still took decades to recognize that it would be even more advantageous to undo this step to the spacial side of Stone duality and to return to the more abstract world of partially ordered sets. It became apparent that topologies could not only serve as a representation for specific ordered structures, but that conversely orders could serve as a general substitute for topological spaces. Indeed, the leap to the spacial side is usually not an easy one – in many cases it cannot be made within classical Zermelo-Fraenkel set theory (ZF). The localic side on the other hand can mimic most of the features of the original topological setting, while being freed from the weight of points which often prevent purely constructive reasoning.

In what follows we embed our specific scenery into the setting of Stone duality. However, it turns out that the special case we consider does not justify to present the theory in its common generality. Hence we give explicit proofs for the object level relationships in our specialized setting and hint at the connections to more abstract versions of Stone duality where appropriate. Other than providing the merit of a more self-contained presentation, this also enables us to work exclusively in ZF, with no additional choice principles whatsoever. In contrast, in

Chapter 5 we will obtain various results that are only valid when Axiom 2.1.9 is assumed to hold. As a general reference on Stone duality, we recommend [Joh82].

The passage from spaces to orders is a particularly simple one: the open set lattice of a topology is already a poset. The class of posets arising in this way are the *spacial locales*.

Definition 3.4.12 A complete lattice L is a *locale* if the following infinite distributive law holds for all $S \subseteq L$ and $x \in L$:

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}.$$

A *point of a locale* is a principal prime ideal of L, i.e. a subset $p \subseteq L$ such that $p = \downarrow \bigwedge p$ and, for any $x \wedge y \in p$, $x \in p$ or $y \in p$. The set of all points of L is denoted $\mathsf{pt}(L)$.

A locale is *spacial* if, for any two elements $x, y \in L$ with $x \not\leq y$, there is a point $p \subseteq L$ such that $x \in p$ and $y \notin p$. L is *spectral* if L is algebraic, its greatest element is compact, and the meet of any two compact elements of L is compact.

We remark that locales are also called *frames*, and that structures of this kind are equivalently characterized as *complete Heyting algebras*.[7]

It is now easy to see that any open set lattice yields a locale, where distributivity follows from the corresponding distributivity of finite intersections over infinite unions. Furthermore, Proposition 3.4.10 (ii), (iii), and (iv) show that, for an algebraic lattice L, $(\sigma(L), \subseteq)$ is even a spectral locale. We shall find that these locales are even more specific than this.

Our starting point for investigating topologies were algebraic lattices, which we have earlier recognized as the model theories of deductive systems of CP logics. The abstraction to (certain) spectral locales brings us back to proof theory. We now characterize the above locales by relating them to Lindenbaum algebras of CP logic, and reobtain topological spaces from this data.

We consider arbitrary meet-semilattices with greatest element, knowing that they are up to isomorphism just the Lindenbaum algebras of CP (Theorem 3.4.5). Furthermore, we already mentioned that the collections of all filters (the order-dual concepts of the ideals) of such semilattices are just the algebraic lattices, which follows immediately from Theorem 3.1.3. We can now give a characterization for the locale of Scott open sets of algebraic lattices:

Theorem 3.4.13 Consider a meet-semilattice S with greatest element and the corresponding algebraic lattice $(\mathsf{Flt}(S), \subseteq)$ of filters of S. The collection of lower sets of S, ordered by subset inclusion, is isomorphic to $\sigma(\mathsf{Flt}(S))$. Every Scott open set lattice of an algebraic lattice is of this form.

[7] A complete Heyting algebra is a Heyting algebra (Definition 2.2.6) that is a complete lattice. The interested reader will find details in [Joh82, GHK+03].

Proof. Theorem 3.1.3 shows the bijective correspondence between the elements of S and the compacts of $\mathsf{Flt}(S)$, since S is dually order-isomorphic to $\mathsf{K}(\mathsf{Flt}(S))$. Proposition 3.4.10 demonstrates that every Scott open set is characterized by the compact elements it contains. Now it is obvious that such sets of compacts correspond to upper sets in the join-semilattice of compacts, and thus to lower sets in its dual meet-semilattice. The other direction is also immediate from the according part of Theorem 3.1.3. □

Hence the spectral locales of the form $\sigma(L)$ for some algebraic lattice L are more precizely characterized as the lower set topologies of meet-semilattices with greatest element, and the Scott topology on L coincides with the Alexandrov topology on $K(L)^{\mathsf{op}}$. Note also that all meets and joins within these locales are really given by the corresponding set operations. By $\sigma_{\mathbf{Alg}}$ we denote the category of all locales isomorphic to the collection of lower sets on some meet-semilattice with greatest element together with *frame homomorphsism*, i.e. functions that preserve finite meets and arbitrary joins.[8]

Next we want to connect up with the common constructions of Stone duality.

Lemma 3.4.14 Consider a meet-semilattice with greatest element S and its locale of lower sets σ. Then the meet-prime elements of σ are exactly the complements of the filters of S.

Proof. Let $F \subseteq S$ be a filter and set $A = S \setminus F \in \sigma$. Now assume there are lower sets $B_1, B_2 \in \sigma$ such that $B_1 \cap B_2 = A$. For a contradiction, assume that there are elements $b_1 \in B_1 \cap F$ and $b_2 \in B_2 \cap F$. Then $b_1 \wedge b_2 \in F$ and $b_1 \wedge b_2 \in B_1 \cap B_2 = A$ – a contradiction. Hence, one of B_1, B_2 contains just the elements of A as required.

Conversely, let $A \in \sigma$ be meet-prime and consider the upper set $F = S \setminus A$. For any two elements $a, b \in F$ it is easy to see that $\downarrow a \cap \downarrow b = \downarrow(a \wedge b)$. Hence, if $a \wedge b \in A$ then $\downarrow a \cup A$ and $\downarrow b \cup A$ are elements of σ with intersection A, which cannot be. Hence $a \wedge b \in F$ as required. □

This gives us all necessary information about the points of these locales (see Definition 3.4.12), since these were defined to be just the principal ideals generated by meet-prime elements. We can thus identify the set of points $\mathsf{pt}(\sigma)$ with the set of all meet-prime elements of σ.[9] Our insights allow us to give a direct description of the topological spaces associated with semilattices.

[8] Typically, a category of locales would rather be described by the dual of this category, which of course has an easy concrete representation: According to Theorem 2.2.5, frame homomorphisms are lower adjoints and thus induce unique upper adjoints in the opposite direction. We have chosen to trade some terminological precision for conciseness of the presentation.

[9] Furthermore, we remark that this guarantees a sufficient supply of prime elements without invoking any additional choice principles, i.e. we are dealing with a class of locales that is spacial in Zermelo-Fraenkel set theory. This contrasts with the class of all spectral locales, which are

Corollary 3.4.15 Let S be a meet-semilattice with greatest element, let L be an algebraic lattice, and let σ be a spectral locale, such that

- S^{op} is isomorphic to $\mathsf{K}(L)$ and
- σ is isomorphic to $\sigma(L)$.

Then the following are homeomorphic:

(i) $(L, \sigma(L))$, the Scott topology on L;

(ii) the topology on $\mathsf{Flt}(S)$ generated from the basic open sets

$$O_a = \{F \in \mathsf{Flt}(S) \mid a \in F\} \qquad \text{for all } a \in S;$$

(iii) the topology on $\mathsf{pt}(\sigma)$ given by the open sets

$$P_A = \{p \in \mathsf{pt}(\sigma) \mid A \notin p\} \qquad \text{for all } A \in \sigma.$$

Proof. Most of the above should be obvious at this stage, so we spare out some details. Suitable bijections between L, $\mathsf{Flt}(S)$, and $\mathsf{pt}(\sigma)$ have been obtained in 3.1.3 and 3.4.14. First we show the homeomorphism between (i) and (ii) (which induces also that (O_a) is indeed a base). For this we only have to note that $O_a = \{F \in \mathsf{Flt}(S) \mid \uparrow a \subseteq F\}$. Using the bijection between (principal) filters and (compact) elements from Theorem 3.1.3, one sees that O_a corresponds to an open set $\uparrow c$, $c \in \mathsf{K}(L)$, of (i). The fact that these subsets are open and form the basis for the Scott topology has been shown in Proposition 3.4.10.

For the homeomorphism between (ii) and (iii), we consider the locale of lower sets of S, which is isomorphic to σ by Theorem 3.4.13. Clearly this affects the topology of (iii) only up to homeomorphism. Now in the locale of lower sets, a point (principal prime ideal) $p = \downarrow B$ is in P_A iff the corresponding meet-prime$B$ does not contain A. But this is the case iff the complement of B intersects A. Hence, by Lemma 3.4.14, P_A corresponds exactly to the collection of those filters of S that contain some element of A, i.e. to the set $\bigcup\{O_a \mid a \in A\}$. But these are precisely the open sets of the topology of (ii). □

With respect to the given preconditions on the relationship between S, L, and σ, note that the various transformations between semilattices, algebraic lattices, and locales established earlier yield a variety of equivalent ways to state that the three given objects describe "the same thing."

To complete the targeted categorical equivalence between the dual category of $\sigma_{\mathbf{Alg}}$ and $\mathbf{\Sigma_{Alg}}$ ($\mathbf{Alg}$, $\mathbf{Sem}_\vee$, ...), one still needs to prove a suitable bijection of

only spacial when the existence of prime ideals is explicitly postulated, i.e. when Axiom 2.1.9 is assumed to hold.

homsets. This correspondence between inverse frame homomorphisms and continuous functions is a basic result of Stone duality which we will obtain later on as a corollary of Theorem 5.3.1. Another possible construction can be found in [Joh82] or [WP, Article "Stone duality"].

3.5 Summary and further results

We provided characterizations of the category of algebraic lattices by means of structures from logic, topology, domain theory, and formal concept analysis. More precisely, we characterized algebraic lattices by certain semilattices, formal contexts, and deductive systems of the conjunctive fragment of propositional logic. The novel category **Cxt** of formal contexts and approximable mappings was used to establish the cartesian closure of these categories, and the categorical constructions needed for this were explicitly given. Other representations referred to special classes of closure systems, Scott topologies, locales, and definite logic programs. An overview of the major equivalences given herein is displayed in Figure 3.2.

Although this treatment is quite comprehensive, one could still add some more equivalent formalisms. Especially, we left out the *coverage technique* of [Joh82] (see also [Sim04]), which represents locales in a syntactical way that relates closely to Scott information systems. Furthermore, we deliberately ignored Scott's earlier approach to presenting domains via *neighborhood systems* [Sco82b], since these structures are not much more than a mixture of the later (token-set based) information systems and continuous closure operators.

3.5.1 Further logics

In this chapter we have also presented a unified treatment of the basic techniques and mechanisms that are applied to join domain theory, algebra, logic, and topology. Algebraic lattices turn out to be the simplest case where such a discussion is feasible, but the given results can be extended to the case of more expressive logics.

For classical propositional logic, one obtains Boolean algebras as Lindenbaum algebras. Given a model of this logic, one finds that a formula $(F \vee G)$ is mapped to true, only if at least one of F and G is. Further investigations reveal that the set of models can indeed be identified with the set of prime filters of a Boolean algebra – a statement that is also true for other propositional logics that allow for disjunction.

Thus one cannot extend the results of this chapter by simply considering Boolean algebras as special cases of meet-semilattices, since the notion of a prime

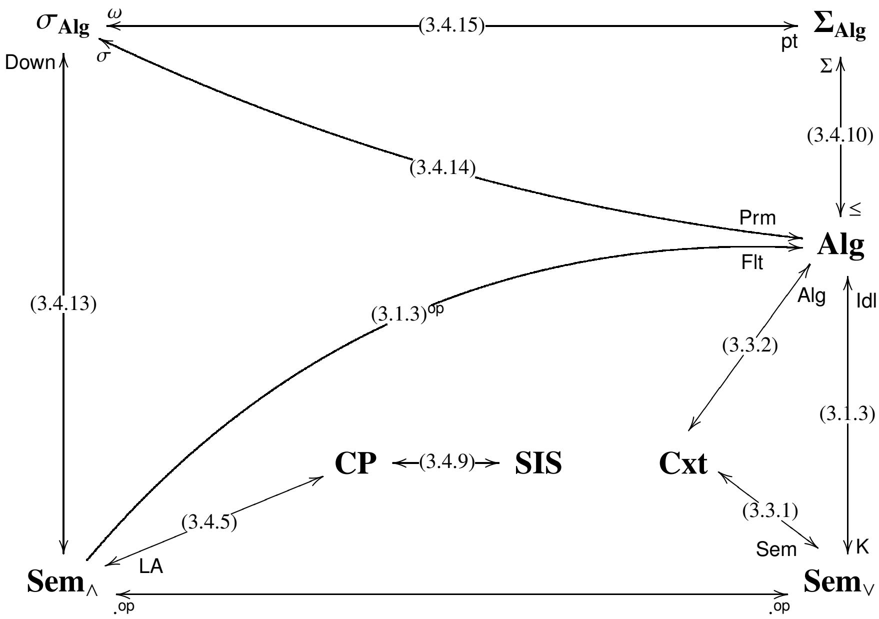


Figure 3.2: Summary of all established equivalences with reference to the corresponding (object-level) statements. Labels at the arrow tips specify the name of the functor that was used in a construction, where Down denotes the construction of the lower set topology from a meet-semilattice, Prm yields the set of principal prime ideals of a locale, ordered by subset inclusion, and ≤ denotes the construction of the specialization order from a topological space.

filter is not feasible in this case. Instead, we need to base our considerations on lattices which are distributive (since this is a basic feature of conjunction and disjunction in logic) and bounded (to account for truth and falsity). Within the framework of Stone duality, such lattices appear as the lattices of compact elements of a spectral locale (Definition 3.4.12). We already noticed that the locales in the above investigations have always been spectral, and thus we immediately obtain a bounded distributive lattice for every meet-semilattice. Another possibility to obtain this lattice is to construct the *free* bounded distributive lattice over a given meet-semilattice with greatest element.

The filters (models) of the semilattice are in bijection with the prime filters of this newly constructed lattice, which generally suggests to base further logical considerations on bounded distributive lattices and their prime filters. As mentioned above, spectral locales arise in this setting as the ideal completions of bounded distributive lattices. On the spacial side, however, one obtains topological spaces that might no longer be described by the Scott topology. Indeed, of all the descriptions given in Corollary 3.4.15, only item (iii) yields a description of the topological space that emerges in the general case.[10] With this framework in mind, we can give an overview of the results for some other logics.

The easiest extension of CP logic is to add logical falsity $\perp$. In terms of proof theory, this allows for additional constraints of the form $\bigwedge X \Vdash \perp$, assuring the *inconsistency* of the finite set X – a construction well-known under the notion of *integrity constraint* in database theory. The according Lindenbaum algebras are meet-semilattices that have both a greatest and a least element. Since this least element is preserved when constructing the free bounded distributive lattice, the prime filters of this lattice now correspond to the *proper* filters of the semilattice (which are exactly those filters that do not contain the least element). The posets of models obtained in this way turn out to be exactly the Scott domains (the bounded complete algebraic cpos), and their Scott topology coincides with the resulting Stone space. This case has originally been studied by Scott and lead to the definition of his information systems [Sco82a].

As another step, one can include disjunction into the formalism to obtain *positive logic* with falsity. This already leads to a substantial complication of the theory: some choice principle like Axiom 2.1.9 is now needed to find sufficiently many models. Lindenbaum algebras now are bounded distributive lattices, and the emerging class of locales are exactly the spectral ones. Ordering prime filters (models) by inclusion, one finds a curious class of dcpos that have been termed *information domains* in [DG90]. Later the direct construction of distributive lattices and locales from the according deductive systems was studied in [CC00] and [CZ00].

Further strengthening of the logic is possible by including some internal negation operation. Intuitionistic negation yields Heyting algebras as Lindenbaum algebras. The resulting topologies are already studied in [Sto37b], though the significance of specialization orders and domain theoretic concepts were not yet recognized at this time. In fact, we are not aware of a treatment that investigates the posets of models that appear in this setting from a domain theoretical perspective. However, also in the light of the next paragraph, one might presume that the order-theoretical features of such posets are very weak.

If classical negation is introduced, thus yielding classical propositional logic, the class of Boolean algebras provides the well-known algebraic semantics. While topological representation via Stone's theorem is rather pleasant in this case, the domain theoretic aspects are quite disappointing: the specialization order of models is discrete. Since the emerging Stone spaces are not discrete, the Scott topology is not an appropriate tool in this situation. Related approaches nevertheless have been taken for the context of negation in logic programming [Sed95, Hit04], but

[10] The according topology is always coarser than the Scott topology, see [AJ94, Proposition 7.2.13].

the domain-theoretic content of these investigations remains to be determined.

For reasons as those just described, internal negation is usually not considered in domain-theoretical studies. However both inconsistency of finite subsets and finite disjunctions can be employed with various restrictions to obtain classes of domains that are more general than the Scott domains. A slight constraint on either the logical ([DG90]) or the localic level ([Abr91]) restricts the obtained class of dcpos (of models) to the coherent algebraic dcpos. However, while this is a well-known concept in domain theory, it results in rather unusual restrictions on the logics (Lindenbaum algebras, locales). Further conditions will lead to SFP-domains [Abr91, Zha91]. On the other hand, conditions that characterize a class of deductive systems that produces exactly the L-domains have been studied in [Zha92].

One result from these considerations is that all the above logics are basically specializations of positive logic – the logic of distributive lattices with greatest element. Conjunctive logic and Scott information systems restrict to certain free distributive lattices, while intuitionistic logic and classical logic focus on subclasses with additional lattice-theoretic operations. This observation allows for the application not only of Stone's but also of Priestley's representation theorem ([DP02, Joh82]) on these structures.

Techniques similar to those described above were also applied to a clausal logic in [RZ01], leading to a characterization of Smyth powerdomains. Subsequently, this logic was extended to non-monotonic reasoning paradigms on hierarchical knowledge [RZ01, Hit04].

Chapter 4

Morphisms in FCA

The theme of this chapter is the extension of the theory of morphisms between formal contexts, both in order to enhance our understanding of the mathematical foundations within this field, and to prepare useful results for Chapter 5. The following exposition largely agrees with [KHZ05].

The struture of this chapter is as follows. In Section 4.1, we study dual bonds and their relationships to direct products of formal contexts and Galois connections. In Section 4.2, dual bonds featuring certain continuity properties will be identified as an important subclass. Section 4.3 will deal with the relationship between scale measures, functional types of dual bonds, and Galois connections, while Section 4.4 is devoted to infomorphisms. In Section 4.5, we summarize some of our results in form of a concept lattice of context-morphisms, which we obtain by attribute exploration. We conclude our results by discussing various possible directions for future research in Section 4.6.

4.1 Dual bonds and the direct product

The construction of concept lattices exploits the fact that the derivation operators $\cdot^I$ form an antitone Galois connection (see Section 2.3). Hence Galois connections naturally are also of special interest when looking for suitable morphisms between concept lattices.[1] In order to represent Galois connections on the level of contexts, functions between the sets of attributes or objects turn out to be too specific. Instead, one makes use of certain relations called *dual bonds* which we shall study in this section. Most of the material before Lemma 4.1.7 can be found in [GW99, Xia93, Gan04].

[1]We will mainly work with antitone Galois connections within this chapter, since these are much more common in FCA than their monotone relatives.

Definition 4.1.1 A *dual bond* between formal contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ is a relation $R \subseteq G \times H$ for which the following hold:

- for every object $g \in G$, g^R (which is equal to $R(g)$) is an extent of $\mathbb{L}$ and
- for every object $h \in H$, h^R (which is equal to $R^{-1}(h)$) is an extent of $\mathbb{K}$.

This definition is motivated by the following result.

Theorem 4.1.2 ([GW99] Theorem 53) Consider a dual bond R between contexts $\mathbb{K}$ and $\mathbb{L}$ as above. The mappings

$$\vec{\phi}_R : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}) : X \mapsto X^R \quad \text{and} \quad \overleftarrow{\phi}_R : \mathsf{B}_o(\mathbb{L}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto Y^R$$

form an antitone Galois connection between the (object) concept lattices of the contexts $\mathbb{K}$ and $\mathbb{L}$.

Conversely, given such an antitone Galois connection $(\vec{\phi}, \overleftarrow{\phi})$, the relation $R_{(\vec{\phi},\overleftarrow{\phi})} = \left\{(g, h) \mid h \in \vec{\phi}(g^{II})\right\} = \left\{(g, h) \mid g \in \overleftarrow{\phi}(h^{JJ})\right\}$ is a dual bond, and these constructions are mutually inverse in the following sense:

$$\vec{\phi} = \vec{\phi}_{R_{(\vec{\phi},\overleftarrow{\phi})}} \qquad \overleftarrow{\phi} = \overleftarrow{\phi}_{R_{(\vec{\phi},\overleftarrow{\phi})}} \qquad R = R_{\vec{\phi}_R,\overleftarrow{\phi}_R}$$

Hence, formal contexts with dual bonds are "equivalent" to complete lattices with antitone Galois connections. However, antitone Galois connections of course cannot be composed, such that none of the above form a category and we cannot make this statement of equivalence formal in the sense of Definition 2.5.5. Of course, some straightforward dualizing will fix the situation, but following this path will not give us much additional insights.

Before proceeding, let us note the following consequence of Lemma 2.3.5.

Lemma 4.1.3 Consider a dual bond R between contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. Then $R(g^{\not{I}\not{I}}) = R(g)$ and $R^{-1}(h^{\not{J}\not{J}}) = R^{-1}(h)$ holds for any $g \in G$, $h \in H$. Especially, $R(g^{\not{I}\not{I}})$ and $R^{-1}(h^{\not{J}\not{J}})$ are extents.

Proof. The inclusion $R(g) \subseteq R(g^{\not{I}\not{I}})$ is obvious for any relation R, since $g \in g^{\not{I}\not{I}}$. For the converse, assume that $h \in R(g^{\not{I}\not{I}})$, i.e. there is some $g' \in g^{\not{I}\not{I}}$ such that $g' \; R \; h$. By Lemma 2.3.5 we conclude $g \in g'^{II}$ which is a subset of $R^{-1}(h)$ since the latter is an extent. This shows $h \in R(g)$ as required. The statement for R^{-1} follows by a similar reasoning. □

Now we want to ask how the dual bonds between two contexts can be represented. Since extents are closed under intersections, the same is true for the set of all dual bonds between two contexts. Thus the dual bonds form a closure system and one might ask for a way to cast this into a formal context which has dual bonds for concepts. An immediate candidate for this purpose is the direct product of the contexts.

Definition 4.1.4 Given contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, the *direct product* of $\mathbb{K}$ and $\mathbb{L}$ is the context $\mathbb{K} \times \mathbb{L} = (G \times H, M \times N, \nabla)$, where $(g, h) \nabla (m, n)$ iff $g\ I\ m$ or $h\ J\ n$.

Proposition 4.1.5 ([Gan04]) The extents of a direct product $\mathbb{K}\times\mathbb{L}$ are dual bonds from $\mathbb{K}$ to $\mathbb{L}$.

Proof. It suffices to show that attribute extents are dual bonds, because any extent is an intersection of attribute extents and intersections of dual bonds are still dual bonds. Thus consider $(m, n) \in M \times N$ and define $R = (m, n)^\nabla$. We find that $R = (m^I \times H) \cup (G \times n^J)$. Thus, for any $g \in G$, $g^R = H$ or $g^R = n^J$, both of which are extents in $\mathbb{L}$. Likewise, for $h \in H$, $h^R = m^I$ or $h^R = H$, such that R is indeed a dual bond. □

However, it is known that the converse of this result is false in the general case, i.e. there are dual bonds which are not extents of the direct product. We give the following counterexample:

Counterexample 4.1.6 Consider the context $\mathbb{K} = (\{a, b, c\}, \{1, 2, 3\}, I)$ with incidence relation I given as follows:

I	1	2	3
a	×		
b		×	
c			×

Obviously, the relation $R = \{(a, a), (b, b), (c, c)\}$ is a dual bond from $\mathbb{K}$ to itself, since all singleton sets are extents. However, we find $R^\nabla = \emptyset$ in $\mathbb{K} \times \mathbb{K}$. Thus $R \neq R^{\nabla\nabla} = \{a, b, c\} \times \{a, b, c\}$ is not an extent of the direct product.

In consequence, the direct product only represents a distinguished subset of all dual bonds. In order to find additional characterizations for these relations, we will use the following result. The notation R^∇ indicated the intent of the relation R considered as a set of objects in the direct product.

Lemma 4.1.7 Consider the contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ and a relation $R \subseteq G \times H$. For any attribute $m \in M$, the following sets are equal:

- $X_1 := R^\nabla(m) = \{n \in N \mid (m, n) \in R^\nabla\}$
- $X_2 := R(m^{\not I})^J = \{h \in H \mid \text{there is } g \in G \text{ with } g \not I\ m \text{ and } (g, h) \in R\}^J$
- $X_3 := \bigcap_{g \in m^{\not I}} R(g)^J$

Furthermore, $R^{\nabla\nabla}(g) = R^\nabla(g^{\not I})^J = \bigcap_{m \in g^{\not I}} R(m^{\not I})^{JJ}$ holds for any object $g \in G$.

Proof. We first show the equality between X_1 and X_2. If $(m,n) \in R^\nabla$ then $(g,h)\nabla(m,n)$ holds for all $(g,h) \in R$. Thus, if $g \not I\, m$ for some $(g,h) \in R$, one certainly has $h\ J\ n$. Hence $n \in X_2$ and we obtain $X_1 \subseteq X_2$. For the other direction consider some $n \in X_2$. Then for all $(g,h) \in R$, $g \not I\, m$ implies $h\ J\ n$. Hence $(m,n) \in R^\nabla$ and $X_2 \subseteq X_1$ as required.

Next observe that X_2 clearly can be expressed as $\left(\bigcup_{g\in m^{\not I}} R(g)\right)^J$. The fact that this is equal to X_3 has been shown as the second part of Theorem 2.3.4.

For the rest of the proof, note that R^∇ is a relation between the sets of objects of the dual contexts $\mathbb{K}^{\mathsf{d}}$ and $\mathbb{L}^{\mathsf{d}}$. Thus we can apply the first part of the lemma on R^∇ to obtain the equality

$$R^{\nabla\nabla}(g) = R^\nabla(g^{\not I})^J = \bigcap_{m\in g^{\not I}} R^\nabla(m)^J.$$

Another application of the above results shows that $R^\nabla(m) = R(m^{\not I})^J$ and we obtain $\bigcap_{m\in g^{\not I}} R^\nabla(m)^J = \bigcap_{m\in g^{\not I}} R(m^{\not I})^{JJ}$ as required. □

Now we can state a characterization theorem for dual bonds in the direct product.

Theorem 4.1.8 Consider the contexts $\mathbb{K} = (G,M,I)$ and $\mathbb{L} = (H,N,J)$ and a relation $R \subseteq G \times H$. The following are equivalent:

(i) R is an extent of the direct product $\mathbb{K}\times\mathbb{L}$.
(ii) For all $g \in G$, $R(g) = R^\nabla(g^{\not I})^J \ \left(= \bigcap_{m\in g^{\not I}} R(m^{\not I})^{JJ}\right)$.
(iii) R is a dual bond and, for all $g \in G$, $\bigcap_{m\in g^{\not I}} R(m^{\not I})^{JJ} = R(g^{\not I\not I})$

Proof. The equivalence of (i) and (ii) follows immediately from Lemma 4.1.7 where we established that $R^\nabla(g^{\not I})^J = \bigcap_{m\in g^{\not I}} R(m^{\not I})^{JJ} = R^{\nabla\nabla}(g)$. Using Lemma 4.1.3 on condition (iii) yields $\bigcap_{m\in g^{\not I}} R(m^{\not I})^{JJ} = R(g)$, which is just condition (ii). □

Another feature of the dual bonds in the direct product allows for the construction of Galois connections other than those considered in Theorem 4.1.2. Given a dual bond R in $\mathbb{K}\times\mathbb{L}$, its intent R^∇ is a dual bond from $\mathbb{K}^{\mathsf{d}}$ to $\mathbb{L}^{\mathsf{d}}$, which induces another antitone Galois connection between the dual concept lattices. This Galois connection appears to have no simple further relationship to the antitone Galois connection derived from R.

Corollary 4.1.9 Consider the contexts $\mathbb{K} = (G,M,I)$ and $\mathbb{L} = (H,N,J)$ and an extent R of the direct product $\mathbb{K}\times\mathbb{L}$. There are two distinguished Galois connections $\phi_R : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L})$ and $\phi_{R^\nabla} : \mathsf{B}_o(\mathbb{K})^{\mathsf{op}} \to \mathsf{B}_o(\mathbb{L})^{\mathsf{op}}$ and each of R, R^∇, ϕ_R and ϕ_{R^∇} uniquely determines the others.

Proof. Just use Theorem 4.1.2 on R and R^{∇}. □

Of course any antitone Galois connection between two posets contravariantly induces another antitone Galois connection, obtained by exchanging both adjoints (see Section 2.2.1). But there appears to be no general way to bijectively construct an additional antitone Galois connection between the order duals of the original posets, which, in the light of Theorem 2.2.8, would mean to establish a bijective relation between closure operators on a complete lattice L and closure operators on L^{op}. The following results on extents of direct products can always be extended to this second Galois connection, but we will often prefer to save space and refrain from stating these explicitly.

4.2 Continuity for dual bonds

In Section 2.4.2, we already considered continuity – the preservation of certain structures in the preimage – for functions between topological spaces. Now continuity is an important concept in many branches of mathematics, and is also of relevance in formal concept analysis. However, we will generally not be dealing with functions but with relations such as dual bonds, and the notion of continuity will be lifted accordingly as follows, which is partially taken from [GW99].

Definition 4.2.1 Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. A relation $R \subseteq G \times H$ is *extensionally continuous* if it *reflects* extents of $\mathbb{L}$, i.e. if for every extent O of $\mathbb{L}$ the preimage $R^{-1}(O)$ is an extent of $\mathbb{K}$.

R is *extensionally object-continuous* (*attribute-continuous*) if it reflects all object extents (attribute extents) of $\mathbb{L}$, i.e. if for every object-extent $O = h^{JJ}$ (attribute-extent $O = n^J$) the preimage $R^{-1}(O)$ is an extent of $\mathbb{K}$ (but not necessarily an object-extent).

A relation is *extensionally closed* from $\mathbb{K}$ to $\mathbb{L}$ if it *preserves* extents of $\mathbb{K}$, i.e. if its inverse is extensionally continuous from $\mathbb{L}$ to $\mathbb{K}$. Extensional object- and attribute-closure are defined accordingly.

The dual definitions give rise to *intensional* continuity and closure properties.

Lemma 4.1.3 above shows that extensional object-continuity and -closure are properties of any dual bond when considered as a relation between one context and the complement of the other. We thus focus on extensional attribute-continuity and -closure in the present section. The other notions will however become important later on in Section 4.3.

Whenever it is clear whether we are dealing with a relation on attributes or on objects, we will tend to omit the additional qualifications "extensionally" and

I	1	2
a	×	
b		×

J	3	4	5
c	×		
d		×	

I	1	2	3
a	×		
b		×	
c			

Figure 4.1: Formal contexts for Counterexamples 4.2.3 (left) and 4.2.4 (right).

"intensionally." We also remark that neither object- nor attribute-continuity is sufficient to obtain full continuity in the general case, as can be seen from R^∇ in Counterexample 4.2.3.

Now we can investigate the interaction between continuity and the representation of dual bonds.

Theorem 4.2.2 Consider a dual bond R from $\mathbb{K} = (G, M, I)$ to $\mathbb{L} = (H, N, J)$. If R is extensionally attribute-continuous from $\mathbb{K}$ to $\mathbb{L}^c$, then R is an extent of $\mathbb{K} \times \mathbb{L}$ and R^∇ is intensionally object-closed from $\mathbb{K}^c$ to $\mathbb{L}$.

Proof. We will first show that $R(g)^J = R^\nabla(g^{\not I})$ holds for arbitrary $g \in G$ $(*)$. Clearly, $R^\nabla(g^{\not I}) \subseteq R(g)^J$, since $n \in R(g)^J$ for any $(m, n) \in R^\nabla$ for which $g \not I\, m$.

For the other direction, assume that there is $n \in R(g)^J$, i.e. all objects which are R-related to g satisfy n. Thus g relates to no objects that do not satisfy n, i.e. $g \notin R^{-1}(n^{\not J})$. Due to attribute-continuity of R, the latter is closed in $\mathbb{K}$ and thus there must be some element $m \in R^{-1}(n^{\not J})^I$ such that $g \not I\, m$. We want to show that $(m, n) \in R^\nabla$ which follows if any pair in R is ∇-related to (m, n). We only need to consider pairs which have a first component g' such that $g' \not I\, m$. But then $g' \notin R^{-1}(n^{\not J})^{II} = R^{-1}(n^{\not J})$ and we find that $n \in R(g')^J$. Hence all pairs $(g', h') \in R$ satisfy (m, n) and we conclude that $(m, n) \in R^\nabla$. Together with the above information that $g \not I\, m$, this finishes the proof of $(*)$.

Now it is immediate that R is an extent of the direct product. Indeed, by property $(*)$, we obtain $R(g)^{JJ} = R^\nabla(g^{\not I})^J$. Now since $R(g) = R(g)^{JJ}$, this yields condition (ii) of Theorem 4.1.8 which establishes the claim.

Finally, note that $(*)$ also shows that the set $R^\nabla(g^{\not I})$ is an intent of $\mathbb{L}$, such that R^∇ is indeed object-closed. □

Of course, analogous results can be obtained for closure by exchanging the roles of $\mathbb{K}$ and $\mathbb{L}$. One may wonder whether similar statements can be proven for dual bonds which are fully continuous and/or closed. However, this is not the case:

Counterexample 4.2.3 Consider the formal contexts $\mathbb{K} = (\{a, b\}, \{1, 2\}, I)$ and $\mathbb{L} = (\{c, d\}, \{3, 4, 5\}, J)$ depicted in Figure 4.1 (left).

Define $R = \{(a, c), (b, d)\}$. All subsets of $\{a, b\}$ are extents of both $\mathbb{K}$ and $\mathbb{K}^c$. Likewise, all subsets of $\{c, d\}$ are extents of $\mathbb{L}$ and $\mathbb{L}^c$. Thus R is trivially closed and continuous in every sense. However, we find that $R^{\nabla} = \{(1,4),(2,3)\}$ is not closed from $\mathbb{K}^c$ to $\mathbb{L}$. Indeed, $\{1,2\}$ is an intent of $\mathbb{K}^c$ but $R^{\nabla}(\{1,2\}) = \{3,4\}$ is not an intent of $\mathbb{L}$, since $\{3,4\}^{JJ} = \{3,4,5\}$.

Other easy counterexamples for this claim can be obtained by exploiting the fact that for any relation the image and preimage of the empty set is necessarily empty. By adding appropriate attributes, one can always assure that the empty set is not an intent in order to find cases where no relation can be intentionally continuous, even if numerous extensionally closed and continuous dual bonds exist.

Another false assumption that one might have is that the conditions given in Theorem 4.2.2 for being an extent of the direct product are not just sufficient but also necessary. However, neither closure nor continuity is needed for a dual bond to be represented in the direct product.

Counterexample 4.2.4 Consider the context $\mathbb{K} = (\{a,b,c\},\{1,2,3\},I)$ depicted in Figure 4.1 (right). Define $R = \{(a,a),(b,b)\}$. We find that $R^{\nabla} = \{(1,2),(2,1)\}$. Thus $R = R^{\nabla\nabla}$ and R is a dual bond which is an extent of the direct product $\mathbb{K}\times\mathbb{K}$. However, R is not even attribute-continuous from $\mathbb{K}$ to $\mathbb{K}^c$, since $R^{-1}(3^{\not I}) = R^{-1}(\{a,b,c\}) = \{a,b\}$ is not closed in $\mathbb{K}$. On the other hand, using that $R = R^{-1}$, we find that R is not attribute-closed from $\mathbb{K}^c$ to $\mathbb{K}$ either.

Although this shows that continuity is not a characteristic feature of all dual bonds in the direct product, we still find that there are many situations where there is a wealth of continuous dual bonds. This is the content of the following theorem.

Theorem 4.2.5 Consider the contexts $\mathbb{K} = (G,M,I)$ and $\mathbb{L} = (H,N,J)$. If

$$\emptyset \text{ is an extent of } \mathbb{K} \qquad \text{or} \qquad \emptyset \text{ is not an extent of } \mathbb{L}^c$$

then the set of all dual bonds which are continuous from $\mathbb{K}$ to $\mathbb{L}^c$ is $\bigcap$-dense in $\mathsf{B}_o(\mathbb{K}\times\mathbb{L})$ and thus forms a basis for the closure system of all dual bonds in the direct product.

If the assumptions also hold with $\mathbb{K}$ and $\mathbb{L}$ exchanged, then the set of all dual bonds which are both continuous from $\mathbb{K}$ to $\mathbb{L}^c$ and closed from $\mathbb{K}^c$ to $\mathbb{L}$ is $\bigcap$-dense as well.

Proof. From Theorem 4.2.2 we know that the above sets of dual bonds are subsets of the extents of the direct product. For density, we recall that the set of all attribute extents $(m,n)^{\nabla}$ is $\bigcap$-dense in the lattice of extents. For every $(m,n)\in M\times N$, we find that $(m,n)^{\nabla} = m^I \times H \cup G \times n^J$. Therefore, for arbitrary extents $O \subseteq H$ we calculate

$$(m,n)^{\nabla^{-1}}(O) = \begin{cases} \emptyset & \text{if } O = \emptyset, \\ G \cup m^I = G & \text{if } n^J \cap O \neq \emptyset, \\ m^I & \text{otherwise.} \end{cases}$$

In each case $(m,n)^{\nabla^{-1}}(O)$ is an extent of $\mathbb{K}$, where we use the initial assumption that $\emptyset$ is an extent of $\mathbb{K}$ if $O = \emptyset$ is an extent of $\mathbb{L}^c$. Thus any $(m,n)^\nabla$ is continuous from $\mathbb{K}$ to $\mathbb{L}^c$ and the attribute extents must form a subset of the set of continuous dual bonds. This shows the required density property.

Using the additional assumptions for the last part of the theorem, this shows that the dual bonds $(m,n)^\nabla$ are also closed from $\mathbb{K}^c$ to $\mathbb{L}$. Hence the continuous and closed dual bonds form a $\bigcap$-dense set as required. □

Note that the previous theorem could of course also be stated using closure in place of continuity. Furthermore it is evident that the dual bonds $(m,n)^\nabla$ are such that the (pre)image of almost any set is an extent. The only exception is the empty set, which is why we need to add the given preconditions. We remark that these conditions are indeed very weak. By removing or adding full rows, any context can be modified in such a way that the empty set either is an extent or not. Since the concept lattices of the context and its complement are not affected by this procedure, one can always enforce the above situation by restricting to appropriate kinds of contexts.

4.3 Functional bonds and scale measures

In FCA, (extensionally) continuous functions have been studied under the name *scale measures*, the importance of which stems from the fact that they can be regarded as a model for concept scaling and data abstraction. As discussed in Chapter 3, topology provides additional interpretations for continuous functions in the context of knowledge representation and reasoning. Furthermore we shall see that continuity between topological spaces coincides with continuity between appropriate contexts (Theorem 5.3.1).

The definition of continuity for functions constitutes a special case of continuity in the relational case as defined above.

Definition 4.3.1 Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. A function $f : G \to H$ is *extensionally continuous* whenever its graph $\{(x, f(x)) \mid x \in G\}$ is an extensionally continuous relation, i.e. if $f^{-1}(O)$ is an extent of $\mathbb{K}$ for any extent O of $\mathbb{L}$.

Extensional attribute- and *object-continuity*, as well as the according intensional properties and closures are defined similarly based on the graph of the function.

This definition agrees with [GW99, Definition 89], where extensionally continuous maps have also been called *scale measures*. Extensional attribute-continuity (and thus intensional object-continuity) is of course redundant, as the following lemma shows.

Lemma 4.3.2 Given contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, a function $f : G \to H$ is extensionally continuous iff it is extensionally attribute-continuous.

Proof. The forward implication is trivial, so assume that f is attribute-continuous. Consider an extent B^J of $\mathbb{L}$. According to Theorem 2.3.4 one has that $B^J = \bigcap_{n\in B} n^J$. We find that $f^{-1}(\bigcap_{n\in B} n^J) = \bigcap_{n\in B} f^{-1}(n^J)$. By attribute-continuity, the latter is an intersection of concepts of $\mathbb{K}$, and thus a concept. □

This statement relies on the fact that attribute extents are $\bigcap$-dense in the object concept lattice and that preimages of functions commute with intersections. On the one hand, this is not true for images of functions, such that extensional attribute-closure does not yield full closure. On the other hand, though object extents are supremum-dense, the respective suprema are not the set-theoretical unions. Hence extensional object-continuity and -closure are reasonable notions as well.

The link from functions to our earlier studies of dual bonds is established through a specific class of dual bonds which can be represented by functions.

Definition 4.3.3 Consider a dual bond R between contexts (G, M, I) and (H, N, J). Then R is *functional* whenever, for any $g \in G$, the extent $R(g)$ is generated by a unique object $f_R(g) \in H$:

$$R(g) = f_R(g)^{JJ}.$$

In this case R is said to *induce* the corresponding function $f_R : G \to H$.

It is obvious that functional dual bonds are uniquely determined by the function they induce. In fact, it is easy to see that R is the least dual bond that contains the graph of the function f_R. However, not for every function will this construction yield a dual bond that is functional. The next result characterizes the functions that are of the form f_R for some functional dual bond R.

Proposition 4.3.4 Consider a context $\mathbb{K} = (G, M, I)$ and a context $\mathbb{L} = (H, N, J)$ for which the map $h \mapsto h^J$ is injective. There is a bijective correspondence between

- the set of all functional dual bonds from $\mathbb{K}$ to $\mathbb{L}$ and
- the set of all extensionally object-continuous functions from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$.

The required bijections consist of the functions

- $R \mapsto f_R$ mapping each functional dual bond to the induced function and

- $f \mapsto R_f$ mapping each object-continuous function to the least dual bond which contains its graph $\{(g, f(g)) \mid g \in G\}$.

Proof. Consider a functional dual bond R from $\mathbb{K}$ to $\mathbb{L}$ and the induced mapping $f = f_R$. For some object $h \in H$, we find that $R^{-1}(h) = f^{-1}(h^{\not J \not J})$ follows from the defining property of f and Lemma 2.3.5. Since R is a dual bond, $R^{-1}(h)$ must be an extent and hence f is extensionally object-continuous in the required sense.

Conversely, if $f : G \to H$ is an object-continuous function from $\mathbb{K}$ to $\mathbb{L}^c$, then a relation $R \subseteq G \times H$ is defined by setting $R(g) = f(g)^{JJ}$ for any $g \in G$. Clearly R maps objects of $\mathbb{K}$ to extents of $\mathbb{L}$. For the converse, consider $h \in H$. As before we find that $R^{-1}(h) = f^{-1}(h^{\not J \not J})$ which is an extent of $\mathbb{K}$ by object-continuity. Thus R is a dual bond. Moreover, it is easy to see that R is the least dual bond that contains the graph of f. Due to the assumptions on $\mathbb{L}$, we have that R is functional inducing the function f and we obtain the required bijection. □

Object-continuity of the functions f_R is not too much of a surprise in the light of Lemma 4.1.3. The fact that this property suffices for the above result demonstrates how specific functional dual bonds really are. In contrast, the properties established in Lemma 4.1.3 are generally not sufficient for a relation to be a dual bond.

Also note that the additional requirements for $\mathbb{L}$, which guarantee that no two functions induce the same dual bond, are again rather weak. Indeed, they are implied by the common assumption that the contexts under consideration are clarified.

We can now go further and characterize the antitone Galois connections that are obtained from functional dual bonds.

Proposition 4.3.5 Consider a context $\mathbb{K} = (G, M, I)$ and a context $\mathbb{L} = (H, N, J)$ for which the map $h \mapsto h^J$ is injective. The bijection between dual bonds and antitone Galois connections given in Theorem 4.1.2 restricts to a bijective correspondence between

- the set of all functional dual bonds from $\mathbb{K}$ to $\mathbb{L}$ and
- the set of all antitone Galois connections from $\mathsf{B}_o(\mathbb{K})$ to $\mathsf{B}_o(\mathbb{L})$ which map object extents of $\mathbb{K}$ to object extents of $\mathbb{L}$.

Proof. Consider a functional dual bond R from $\mathbb{K}$ to $\mathbb{L}$ and the antitone Galois connection $(\vec{\phi}_R, \overleftarrow{\phi}_R)$ as constructed in Theorem 4.1.2. We claim that $\vec{\phi}_R$ maps object extents to object extents. Thus consider $\vec{\phi}_R(g^{II})$ for some $g \in G$ and let f_R be the function induced by R. The set $R^{-1}(f_R(g))$ contains g and is an extent since R is a dual bond. Consequently $g^{II} \subseteq R^{-1}(f_R(g))$. But this shows that $f_R(g) \in \vec{\phi}_R(g^{II})$ since the latter is equal to $\bigcap\{R(x) \mid x \in g^{II}\}$. Therefore we have $f_R(g)^{JJ} \subseteq \vec{\phi}_R(g^{II})$.

The opposite inclusion follows, since $\vec{\phi}_R(g^{II})$ is an intersection of a collection of sets which includes $f_R(g)^{JJ} = R(g)$. Thus $\vec{\phi}_R(g^{II}) = f_R(g)^{JJ}$, which is an object extent of $\mathbb{L}$ as required.

Now let $(\vec{\phi}, \overleftarrow{\phi})$ be a Galois connection such that $\vec{\phi}$ maps object extents to object extents. There is a unique function $f : G \to H$ for which $\vec{\phi}(g^{II}) = f(g)^{JJ}$ hold for arbitrary $g \in G$. Let $R = R_{(\vec{\phi}, \overleftarrow{\phi})}$ be the dual bond induced by $(\vec{\phi}, \overleftarrow{\phi})$ as in Theorem 4.1.2. But then $R(g) = \vec{\phi}(g^{II}) = f(g)^{JJ}$, for arbitrary $g \in G$, such that R is indeed functional. □

In the light of the previous proposition we give a definition for the corresponding property of Galois connections.

Definition 4.3.6 Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ and a (monotone or antitone) Galois connection $\phi = (\vec{\phi}, \overleftarrow{\phi})$ between $\mathsf{B}_o(\mathbb{K})$ and $\mathsf{B}_o(\mathbb{L})$.

Then ϕ is *functional* (from $\mathbb{K}$ to $\mathbb{L}$) if $\vec{\phi}$ maps object extents to object extents and, for any $g \in G$ there is a unique object $f_{\vec{\phi}}(g)$ such that

$$\vec{\phi}(g^{II}) = f_{\vec{\phi}}(g)^{JJ}.$$

In this case, ϕ is said to *induce* the function $f_{\vec{\phi}} : G \to H$.

Proposition 4.3.5 shows rather natural classes of dual bonds and Galois connections, respectively. However, functional dual bonds do not generally arise as extents of the direct product. Moreover, the corresponding class of extensionally object-continuous functions as described in Proposition 4.3.4 appears to be unidentified. As Theorem 4.3.8 below shows, the more common class of extensionally continuous functions still allows for a nice characterization in terms of dual bonds. It will be helpful to first state the following lemma.

Lemma 4.3.7 Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. If R is a functional dual bond from $\mathbb{K}$ to $\mathbb{L}$ then we find that for any extent O of $\mathbb{L}^{\mathsf{c}}$

$$R^{-1}(O) = f_R^{-1}(O).$$

Proof. Let O be an arbitrary extent of $\mathbb{L}^{\mathsf{c}}$. The inclusion $R^{-1}(O) \supseteq f_R^{-1}(O)$ is obvious, since R contains the graph of f_R.

For the converse note that $R^{-1}(O)$ is just the union of the sets $R^{-1}(h)$ for all $h \in O$. As noted in the proof of Proposition 4.3.4, we have $R^{-1}(h) = f_R^{-1}(h^{\not J \not J})$ for arbitrary $h \in H$. But since O is an extent of $\mathbb{L}^{\mathsf{c}}$, $f_R^{-1}(h^{\not J \not J}) \subseteq f_R^{-1}(O)$ for all $h \in O$. Hence we obtain $R^{-1}(O) \subseteq f_R^{-1}(O)$ as required. □

Theorem 4.3.8 Consider a context $\mathbb{K} = (G, M, I)$ and a context $\mathbb{L} = (H, N, J)$ for which the map $h \mapsto h^J$ is injective. The bijection given in Proposition 4.3.4 restricts to a bijective correspondence between

- the set of all extensionally continuous functions from $\mathbb{K}$ to $\mathbb{L}^c$ and
- the set of all functional dual bonds from $\mathbb{K}$ to $\mathbb{L}$ that are continuous from $\mathbb{K}$ to $\mathbb{L}^c$.

Especially, every dual bond R_f that is induced by a continuous function from $\mathbb{K}$ to $\mathbb{L}^c$ is an extent of the direct product $\mathbb{K} \times \mathbb{L}$.

Proof. Given a function f which is continuous from $\mathbb{K}$ to $\mathbb{L}^c$, we must show that the dual bond R_f as specified in Proposition 4.3.4 is also continuous. From the same proposition we know that $f = f_{R_f}$ and so we can apply Lemma 4.3.7 to show that $R_f^{-1}(O) = f^{-1}(O)$ for any extent O of $\mathbb{L}^c$. Continuity of R_f then follows from continuity of f.

Conversely, consider the function f_R for any functional dual bond R that is continuous in the above sense. Using Lemma 4.3.7 again, we find that $R^{-1}(O) = f_R^{-1}(O)$ for every extent O of $\mathbb{L}^c$ and hence obtain continuity of f_R.

Finally, to show that R_f is an extent of the direct product, one can apply Theorem 4.2.2 and continuity of R_f. □

Thus we find that the extensionally continuous functions, or scale measures, are a rather specific kind of dual bond. Again we must be careful: It is certainly not the case that all functional dual bonds which are extents in the direct product are continuous. Just consider the context $\mathbb{K} = (\{g\}, \{m\}, \{(g, m)\})$. The relation $R = \{(g, g)\}$ is an extent of the direct product $\mathbb{K} \times \mathbb{K}$ and it is functional with f_R being the identity. However, the preimage of the empty set (which is closed in $\mathbb{K}^c$) is not an extent of $\mathbb{K}$.

As a dual bond, every continuous function naturally induces an antitone Galois connection – Propositions 4.3.4 and 4.3.5 discussed the according constructions for object-continuous functions. Due to their special structure, continuous functions can additionally be used to derive another monotone Galois connection and it should not come as a surprise that these entities determine each other uniquely under some mild assumptions.

Theorem 4.3.9 Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, and a function $f : G \to H$ which is continuous from $\mathbb{K}$ to $\mathbb{L}^c$.

(i) An antitone Galois connection $\phi_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L})$ is given by the mappings

$$\vec{\phi}_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}) : X \mapsto \bigcap\{f(x)^{JJ} \mid x \in X\} \quad \text{and}$$

$$\overleftarrow{\phi}_f : \mathsf{B}_o(\mathbb{L}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto \bigcap\{f^{-1}(y^{\not J \not J}) \mid y \in Y\}.$$

(ii) A monotone Galois connection $\psi_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ is given by the mappings

$\vec{\psi}_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}}) : X \mapsto f(X)^{\rotatebox{180}{J}\rotatebox{180}{J}}$ and

$\overleftarrow{\psi}_f : \mathsf{B}_o(\mathbb{L}^{\mathsf{c}}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto f^{-1}(Y).$

Moreover, if $\mathbb{L}$ is such that $h \mapsto h^J$ is injective, the above mappings provide bijective correspondences between

- the set of all extensionally continuous functions from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$,
- the set of all antitone Galois connections $\mathsf{B}_o(\mathbb{K})$ to $\mathsf{B}_o(\mathbb{L})$ that are functional (from $\mathbb{K}$ to $\mathbb{L}$) and for which the induced function is continuous from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$,
- the set of all monotone Galois connections $\mathsf{B}_o(\mathbb{K})$ to $\mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ that are functional (from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$).

Proof. We observe that $\vec{\phi}_f(X) = X^{R_f}$ and $\overleftarrow{\phi}_f(Y) = Y^{R_f}$ such that (i) is an immediate consequence of Theorem 4.1.2 and Proposition 4.3.4. The according bijection follows from Propositions 4.3.4 and 4.3.5.

For part (ii), we repeat the proof given in [GW99, Propositions 118 and 119]. Due to continuity $\overleftarrow{\psi}_f = f^{-1}$ is a function between the specified object- concept lattices. Like the preimage of any function, it preserves all intersections,which are exactly the infima in the given lattices. Thus $\overleftarrow{\psi}_f$ is the upper adjoint of some monotone Galois connection. The lower adjoint of $\overleftarrow{\psi}_f$ then is defined to be the function

$$X \mapsto \bigcap\left\{Y^{\rotatebox{180}{J}\rotatebox{180}{J}} \mid X \subseteq f^{-1}(Y^{\rotatebox{180}{J}\rotatebox{180}{J}})\right\} = \bigcap\left\{Y^{\rotatebox{180}{J}\rotatebox{180}{J}} \mid f(X) \subseteq Y^{\rotatebox{180}{J}\rotatebox{180}{J}}\right\} = f(X)^{\rotatebox{180}{J}\rotatebox{180}{J}} = \vec{\psi}_f(X).$$

Consequently $\vec{\psi}_f$ is adjoint to $\overleftarrow{\psi}_f$ as required.

To show that $\vec{\psi}_f$ maps object extents of $\mathbb{K}$ to object extents of $\mathbb{L}^{\mathsf{c}}$ consider some arbitrary $g \in G$. $f^{-1}(f(g)^{\rotatebox{180}{J}\rotatebox{180}{J}})$ is an extent of $\mathbb{K}$ which contains g and hence g^{II}. Thus $f(g^{II}) \subseteq f(g)^{\rotatebox{180}{J}\rotatebox{180}{J}}$ and therefore $\vec{\psi}_f(g^{II}) \subseteq f(g)^{\rotatebox{180}{J}\rotatebox{180}{J}}$. But since $f(g) \in f(g^{II})$ this shows $\vec{\psi}_f(g^{II}) = f(g)^{\rotatebox{180}{J}\rotatebox{180}{J}}$ as required. Now it is easy to see that if $h \mapsto h^J$ is injective, then so are $h \mapsto h^{JJ}$, $h \mapsto h^{\rotatebox{180}{J}}$, and $h \mapsto h^{\rotatebox{180}{J}\rotatebox{180}{J}}$. Injectivity of $h \mapsto h^{\rotatebox{180}{J}\rotatebox{180}{J}}$ entails that $(\vec{\psi}_f, \overleftarrow{\psi}_f)$ is functional.

For the converse of the claimed bijection, consider any monotone Galois connection $(\vec{\psi}, \overleftarrow{\psi}) : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ which is functional in the above sense, and let f be the induced function. Given some extent X of $\mathbb{K}$ we calculate

$$\begin{aligned}\vec{\psi}(X) &= \vec{\psi}\left(\bigvee\{x^{II} \mid x \in X\}\right) = \bigvee\{\vec{\psi}(x^{II}) \mid x \in X\} \\ &= \bigvee\{f(x)^{\rotatebox{180}{J}\rotatebox{180}{J}} \mid x \in X\} = \left(\bigcup\{f(x)^{\rotatebox{180}{J}\rotatebox{180}{J}} \mid x \in X\}\right)^{\rotatebox{180}{J}\rotatebox{180}{J}} = f(X)^{\rotatebox{180}{J}\rotatebox{180}{J}},\end{aligned}$$

where we used that $\vec{\psi}$ preserves suprema and that f represents the value of $\vec{\psi}$ on object extents. But this shows that $\vec{\psi}$ is indeed the mapping $\vec{\psi}_f$ induced by f as above.

As an extension to the proof from [GW99], we also show explicitly that the function f is continuous from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$, which does not seem to be entirely obvious. Thus consider some extent Y of $\mathbb{L}^{\mathsf{c}}$ and observe that

$$\begin{aligned}\vec{\psi}\left(f^{-1}(Y)^{II}\right) &= \vec{\psi}\left(\bigvee\{g^{II} \mid g \in f^{-1}(Y)\}\right) \\ &= \bigvee\{\vec{\psi}(g^{II}) \mid g \in f^{-1}(Y)\} = \bigvee\{f(g)^{\not{J}\not{J}} \mid g \in f^{-1}(Y)\},\end{aligned}$$

which is clearly a subset of the extent Y. Now for every $g' \in f^{-1}(Y)^{II}$, we find $f(g') \in \vec{\psi}(f^{-1}(Y)^{II})$ and hence $f(g') \in Y$ as required. □

Part (ii) of the previous theorem and the according bijections are known (see [GW99, Propositions 118 and 119]). Note that the two Galois connections from the above result are not obtained from each other by some simple dualizing. This is also evident when comparing the different side conditions in both cases: functional monotone Galois connections always relate to continuous functions, while continuity has to be required explicitly for functional antitone Galois connections. To further explain the situation, we can dualize $\mathbb{L}$ to obtain the following result:

Corollary 4.3.10 Given contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, there is a bijection between

- the set of antitone Galois connections $\mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L})$ which map object extents to attribute extents and
- the set of functions $f : G \to N$ which are extensionally continuous from $\mathbb{K}^{\mathsf{d}}$ to $\mathbb{L}$.

4.4 Infomorphisms

Infomorphisms are a special kind of morphism between formal contexts that have been considered quite independently in rather different research disciplines. The name "infomorphism" which we shall use in the following has been coined in the context of *information flow theory* [BS97]. Literature on Chu spaces means the same when speaking about "Chu mappings" and *institution theory* [GB92] refers to the according definition as the "Satisfaction condition" without naming the emerging morphisms at all. In FCA, the antitone version of these morphisms was studied under the name *(context-)Galois connection* [Xia93, Gan04].

Probably the most decisive feature of informorphisms is self-duality, which is an immediate consequence of their symmetric definition. Some of the relationships of infomorphisms to Galois connections are known, but our results from the above sections will allow to reveal a more complete picture.

Definition 4.4.1 Given contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, an *infomorphism* from $\mathbb{K}$ to $\mathbb{L}$ is a pair of mappings $\vec{f} : G \to H$ and $\overleftarrow{f} : N \to M$ such that

$$g \; I \; \overleftarrow{f}(n) \qquad \text{if and only if} \qquad \vec{f}(g) \; J \; n$$

holds for arbitrary $g \in G$, $n \in N$.

We first establish the following basic facts.

Lemma 4.4.2 Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. The infomorphisms from $\mathbb{K}$ to $\mathbb{L}$ are exactly the infomorphisms from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$.

Given such an infomorphism $(\vec{f}, \overleftarrow{f})$ and sets $O \subseteq G$, $A \subseteq N$, we find that

$$\vec{f}^{-1}(A^J) = \overleftarrow{f}(A)^I, \;\; \vec{f}^{-1}(A^{\not J}) = \overleftarrow{f}(A)^{\not I}, \;\; \overleftarrow{f}^{-1}(O^I) = \vec{f}(O)^J \;\; \text{and} \;\; \overleftarrow{f}^{-1}(O^{\not I}) = \vec{f}(O)^{\not J}.$$

Especially, $\vec{f}$ is extensionally continuous from $\mathbb{K}^{(\mathsf{c})}$ to $\mathbb{L}^{(\mathsf{c})}$ and $\overleftarrow{f}$ is intensionally continuous from $\mathbb{L}^{(\mathsf{c})}$ to $\mathbb{K}^{(\mathsf{c})}$.

Proof. The first statement is immediate from the definition of infomorphisms. Now for some $n \in N$ we find that $g \in \vec{f}^{-1}(n^J)$ iff $\vec{f}(g) \; J \; n$ iff $g \; I \; \overleftarrow{f}(n)$ iff $g \in \overleftarrow{f}(n)^I$. This shows that $\vec{f}^{-1}(n^J) = \overleftarrow{f}(n)^I$. Now for arbitrary sets $A \subseteq N$, $A^J = \bigcap_{n \in A} n^J$ and we can calculate

$$\begin{aligned} \vec{f}^{-1}(A^J) &= \vec{f}^{-1}\left(\textstyle\bigcap_{n \in A} n^J\right) &&= \textstyle\bigcap_{n \in A} \vec{f}^{-1}(n^J) \\ &= \textstyle\bigcap_{n \in A} \overleftarrow{f}(n)^I &&= \left(\textstyle\bigcup_{n \in A} \overleftarrow{f}(n)\right)^I &= \overleftarrow{f}(A)^I \end{aligned}$$

The other cases follow by dualization and/or complementation of this reasoning. □

Using these continuity properties, we can already specify a number of possible Galois connections that could be constructed from an infomorphism. We remark that continuity between two contexts is in general not equivalent to continuity between the respective complements, so that infomorphisms really induce some Galois connections that are not available in the case of continuous functions.

From Theorem 4.3.8, we know that we can obtain continuous dual bonds from both $\vec{f}$ and $\overleftarrow{f}$. Since these relations are extents and intents, respectively, in the direct product, one may ask whether they belong to the same concepts or not. The following proposition shows the expected result.

Proposition 4.4.3 Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. Let $(\vec{f}, \overleftarrow{f})$ be an infomorphism from $\mathbb{K}$ to $\mathbb{L}$ and define relations $R \subseteq G \times H$ and $S \subseteq M \times N$ by setting

$$R(g) = \vec{f}(g)^{JJ} \quad \text{and} \quad S^{-1}(n) = \overleftarrow{f}(n)^{\not I \not I}.$$

Then R is a dual bond from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}$ which is an extent of $\mathbb{K}^{\mathsf{c}} \times \mathbb{L}$ and we have $R^{\nabla} = S$.

Furthermore, R is extensionally continuous from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$ and S^{-1} is intensionally continuous from $\mathbb{L}^{\mathsf{c}}$ to $\mathbb{K}$.

Proof. Since $\vec{f}$ is continuous from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$ (Lemma 4.4.2), the fact that R is an extent of $\mathbb{K}^{\mathsf{c}} \times \mathbb{L}$ and continuous in the required sense follows from Theorem 4.3.8. S^{-1} is obtained accordingly from $\overleftarrow{f}$ and thus is a dual bond from $\mathbb{L}^{\mathsf{d}}$ to $\mathbb{K}^{\mathsf{cd}}$ which is continuous as required.

As already observed in the proof of Proposition 4.3.4, the definition of S^{-1} yields that $S(m) = \overleftarrow{f}^{-1}(m^{II})$ for arbitrary $m \in M$. Thus $S(m) = \bigcup_{g \in m^I} \overleftarrow{f}^{-1}(g^I)$ which is equal to $\bigcup_{g \in m^I} \vec{f}(g)^J$ by Lemma 4.4.2. Due to S^{-1} being a dual bond from $\mathbb{L}^{\mathsf{d}}$ to $\mathbb{K}^{\mathsf{cd}}$, $S(m)$ is an intent of $\mathbb{L}$. Hence the above union is equal to $\left(\bigcup_{g \in m^I} \vec{f}(g)^{JJ}\right)^J$ which is just $R(m^I)^J$. By Lemma 4.1.7, $R(m^I)^J = R^{\nabla}(m)$ such that we find $S(m) = R^{\nabla}(m)$ and thus $S = R^{\nabla}$. □

Observe that the above construction of R (and S) relies only on the continuity of $\vec{f}$ from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$ (and the according continuity of $\overleftarrow{f}$). One can also construct a dual bond based on the continuity properties of these functions between the non-complemented contexts. However, Proposition 4.4.3 does not imply any relationship between these two dual bonds beyond the obvious fact that they induce the same infomorphism.

We already know that the dual bonds that are induced by (one part of) an infomorphism have rather specific properties. The next result shows that these features are sufficient for characterizing the respective dual bonds.

Proposition 4.4.4 Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ and let R be a dual bond from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}$ such that both R and $R^{\nabla-1}$ are functional. If R is extensionally continuous then the functions induced by R and $R^{\nabla-1}$ constitute an infomorphism from $\mathbb{K}$ to $\mathbb{L}$.

Proof. Denote the functions induced by R and $R^{\nabla-1}$ by $\vec{f}$ and $\overleftarrow{f}$, respectively, and consider some $n \in N$. We calculate

$$\overleftarrow{f}(n)^{\not I} = R^{\nabla-1}(n)^{\not I} = R^{-1}(n^{\not J})^{\not I \not I} = R^{-1}(n^{\not J}),$$

where the first and second equalities are consequences of Proposition 4.4.3 and Lemma 4.1.7, respectively, and the last equality uses continuity of R. Clearly $\vec{f}^{-1}(n^{\not J}) \subseteq R^{-1}(n^{\not J})$. For the other direction, assume that $g \in R^{-1}(n^{\not J})$. Then there is some $h \not J n$ with $g\,R\,h$, i.e. $h \in \vec{f}(g)^{JJ}$. But then $h^J \supseteq \vec{f}(g)^J$ and therefore $\vec{f}(g) \not J n$. This shows $g \in \vec{f}^{-1}(n^{\not J})$ such that the latter is equal to $R^{-1}(n^{\not J})$. In summary, we thus obtain $\overleftarrow{f}(n)^{\not I} = \vec{f}^{-1}(n^{\not J})$ which is equivalent to the statement

$$g \not I \overleftarrow{f}(n) \quad \text{iff} \quad \vec{f}(g) \not J n,$$

which states that $(\vec{f}, \overleftarrow{f})$ is an infomorphism as claimed. □

Note that, according to Lemma 4.3.2, extensional continuity of a functional dual bond R is equivalent to extensional attribute-continuity. This in turn implies intensional object-closure of R^{∇} (Theorem 4.2.2) which, since $R^{\nabla -1}$ is also functional, implies closure of R^{∇}. Thus our assumptions are perfectly symmetrical. Furthermore, Propositions 4.4.3 and 4.4.4 induce a bijection between infomorphism and the described class of dual bonds.

Now that we understand how infomorphisms are characterized in terms of dual bonds, we can specify their relationship to Galois connections.

Theorem 4.4.5 Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, and an infomorphism $f = (\vec{f}, \overleftarrow{f})$ from $\mathbb{K}$ to $\mathbb{L}$.

- An antitone Galois connection $\phi_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ is given by the mappings

  $\vec{\phi}_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}}) : X \mapsto \bigcap\{\vec{f}(x)^{\not J \not J} \mid x \in X\} \;= \bigcap\{\overleftarrow{f}^{-1}(x^{\not I})^{\not J} \mid x \in X\}$ and

  $\overleftarrow{\phi}_f : \mathsf{B}_o(\mathbb{L}^{\mathsf{c}}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto \bigcap\{\vec{f}^{-1}(y^{JJ}) \mid y \in Y\} = \bigcap\{\overleftarrow{f}(y^{J})^{I} \mid y \in Y\}$.

  Moreover, further three antitone Galois connections $\phi_f^{\mathsf{c}} : \mathsf{B}_o(\mathbb{K}^{\mathsf{c}}) \to \mathsf{B}_o(\mathbb{L})$, $\phi_f^{\mathsf{d}} : \mathsf{B}_o(\mathbb{K}^{\mathsf{d}}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{cd}})$ and $\phi_f^{\mathsf{cd}} : \mathsf{B}_o(\mathbb{K}^{\mathsf{cd}}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{d}})$ are defined similarly, using the complemented incidence relations $(\,^{\mathsf{c}})$ and exchanging $\vec{f}$ and $\overleftarrow{f}$ $(\,^{\mathsf{d}})$, respectively.

- A monotone Galois connection $\psi_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L})$ is given by the mappings

  $\vec{\psi}_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}) : X \mapsto \vec{f}(X)^{JJ} \;=\; \overleftarrow{f}^{-1}(X^{I})^{J}$ and

  $\overleftarrow{\psi}_f : \mathsf{B}_o(\mathbb{L}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto \vec{f}^{-1}(Y) \;=\; \overleftarrow{f}(Y^{J})^{I}$.

  Another monotone Galois connection $\psi_f^{\mathsf{c}} : \mathsf{B}_o(\mathbb{K}^{\mathsf{c}}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ is defined similarly, but with all incidence relations complemented.

Proof. The fact that the above mappings consitute Galois connections between the given concept lattices is an immediate consequence from Theorem 4.3.9 together with the continuity properties of infomorphisms as established in Proposition 4.4.3.

We have to show that the claimed equalities hold. For ϕ_f the equalities are obtained by applying Lemma 4.4.2 to the sets of objects $\{x\}$ $(x \in X)$ and y^J $(y \in Y)$, respectively. Likewise, the equalities within the definition of ψ_f follow by using Lemma 4.4.2 on X and Y^J. □

We remark that Proposition 4.4.3 shows that the antitone Galois connections ϕ_f^{d} and ϕ_f^{cd} can be constructed as in Corollary 4.1.9 from the two dual bonds induced by the function $\vec{f}$. Especially, Corollary 4.1.9 does not yield any further Galois connections.

4.5 A concept lattice of morphisms

The above considerations show that scale measures and infomorphisms can be identified with special types of dual bonds, and thus that part of this work can also be regarded as a study of various attributes of dual bonds and of the implications between them. The resulting concept lattice of context-morphisms is represented by the *nested line diagram*[2] in Figure 4.2.

To see that the information represented in this concept lattice is indeed correct, one can compute the induced set of implications between its attributes to obtain the following collection of inference rules:

attr.-continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	$\Rightarrow$	extent of $\mathbb{K} \times \mathbb{L}$	Theorem 4.1.8
continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	$\Rightarrow$	attr.-continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	Definition 4.2.1
infomorphism $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	$\Rightarrow$	continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$, functional $\mathbb{K} \to \mathbb{L}$	Proposition 4.4.4
functional $\mathbb{K} \to \mathbb{L}$, attr.-cont. $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	$\Rightarrow$	continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	Lemma 4.3.2
attr.-closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	$\Rightarrow$	extent of $\mathbb{K} \times \mathbb{L}$	Theorem 4.1.8
closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	$\Rightarrow$	attr.-closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	Definition 4.2.1
infomorphism $\mathbb{L} \to \mathbb{K}^{\mathsf{c}}$	$\Rightarrow$	closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$, functional $\mathbb{L} \to \mathbb{K}$	Proposition 4.4.4
functional $\mathbb{L} \to \mathbb{K}$, attr.-closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	$\Rightarrow$	closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	Lemma 4.3.2

As usual, collections of attributes on either side of the implications are comprehended as conjunctions. As the last column documents, each of these implications has indeed already been established within this document.

Conversely, we claim that no further implications between conjunctions of attributes hold for the considered properties. To substantiate this claim, we conducted an *attribute exploration* (see [GW99, pp. 85]) for the attributes used in Figure 4.2 – a task that was greatly simplified through the use of the free software

[2]The concept lattice represented by a nested line diagram consists of the boldfaced nodes, where connections between boxes represent parallel connections between boldfaced nodes at corresponding positions wrt. the background structure. See [GW99, pp. 75].

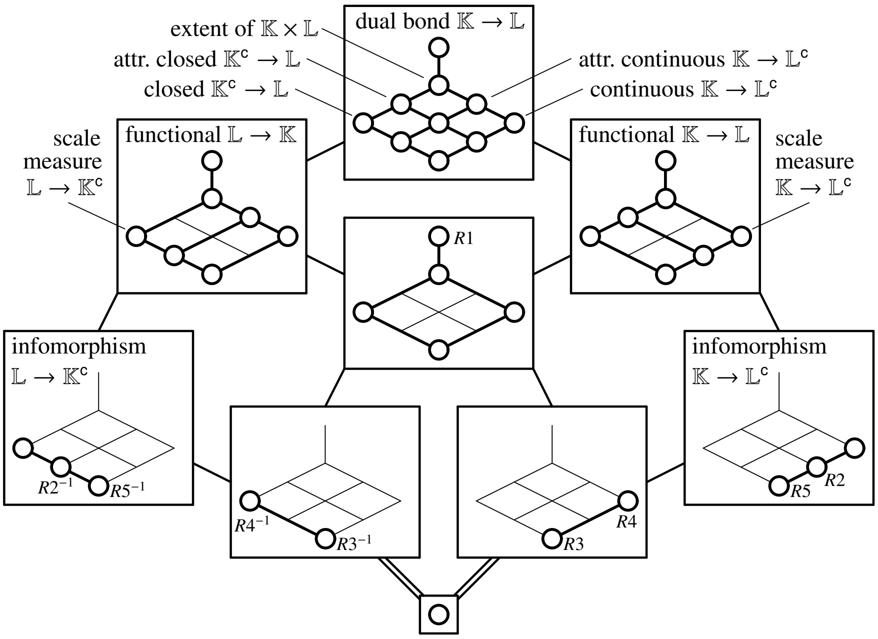


Figure 4.2: The concept lattice of the discussed properties of dual bonds, displayed as a nested line diagram. The included attributes are defined in Definition 4.1.1 (dual bond), 4.1.4 ($\mathbb{K} \times \mathbb{L}$), 4.2.1 (continuity and closure), and 4.3.3 (functionality). The attributes "scale measure" and "infomorphism" refer to the dual bonds described in Theorem 4.3.8 and Proposition 4.4.4, respectively, and thus imply functionality. The labels $R1$ to $R5$ and $R2^{-1}$ to $R5^{-1}$ denote the objects of the formal context in Figure 4.3.

ConExp.[3] After reducing the resulting collection of objects, we obtained the dual bonds and formal context displayed in Figure 4.3. To check that each of the given objects indeed has the specified attributes, first note that the attributes of $R2^{-1}$ to $R5^{-1}$ are determined by the properties of their inverted variants. Thus it remains to verify the attributes for $R1$ to $R5$. Considering the fact that the above implications have already been shown, this task reduces to a small number of straightforward computations, which we will not include here.

Finally, we want to remark that the conjunctive implications considered in FCA cannot describe all possible relationships between the attributes of a context. In particular, it could still occur that some properties are just disjunctions of others, i.e. that some suprema in the concept lattice are computed as simple set-unions. Counterexample 4.2.4 demonstrates the reasoning that is necessary to exclude such cases explicitly. We refrain from giving similar counterexamples for each of the 40 concepts in Figure 4.2, since it is rather evident that all of them are indeed

[3] *Concept Explorer*: `http://sourceforge.net/projects/conexp`

	extent of $\mathbb{K} \times \mathbb{L}$	functional $\mathbb{K} \to \mathbb{L}$	infomorph. $\mathbb{K} \to \mathbb{L}^c$	functional $\mathbb{L} \to \mathbb{K}$	infomorph. $\mathbb{L} \to \mathbb{K}^c$	attr.-cont. $\mathbb{K} \to \mathbb{L}^c$	continuous $\mathbb{K} \to \mathbb{L}^c$	attr.-closed $\mathbb{K}^c \to \mathbb{L}$	closed $\mathbb{K}^c \to \mathbb{L}$
$R1$		×		×					
$R2$	×	×	×			×	×	×	
$R3$	×	×	×	×		×	×	×	×
$R4$	×	×	×	×		×	×		
$R5$	×	×	×			×	×	×	×
$R2^{-1}$	×			×	×	×		×	×
$R3^{-1}$	×	×		×	×	×	×	×	×
$R4^{-1}$	×	×		×	×			×	×
$R5^{-1}$	×			×	×	×	×	×	×

$\mathbb{K}_1$	1	2
a	×	
b		×
c		

$\mathbb{K}_2$	1	2	3	4
a	×		×	
b		×	×	
c			×	×

$\mathbb{K}_3$	1	2	3	4	5
a		×		×	
b	×			×	
c	×	×			×

$\mathbb{K}_4$	1	2
a	×	
b		×

$\mathbb{K}_5$	1	2	3	4
a	×		×	
b		×	×	

$R1$: R from Counterexample 4.1.6
$R2 : \mathbb{K}_4 \to \mathbb{K}_1 \quad R2 = \{(a,a),(b,b)\}$
$R3 : \mathbb{K}_5 \to \mathbb{K}_4 \quad R3 = \{(a,a),(b,b)\}$
$R4 : \mathbb{K}_3 \to \mathbb{K}_2 \quad R4 = \{(a,a),(b,b),(c,c)\}$
$R5 : \mathbb{K}_5 \to \mathbb{K}_4 \quad R5 = \{(a,a),(b,a)\}$

Figure 4.3: A formal context for the concept lattice from Figure 4.2 and the definition of the dual bonds that consitute its set of objects.

object-concepts of appropriate dual bonds.

4.6 Conclusion and future work

In spite of the rather complete picture of the mutual relationships between dual bonds, scale measures and infomorphisms obtained in our considerations, there are many other aspects of the theory of morphisms in FCA which have not been considered within this chapter, and which indicate possible directions for future research. As mentioned in the introduction, the use of morphisms to model knowledge transfer and information sharing may employ methods from category theory [KHES04]. But not all of the above morphisms immediately yield categories of contexts, especially since antitone Galois connections cannot be composed in an obvious way. As a solution, one can dualize one context and consider *bonds* which yield monotone Galois connections that can be composed easily [GW99]. But one can also restrict to special classes of dual bonds: scale measures, infomorphisms, and dual bonds that are both closed and continuous all allow for rather obvious associative composition operations. In the following Chapter 5, we will further investigate these morphisms in conjunction with formal contexts that represent deductive systems of a logic.

Besides these (onto-)logical and categorical investigations, there are also further questions related to lattice theory. We already gave characterizations for the Galois connections that are induced by certain types of dual bonds, especially in the functional case (Proposition 4.3.5, Theorem 4.3.9, and Theorem 4.4.5). For many other types of dual bonds corresponding descriptions are missing. Likewise, although dual bonds are closed under intersections, no (non-canonical) context that has all dual bonds as extents is known to us.

In FCA, the concept lattice of the direct product $\mathbb{K} \times \mathbb{L}$ is known as the *tensor product* of the lattices $\mathsf{B}_o(\mathbb{K})$ and $\mathsf{B}_o(\mathbb{L})$. Theorem 4.2.5 showed that the study of dual bonds can also yield additional results on the tensor product, but further relationships between both subjects have not been investigated yet. As shown in [Xia93, Satz 15], infomorphisms can be represented by a concept lattice as well, but the role of this structure in the light of our present investigations still needs to be explored.

Finally, there are many other results in [Xia93, GW99, Gan04] which we did not discuss here. It would be a useful endeavor to compile the available knowledge from these publications in a systematic way and to investigate what additional insights are obtained in the sum.

Chapter 5

Categories of Logics

In this chapter, we bring together the ideas of Chapter 3 and Chapter 4, starting from the standard FCA based representation of deductive systems and using Stone duality to connect it with topological spaces. This enables us to compare morphisms from FCA, topology and algebra, and the resulting unified perspective on morphisms allows us to state the central Theorems 5.3.1 and 5.4.2.

Our exposition builds on the algebraic view on logics that was articulated in Chapter 3, especially in Section 3.5.1. In particular, we adhere to a purely semantic perspective, which is based on the Lindenbaum algebras of a logic rather than on syntactic descriptions of formulae. Not only will this strategy save the effort of introducing syntax and proof theory of various logics, but it will also emphasize that our investigations are exclusively on a semantic level. Due to this circumstance, we can consider elements of a Lindenbaum algebra (i.e. formulae up to semantic equivalence) instead of some syntactically defined set of logical formulae.

The logics that we consider are those of Section 3.5.1, i.e. conjunctive, positive, intuitionistic and classical logic, and their respective Lindenbaum algebras (meet-semilattices with greatest element, bounded distributive lattices, Heyting algebras and Boolean algebras). For our present purposes these logics often behave similarly, and we will use the terms "deductive system" and "Lindenbaum algebra" without further qualification whenever a statement can be made for all of these four logics.

The structure of this chapter is as follows. In Section 5.1, we introduce a representation of deductive systems via FCA and apply methods from Stone duality to relate it to topological spaces. Afterwards in Section 5.2, we study a new notion of logical consequence relation that is used to define categories of logical contexts. The central results of this chapter are given in Section 5.3 and Section 5.4, where we connect consequence relations to continuous functions/scale measures and coherent maps/infomorphisms, respectively. Section 5.5 briefly discusses possible future work.

5.1 Logic and FCA

Though the representation of conjunctive propositional logic in Section 3.3 is new, other means of presenting logics in FCA are well-known. Our goal in this section is to identify an appropriate way of using FCA in the representation of Lindenbaum algebras. The approach discussed in Section 3.3 does not suit this task, since it is based on rather specific relationships between semilattices and closures (Corollary 3.1.5) that cannot readily be transfered to lattices. In addition, the non-standard interpretation of formal contexts used in Chapter 3 would not allow us to make use of results from Chapter 4.

Hence our method of representing logics with FCA must be different, but nonetheless will turn out to be rather simple. Recall that the (deductive systems of) propositional logics considered in Section 3.5.1 induce a bounded distributive lattice as (the free extension of) their Lindenbaum algebra, and that a model is a prime filter in this lattice. A model of an element of a lattice is a prime filter containing this element.

The only case where Lindenbaum algebras are not lattices is CP logic, where we obtain meet-semilattices with greatest element. The free bounded distributive lattice L obtained from such a semilattice S is just the collection of all finite (possibly empty) unions of lower sets of the form $\downarrow s$, $s \in S$. This has some easy consequences:

- There is a bijection between lower sets of S and ideals of L.
- There is a bijection between filters of S and prime filters of L.
- All filters of L are prime.

This leads us to the following notational convention:

Notation 5.1.1 Consider a meet-semilattice S with greatest element. An *ideal* of S is a lower set of S. $\mathsf{Flt}(S)$ denotes the set of all filters and $\mathsf{Idl}(S)$ denotes the set of all ideals (lower sets) of S. A *model* of (an element of) S is just a filter of S (that contains the element). A *prime filter* of S is a filter of S.

This notation allows us to simplify the following presentation and to give a more compact treatment of the different logics.

The next result defines a representation of deductive systems via formal contexts and details its relationship to the concepts of Stone duality.

Proposition 5.1.2 Let M be the Lindenbaum algebra of some deductive system and let G be the set of its models. A formal context $\mathbb{K} = (G, M, I)$, defined by setting $g\ I\ m$ if and only if g is a model of m.

(i) $\omega = \mathsf{Idl}(M)$ is a spectral locale and the sub-poset of its points $\mathsf{pt}(\omega)$ is isomorphic to G, ordered by inclusion.

(ii) Let Ω be the topological space on G that is generated by the basic open sets

$$P_a = \{p \in G \mid a \in p\} \qquad a \in M.$$

Then Ω is homeomorphic to the topological space on $\mathsf{pt}(\omega)$ that is generated by the basic open sets

$$P_a = \{p \in \mathsf{pt}(\omega) \mid a \notin p\} \qquad a \in M.$$

(iii) ω is isomorphic to the open set lattice of Ω.

(iv) The attribute concept lattice $\mathsf{B}_a(\mathbb{K})$ is isomorphic to $\mathsf{Flt}(M)$, the set of all filters of M ordered by inclusion.

(v) The object concept lattice $\mathsf{B}_o(\mathbb{K}^c)$ is isomorphic to the lattice of closed subsets of Ω.

(vi) The attribute concept lattice $\mathsf{B}_a(\mathbb{K}^c)$ is isomorphic to $\mathsf{Idl}(M)$, the set of all ideals of M ordered by inclusion.

Proof. For the case of CP logic, (i) follows from Proposition 3.4.10 and Lemma 3.4.14. For logics with disjunction, first note that $\mathsf{Idl}(M)$ is a spectral locale for any distributive lattice M as an immediate consequence of Theorem 3.1.3 and Definition 3.4.12. Furthermore, as a prime filter of M, any model determines a unique prime ideal of L as its complement. The prime ideals are the prime elements of the ideal completion of L and thus generate lower sets which are principal prime ideals. Since these relationships are bijections, we can indeed identify the points of ω with the models of M.

for CP logic, (ii) has been shown in Corollary 3.4.15. Analogously, the required homeomorphism in the disjunctive case is established by observing that the bijection of (i) restricts to a bijection of basic open sets. Indeed, an element $a \in M$ is not contained in a point p iff it is not contained in the principal of p (which is a prime ideal of M) iff it is contained in the complement of this principal, which in turn is just the model that corresponds to p.

Item (iii) uses the fact that the open set lattice of the space of points from (ii) is isomorphic to the original locale. This is a basic result of Stone duality, so we will not give further details (see [Joh82] or [WP, Article "Stone duality"]).

For (iv), one again considers two cases: If M is a Lindenbaum algebra of CP logic, one needs to establish a bijection between the intents of $\mathbb{K}$ and the filters of the meet-semilattice M. This is immediate: any filter of M is itself a model with the filter as its object intent, and the collection of filters is closed under intersections.

For the case of logics with disjunction, one only has to show that the filters of M are exactly the intersections of prime filters of M, since the latter obviously constitute the object intents of $\mathbb{K}$. Clearly, any such intersection is indeed a filter. The converse follows from the fact that, given a filter F of M and an element $a \in M \setminus F$, there is a prime filter P of M with $F \subseteq P$ and $a \notin P$. This is a direct consequence of Axiom 2.1.9 applied on F and the ideal $\downarrow a$.

According to (ii), attribute concepts of $\mathbb{K}^c$ correspond to closed sets of Ω, from which (v) follows immediately.

Finally, by (v) and Theorem 2.3.4, $\mathsf{B}_a(\mathbb{K}^c)$ is isomorphic to the open set lattice of Ω. By (iii) this is isomorphic to ω, which in turn is isomorphic to the ideal completion $\mathsf{Idl}(M)$ of M. This establishes (vi). □

Filters as in (iv) are also called *theories* of the logic, since they are deductively closed sets of formulae.

Proposition 5.1.2 demonstrates that appropriate formal contexts can represent deductive systems of propositional logics quite faithfully. This approach to regard the semantical consequence relation between models (objects) and formulae (attributes) as a formal context has also been considered in Institution Theory. There, however, one introduces additional categorical machinery which goes beyond singular deductive systems. Our present interest also is in possible categories of deductive systems, but nonetheless does not justify to introduce institution theory in greater depth. In order to still be able to differentiate between the considered logics, we introduce the following notation.

Definition 5.1.3 A *conjunctive logical context* is a context that represents a deductive system of conjunctive logic as in Proposition 5.1.2. The class of all conjunctive logical contexts is denoted $\mathfrak{C}$.

Similarly, the classes of positive, intuitionistic, and classical logical contexts are denoted $\mathfrak{D}$, $\mathfrak{I}$, and $\mathfrak{N}$, respectively.

Observe that the above classes of logical contexts derive their name from the characteristic logical connectives conjunction, disjunction, implication, and negation. In order to unify our treatment of these different logics, we introduce the following important notational convention.

Notation 5.1.4 In the following, $\mathfrak{L}$ denotes the class of all logical contexts for an arbitrary but fixed logic, i.e. $\mathfrak{L}$ is one of $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{I}$, and $\mathfrak{N}$.

Given a logical context $\mathbb{K} \in \mathfrak{L}$, $\mathsf{LA}(\mathbb{K})$ denotes the Lindenbaum algebra induced by the deductive system, i.e. the set of attributes of $\mathbb{K}$, ordered by subset inclusion of the associated attribute extents. Furthermore, $\Omega(\mathbb{K})$ denotes the according topological space Ω as defined in Proposition 5.1.2.

Thus $\mathfrak{L}$ is used whenever a statement can be made for arbitrary but fixed class of logical contexts.

5.2 Consequence relations

In the light of Chapter 4, the representation of deductive systems via contexts yields a broad range of possible morphisms between them. However, for our current discussion we are particularly interested in those relationships between deductive systems that have a clear logical interpretation. The present section will thus turn to *consequence relations* as they typically arise in proof theory.

Definition 5.2.1 Given contexts $\mathbb{K}$ and $\mathbb{L}$, a *consequence relation* from $\mathbb{K}$ to $\mathbb{L}$ is a dual bond from $\mathbb{K}^{\mathsf{cd}}$ to $\mathbb{L}^{\mathsf{d}}$.

A consequence relation is *continuous* (*closed*) whenever the associated dual bond is extensionally continuous from $\mathbb{K}^{\mathsf{cd}}$ to $\mathbb{L}^{\mathsf{cd}}$ (extensionally closed from $\mathbb{K}^{\mathsf{d}}$ to $\mathbb{L}^{\mathsf{d}}$).

This definition deserves some explanation. The purpose of consequence relations in the above sense is to provide a meaningful generalization of the syntactical entailments that are typically considered in proof theory. Hence we are dealing with dualized contexts to obtain relations between formulae instead of models. Definition 3.4.1 gave a particularly simple case of such a relation in the case of CP logic, which we introduced in terms of deduction rules as is common in proof theory. Such rules usually describe deductions on both the left and the right hand side of the turnstile $\vdash$, but in many cases both sides are not mixed. In the given example, rules (R) and (Cut) are the only cases where a formula appears on both sides. However, since we are interested in relationships between different deductive systems (with different underlying languages), reflexivity (R) of consequence relations is not a meaningful property in our case. Likewise, it is known from proof theory that it is often possible to find sound and complete rule systems without (Cut).

The other rules can often be grouped into left and right rules, according to the side on which they introduce additional conclusions. In the example of CP logic, all rules other than (R) and (Cut) are right rules, which already suffices for the primitive conjunctive structure of this logic. Now one finds that the purpose of the left and right rules is to compute certain kinds of deductive closures on either side of a formula. For an example we consider positive logic. From $F \vdash G$ one would certainly want to conclude $F \vdash (G \vee H)$. If, additionally, $F \vdash H$ was given, one could infer $F \vdash (G \wedge H)$ as well. The closure just sketched can generally be described as follows: If all models of the set of consequences $\vdash (F)$ of a formula F are also models of a formula H, then H is a consequence of F as well. Recognizing this as the deductive closure of a logical context, we derive the forward condition of the aforementioned dual bond. In fact, we just gave a logical explanation why theories are filters in the Lindenbaum algebra.

The reasoning for the other side is dual. If $G \vdash F$ then we can also infer $(G \wedge H) \vdash F$, whereas $(G \vee H) \vdash F$ only follows when $H \vdash F$ is given as well. The general relationship is as follows: If a formula G is false in all models that falsify each of the premises $\vdash^{-1}(F)$ of F, then $G \vdash F$ also holds true. This introduces the complement of the satisfaction relation and yields the backwards condition of the dual bonds in Definition 5.2.1.

In [JKM99], a similar setting is considered for the case of positive logic. There, however, deductive systems are represented by means of syntactical consequence relations instead of contexts, and relations between these systems are described in a classical rule-based style. This formulation allows to drop reflexivity from deductive systems, which in turn requires to restrict the morphisms between objects to those for which such non-reflexive entailments act as an identity.[1]

Our notion of consequence relation requires closure only for the set of consequences of a single formula. Expressions of the form "$F_1, F_2 \vdash G_1, G_2$" that are typically found in proof theory at first sight do not seem to be covered by this definition. However, it is readily seen that this is not a real restriction. Indeed, one could use (possibly infinite) sequences of formulae as attributes of a context and extend the model theory accordingly. Sequences on the left and on the right of a consequence relation are typically interpreted as conjunctions and disjunctions, respectively, and thus represent a form of syntactical extension of the logic that we can easily take into account when constructing extended logical contexts. Anyway, it is well-known that these modifications are merely of proof theoretical convenience and do not affect the expressivity of the logic.

We are now interested in continuity and closure of consequence relations. Note that these notions are defined with respect to different contexts (either complemented or not). This choice of terminology will be justified in Section 5.3. The following technical lemma gives a simple criterion for showing closure.

Lemma 5.2.2 Consider a consequence relation S between contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, and an intent $A \in \mathsf{B}_a(\mathbb{K})$ of $\mathbb{K}$. The following are equivalent:

(i) $S(A)$ is an intent of $\mathbb{L}$.

(ii) For every $b \in S(A)^{JJ}$, there is a set $X \subseteq A$ such that $b \in S(X)^{JJ}$ and there is $a \in A$ with $X \subseteq a^{II}$.

Proof. Clearly, if $S(A)$ is an extent, then for every $b \in S(A)^{JJ}$, there is an element $a \in A$ such that $b \in S(a)$. Since $a \in a^{II}$, this shows that (i) implies (ii).

[1] In general, this process amounts to *splitting of idempotents* of a category with the common reflexive calculi as objects.

For the converse, let a be as in (ii). By Lemma 2.3.5, $a \in x^{\not{I}\not{I}}$ for all $x \in X$. Given $x \in X$, $S^{-1}(y)$ is an intent of $\mathbb{K}^c$ for all $y \in S(x)$, such that we find $a \in S^{-1}(y)$ for all $y \in S(x)$, i.e. $S(x) \subseteq S(a)$. This yields $S(X) \subseteq S(a)$ which implies $b \in S(a)$ since the latter is an intent of $\mathbb{L}$. □

Intuitively, the previous lemma states that closure of the image of a set A can be reduced to the closures of the images of single elements a, which are guaranteed by the definition of a consequence relation. The next theorem shows how to put this insight into practice.

Theorem 5.2.3 Consider logical contexts $\mathbb{K} = (G, M, I)$, $\mathbb{L} = (H, N, J) \in \mathfrak{L}$. Then all consequence relations from $\mathbb{K}$ to $\mathbb{L}$ are closed and continuous.

Especially, the consequence relations between $\mathbb{K}$ and $\mathbb{L}$ are exactly the intents of the direct product $\mathbb{K}^c \times \mathbb{L}$.

Proof. Let S be a consequence relation from $\mathbb{K}$ to $\mathbb{L}$. To show closure, consider an intent A of $\mathbb{K}$. In Proposition 5.1.2, we noticed that intents of the logical context $\mathbb{L}$ correspond to filters of the associated Lindenbaum algebra $\mathsf{LA}(\mathbb{L})$. We conclude that the closure $\cdot^{JJ}$ maps each subset of N to the least filter in which it is contained, which corresponds to the upper closure of the closure under finite meets. Consequently, given an element $b \in S(A)^{JJ}$, there must be a finite set $Y \subseteq S(A)$ such that $b \in Y^{JJ}$. Moreover, there is a finite set $X \subseteq A$ with $Y \subseteq S(X)$ and thus $b \in S(X)^{JJ}$. Each of the considered logics has finite conjunctions (meets in the Lindenbaum algebra), such that there is a formula $\bigwedge X$ which is in the extent-closure of X, hence in A, and which has the property $X = (\bigwedge X)^{II}$. Applying Lemma 5.2.2 finishes the first part of the proof.

For continuity, note that S is continuous if and only if S^{-1} is a closed consequence relation from $\mathbb{L}^c$ to $\mathbb{K}^c$, which again allows for the application of Lemma 5.2.2. Consider an intent B of $\mathbb{L}^c$. Proposition 5.1.2 states that B is an ideal of $\mathsf{LA}(\mathbb{L})$. As before, we consider some element $a \in S^{-1}(B)^{\not{I}\not{I}}$.

For the case that the considered logics have disjunction, there is again a finite set $Y \subseteq B$ such that $a \in S^{-1}(Y)^{\not{I}\not{I}}$. Thus the element $\bigvee Y$ allows for an application of Lemma 5.2.2, establishing closure of S for logical contexts from $\mathfrak{D}$, $\mathfrak{I}$, and $\mathfrak{N}$.

For the case of conjunctive logics, $S^{-1}(B)^{\not{I}\not{I}}$ is just the lower closure of $S^{-1}(B)$. Thus there is some element $x \in S^{-1}(B)$ such that $x \in \uparrow a$. But now one can take the singleton $\{x\}$ in place of the above set X to apply Lemma 5.2.2.

Finally, by (the dual of) Theorem 4.2.2, one immediately finds that consequence relations are intents of the given direct product. The converse follows by Proposition 4.1.5. □

Note that most cases of the previous theorem were proven similarly for all propositional logics we considered. Thus one could even extend the statement to arbitrary combinations of logical contexts for positive, intuitionistic or classical logic. In contrast, continuity for conjunctive logic was shown in a different way, and indeed consequence relations from some more expressive logic to conjunctive logic are not continuous in general.

The collection of all consequence relations between two logical contexts is a closure system that is described more precisely as the concept lattice of the direct product of Theorem 5.2.3. Another computationally feasible description of the associated closure operator is given in the form of rule-based calculi as known in proof theory. Since we are tempted to view morphisms between deductive systems as logical relationships between two specifications, this is a particularly pleasing situation, since it allows for "programming of morphisms." Indeed, whatever amount of information about the interrelations between two deductive systems is available, it always yields a canonical consequence relation which can be used for deduction. This situation is not entirely unlike that of definite logic programming, where users provide collections of valid inference rules, which are then used to automatically approximate a Scott continuous closure operator on a (uncountable) algebraic lattice (see Chapter 3).

Another useful property – also from the practical perspective of specification of and computation with morphisms – is the possibility of composing consequence relations. Of course, our primary interest lies in the fact that this provides us with categories for further investigations.

Corollary 5.2.4 The collection of all conjunctive logical contexts $\mathfrak{C}$ together with consequence relations constitutes a category **CCxt**. In a similar fashion, the classes $\mathfrak{D}$, $\mathfrak{I}$, and $\mathfrak{N}$ of all positive, intuitionistic, and classical logical contexts, respectively, yield categories **DCxt**, **ICxt**, and **NCxt**.

Proof. Composition of morphisms is given by relational composition, which is clearly associative. The fact the composition of consequence relations again yields consequences follows from the continuity and closure properties established in Theorem 5.2.3.

It is not too hard to see that identity morphisms are given as the least consequence relations containing the identity relation on the set of formulae. These can for example be obtained by applying the intent-closure of the direct product to the relational identities. □

Observe that the previous result is certainly not valid for consequence relations or dual bonds between arbitrary contexts.

To further enhance our understanding of the above categories, we want to describe their respective morphisms in order-theoretical terms. Since consequence relations are dual bonds, one can immediately view them in terms of antitone

Galois connections. However, given logical contexts $\mathbb{K}$ and $\mathbb{L}$ that induce Lindenbaum algebras $\mathsf{LA}(\mathbb{K})$ and $\mathsf{LA}(\mathbb{L})$, a consequence relation between the corresponding logical contexts yields an antitone Galois connection from the lattice of ideals of $\mathsf{LA}(\mathbb{K})$ to the lattice of filters of $\mathsf{LA}(\mathbb{L})$ (Proposition 5.1.2). But this is not a very enlightening description of the above categories, since the composition of such Galois connections is not at all obvious.

On the other hand, the composition of consequence relations via relational product is compatible with mappings of the form $X \mapsto S(X)$ rather than with the Galois connections $X \mapsto X^S$. Using Theorem 5.2.3, we obtain the following alternative interpretation of consequence relations.

Proposition 5.2.5 Consider logical contexts $\mathbb{K} = (G, M, I)$, $\mathbb{L} = (H, N, J) \in \mathfrak{L}$. There is a bijection between

(i) the consequence relations from $\mathbb{K}$ to $\mathbb{L}$, and

(ii) the pairs of maps $\phi : \mathsf{Flt}(\mathsf{LA}(\mathbb{K})) \to \mathsf{Flt}(\mathsf{LA}(\mathbb{L}))$ and $\psi : \mathsf{Idl}(\mathsf{LA}(\mathbb{L})) \to \mathsf{Idl}(\mathsf{LA}(\mathbb{K}))$ for which

$$Y \cap \psi(X) = \emptyset \quad \text{iff} \quad \phi(Y) \cap X = \emptyset.$$

Furthermore, the mapping ψ of (ii) preserves finite infima.

Proof. Given a consequence relation S from $\mathbb{K}$ to $\mathbb{L}$, Theorem 5.2.3 yields mappings $S : \mathsf{Flt}(\mathsf{LA}(\mathbb{K})) \to \mathsf{Flt}(\mathsf{LA}(\mathbb{L}))$ and $S^{-1} : \mathsf{Idl}(\mathsf{LA}(\mathbb{L})) \to \mathsf{Idl}(\mathsf{LA}(\mathbb{K}))$, where we use the bijections between concept lattices and sets of filters/ideals established in Proposition 5.1.2. Now for arbitrary sets $X \subseteq M$, $Y \subseteq N$, one clearly finds that $X \cap S^{-1}(Y) = \emptyset$ iff $S(X) \cap Y = \emptyset$. Indeed, if there is $x \in X \cap S^{-1}(Y)$ then $S(x) \cap Y \neq \emptyset$, and the other direction is symmetrical.

For the converse, consider mappings ϕ and ψ as in (ii). A relation $S \subseteq M \times N$ is defined as $S(a) := \phi(\uparrow a)$. Clearly, $S(a)$ is a filter of $\mathsf{LA}(\mathbb{L})$ and thus, according to Proposition 5.1.2, an intent of $\mathbb{L}$. Conversely, given an element $b \in N$, one calculates $S^{-1}(b) = \{a \in M \mid \phi(\uparrow a) \cap \downarrow b \neq \emptyset\} = \{a \in M \mid \uparrow a \cap \psi(\downarrow b) \neq \emptyset\} = \psi(\downarrow b)$, where the second equality uses the precondition on the relationship of ϕ and ψ. Again, it is immediate that this is an ideal, and thus, again by Proposition 5.1.2, an intent of $\mathbb{K}^{\mathsf{c}}$. Now it is easy to see that the given constructions yield the claimed bijection.

Finally, for preservation of finite infima, note that infima of ideals are computed as intersections, and consider a finite collection $\mathcal{I}$ of ideals. Clearly we have $S^{-1}(\bigcap \mathcal{I}) \subseteq \bigcap\{S^{-1}(X) \mid X \in \mathcal{I}\}$. Thus suppose there is some element $a \in \bigcap\{S^{-1}(X) \mid X \in \mathcal{I}\}$ with $a \notin S^{-1}(\bigcap \mathcal{I})$. Then the filter $S(a)$ contains an element out of every ideal in $\mathcal{I}$. But then the finite meet of these elements is in $S(a)$ and in $\bigcap \mathcal{I}$ and thus $a \in S^{-1}(\bigcap \mathcal{I})$, showing the required contradiction. □

We immediately recognize the above relationship of ϕ and ψ as a typical adjunction condition, similar to those encountered for Galois connections or infomorphisms. Indeed, considering disjointness of filters and ideals as incidence relations of formal contexts, the pairs of maps (ϕ, ψ) are precisely the infomorphisms. Many further characterizations can be obtained along the lines of Theorem 4.4.5, but we will not go into the details.

5.3 Continuous functions

Choosing our terminology with care, continuity of consequence relations of course is closely related to continuity between the induced Stone spaces. However, topological continuity applies to functions only, such that we need to restrict to more specific consequence relations. Recall that in 5.1.4 we introduced $\Omega(\mathbb{K})$ as the topological space associated with a logical context $\mathbb{K}$.

Theorem 5.3.1 Consider logical contexts $\mathbb{K} = (G, M, I)$, $\mathbb{L} = (H, N, J) \in \mathfrak{L}$. The following collections of morphisms correspond bijectively:

(i) the topologically continuous functions from $\Omega(\mathbb{K})$ to $\Omega(\mathbb{L})$,

(ii) the extensionally continuous functions from $\mathbb{K}^c$ to $\mathbb{L}^c$,

(iii) the functional dual bonds from $\mathbb{K}^c$ to $\mathbb{L}$ that are extensionally continuous from $\mathbb{K}^c$ to $\mathbb{L}^c$,

(iv) the consequence relations S from $\mathbb{K}$ to $\mathbb{L}$, for which $S(g^I)$ has a least model for any $g \in G$,

(v) the consequence relations S from $\mathbb{K}$ to $\mathbb{L}$, for which $S(g^I)$ is a prime filter of $\mathsf{LA}(\mathbb{L})$ for any $g \in G$,

(vi) the frame homomorphisms from $\mathsf{Idl}(\mathsf{LA}(\mathbb{L}))$ to $\mathsf{Idl}(\mathsf{LA}(\mathbb{K}))$.

Proof. For the bijection of (i) and (ii), note that the preimage of a function preserves open sets iff it preserves closed sets, since preimages commute with complementations of sets. Since the closed sets of the associated topologies are exactly the extents of the complemented contexts (Proposition 5.1.2), this coincides with extensional continuity and the required bijection is just the identity map.

The bijection between (ii) and (iii) has been shown in Theorem 4.3.8.

For the correspondence with (iv), recall that by Theorem 4.2.2 any dual bond R as in (iii) is an extent of the direct product $\mathbb{K}^c \times \mathbb{L}$. We claim that the required bijection is a restriction of the bijection of extents and intents in the direct product. Indeed, for a dual bond R as in (iii), $R(g) = R^\nabla(g^I)^J$ by Theorem 4.1.8, and, since R is functional, there is a unique $h \in H$ such that $R(g) = h^{JJ}$. But then h clearly is the least model of $R^\nabla(g^I)$.

For the converse, consider any consequence relation S as in (iv). By a similar reasoning as before, if $S(g^I)$ has a least model h, then $h^{JJ} = S(g^I)^J = S^\nabla(g)$. Since h must be unique, this shows that S^∇ is functional. Extensional attribute-continuity of S^∇ follows by applying Theorem 4.2.2 on the dual bond S^{-1} from $\mathbb{L}^{\mathsf{d}}$ to $\mathbb{K}^{\mathsf{cd}}$, where the required extensional attribute-continuity of S^{-1} follows from Theorem 5.2.3. Finally, using Lemma 4.3.7, we conclude that extensional attribute-continuity of S^∇ is equivalent to extensional attribute-continuity of the induced function f. Thus, as detailed in Lemma 4.3.2, f is extensionally continuous from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$, and by Theorem 4.3.8, S^∇ must be continuous in the same sense.

The bijection of (iv) and (v) is again the identity mapping. For CP logic this is trivial: every filter is prime and every filter is an object intent (the least model of which is the filter itself). Additionally, for the case of logics with disjunction, if $S(g^I)$ is a prime filter, then this prime filter is clearly the least model of $S(g^I)$. Conversely, recall that, in the proof of Proposition 5.1.2, we already found that – using Axiom 2.1.9 – every filter is an intersection of prime filters. But if the filter $S(g^I)$ has a least model h, then the intersection of all models of $S(g^I)$ is h, i.e. $S(g^I) = h$ is prime.

Finally, let S be a consequence relation as in (v). We claim that the mapping $S^{-1} : \mathsf{Idl}(\mathsf{LA}(\mathbb{L})) \to \mathsf{Idl}(\mathsf{LA}(\mathbb{K}))$ from Proposition 5.2.5 is a frame homomorphism. In this proposition we already showed that S^{-1} preserves finite infima.

For preservation of suprema, consider an arbitrary collection $\mathcal{I}$ of ideals. The case of CP logic again is trivial, since suprema of lower sets are just unions, which are certainly preserved by S^{-1}. Thus assume that the given logics support disjunction. Certainly, $\bigvee\{S^{-1}(X) \mid X \in \mathcal{I}\} \subseteq S^{-1}(\bigvee \mathcal{I})$. For the converse, suppose there is some element $a \in S^{-1}(\bigvee \mathcal{I})$ such that $a \notin \bigvee\{S^{-1}(X) \mid X \in \mathcal{I}\}$. Using Axiom 2.1.9, one finds a prime filter g such that $\uparrow a \in g$ and $\bigvee\{S^{-1}(X) \mid X \in \mathcal{I}\} \cap g = \emptyset$. By our preconditions on S, $S(g)$ is a prime filter that contains $S(a)$; thus there is $b \in S(g)$ with $b \in \bigvee \mathcal{I}$, $a\ S\ b$, and $b \notin \bigcup \mathcal{I}$. It is easy to see that $\bigvee \mathcal{I}$ is the collection of all finite joins of elements from $\bigcup \mathcal{I}$, such that $b = \bigvee B$ for some finite set $B \subseteq \bigcup \mathcal{I}$. But then, since $S(g)$ is prime, $B \cap S(g) \neq \emptyset$ and thus g and $\bigvee\{S^{-1}(X) \mid X \in \mathcal{I}\}$ cannot be disjoint, which yields the required contradiction and shows that S^{-1} is indeed a frame homomorphism.

For the other direction, consider some frame homomorphism $\psi : \mathsf{Idl}(\mathsf{LA}(\mathbb{L})) \to \mathsf{Idl}(\mathsf{LA}(\mathbb{K}))$. A consequence relation S from $\mathbb{K}$ to $\mathbb{L}$ is defined as in Proposition 5.2.5 by setting $S^{-1}(b) = \psi(\downarrow b)$. Once more we can restrict attention to logics with disjunction, since all consequence relations of CP logics have the required property. Now for any $g \in G$, $S(g^I)$ is a filter of $\mathsf{LA}(\mathbb{L})$. To show that it is prime, assume that there is some join $(b \vee b') \in S(g^I)$. Then $(b \vee b')$ is the principal of the supremum of $\downarrow b$ and $\downarrow b'$, and, since ψ preserves suprema, $S^{-1}(b \vee b') = \psi(\downarrow(b \vee b')) = \psi(\downarrow b) \vee \psi(\downarrow b') = S^{-1}(b) \vee S^{-1}(b')$. Thus there is a finite join of elements from $S^{-1}(b)$ and $S^{-1}(b')$ which is contained in g^I, since g^I

meets $S^{-1}(b \vee b')$. But then, due to g^I being a prime filter, some element of this finite join is in g^I as well, and thus either b or b' is in $S(g^I)$. This finishes the proof. □

We remark that, frame homomorphisms being lower adjoints of monotone Galois connections, the morphism of (vi) can also be related directly to continuous functions by applying Theorem 4.3.8. An advantage of the above proof is that it shows preservation of suprema in a self-contained way that emphasizes the specific logical setting. Yet it is interesting to note that Theorem 4.3.8 actually is a generalization of constructions that are at the basis of Stone duality. The spirit of these observations relates to [Ern04], although the details of this relationship remain to be investigated.

The previous proof also once more exposed the extreme simplicity of CP logic. Especially, part of the above conditions on consequence relations are trivially satisfied for these logics. As an easy corollary, we derive the expected equivalence of **CCxt**, and the categories **Alg**, **Cxt**, and especially $\mathbf{\Sigma_{Alg}}$, which we studied in Chapter 3.

Corollary 5.3.2 The category **CCxt** of conjunctive logical contexts and consequence relations is equivalent to the category $\mathbf{\Sigma_{Alg}}$ of Scott topologies on algebraic lattices and continuous functions.

Proof. We sketch the proof by specifying the required correspondences on objects and morphisms. Mappings between objects have been provided in Proposition 5.1.2 and Corollary 3.4.15. Explicitly, every context $\mathbb{K} \in \mathfrak{C}$ is mapped to the topological space $\Omega(\mathbb{K})$, which, by Corollary 3.4.15 is a Scott topology on an algebraic lattice. Conversely, any such Scott topology is assigned to the logical context which is obtained by considering the lattice of its $\cup$-prime compact opens as a Lindenbaum algebra.

Theorem 5.3.1 provides the required bijections continuous functions and consequence relations. Indeed, every consequence relation between conjunctive logical contexts is easily seen to have the property of (iv), since every theory of CP logic has a least model. □

Using Theorem 3.3.9, one infers that the category **CCxt** is cartesian closed. However, the proof of the previous proposition relies on very specific properties of conjunctive logic, namely on the fact that every theory admits a least model. For more expressive logics, one can readily find examples for consequence relations that do not meet the requirements of Theorem 5.3.1. In consequence, the categories **DCxt**, **ICxt**, and **NCxt** are indeed different from their well-known categories of topological spaces and continuous functions.

Using Theorem 5.2.3, one also recognizes the direct product construction as a way to construct function spaces for **CCxt** which is different from our efforts in Section 3.3. On the other hand, the representations of algebraic lattices through the categories **CCxt** and **Cxt** differ in a fundamental way, so we do not obtain an alternative for the function space construction of **Cxt** (Proposition 3.3.6).

5.4 Infomorphisms and coherent maps

In Institution Theory, infomorphisms are commonly considered as morphisms between logical contexts. In Chapter 4, these were already characterized as a special case of extensionally continuous map. In the light of the previous section, one should thus ask how infomorphism are described in terms of consequence relations and topologically continuous functions. On the topological side, we encounter the following type of functions.

Definition 5.4.1 A function $f : X \to Y$ between topological spaces X and Y is *coherent* whenever f is continuous and f^{-1} preserves compact open sets.

Since preimages of functions commute with set complements, it is clear the coherent maps can as well be characterized as the maps that preserve co-compact (compact in the dual order) closed sets.

Now we can state the main result of this section.

Theorem 5.4.2 Consider logical contexts $\mathbb{K} = (G, M, I)$, $\mathbb{L} = (H, N, J) \in \mathfrak{L}$. The following collections of morphisms correspond bijectively:

(i) the infomorphisms from $\mathbb{K}$ to $\mathbb{L}$,

(ii) the consequence relations S from $\mathbb{K}$ to $\mathbb{L}$, for which $S(g^I)$ has a least model for any $g \in G$, and for which $S^{-1}(b)$ is a principal ideal for any $b \in N$,

(iii) frame homomorphisms $\mathsf{Idl}(\mathsf{LA}(\mathbb{L})) \to \mathsf{Idl}(\mathsf{LA}(\mathbb{K}))$ that map principal ideals to principal ideals,

(iv) the functions $f : \mathsf{LA}(\mathbb{L}) \to \mathsf{LA}(\mathbb{K})$ which are

   (a) homomorphisms of meet-semilattices with greatest element if $\mathfrak{L} = \mathfrak{C}$,

   (b) homomorphisms of bounded distributive lattices if $\mathfrak{L} = \mathfrak{D}$ or $\mathfrak{L} = \mathfrak{D}$,

   (c) homomorphisms of Boolean algebras if $\mathfrak{L} = \mathfrak{N}$.

Furthermore, if $\mathfrak{L} \neq \mathfrak{C}$, then there is a bijection between the above classes of morphisms and

(v) the coherent maps from $\Omega(\mathbb{K})$ to $\Omega(\mathbb{L})$.

Proof. Given an infomorphism $(\vec{f}, \overleftarrow{f})$ from $\mathbb{K}$ to $\mathbb{L}$, it was shown in Proposition 4.4.3 that the relation $S \subseteq M \times N$, defined by setting $S^{-1}(n) = \overleftarrow{f}(n)^{\chi\chi}$, is an intent of the direct product $\mathbb{K}^c \times \mathbb{L}$, and thus a consequence relation from $\mathbb{K}$ to $\mathbb{L}$ (Theorem 5.2.3). Proposition 4.4.3 also states that S^∇ is a functional dual bond from $\mathbb{K}^c$ to $\mathbb{L}$ which is extensionally continuous from $\mathbb{K}^c$ to $\mathbb{L}^c$. By Theorem 5.3.1 this yields the required least models for S. To show that $S^{-1}(b)$ is a principal ideal, just note that this is equal to S^{-1} being functional, which is immediate from its definition and the fact that no two attributes in a logical context generate the same intent.

For the converse, consider a consequence relation S as in (ii). By Theorem 5.3.1, S^∇ is functional from $\mathbb{K}^c$ to $\mathbb{L}$ and extensionally continuous from $\mathbb{K}^c$ to $\mathbb{L}^c$. Likewise, the requirement for preservation of principal ideal yields functionality of S^{-1}. By Proposition 4.4.4, the induced functions thus constitute an infomorphism from $\mathbb{K}$ to $\mathbb{L}$, which establishes the bijection between (i) and (ii).

For the bijection of (ii) and (iii), observe that the frame homomorphism S^{-1} constructed in Theorem 5.3.1, inherits the property of preserving principal ideals from S^{-1}. Similar remarks hold for the inverse construction of consequence relations from frame homomorphisms given in Theorem 5.3.1.

The bijection with (iv) requires a case distinction, since the structure of the Lindenbaum algebras depends on the chosen logics. Nonetheless, the constructions are similar in all cases: given a frame homomorphism ψ as in (iii), a function f as in (iv) is obtained as the restriction of ψ to principal ideals. Since binary meets in $\mathsf{LA}(\mathbb{L})$ correspond to binary meets of the according principal ideals, f preserves binary meets. Since preservation of the greatest element is also obvious, this shows (a). If $\mathsf{LA}(\mathbb{L})$ is a lattice, analogous reasoning shows preservation of binary joins and the least element, thus establishing (*b*). For (c) one just has to note the morphisms of bounded lattices between Boolean algebras are exactly the Boolean morphisms, i.e. must preserve negation as well. Indeed, given $b \in \mathsf{LA}(\mathbb{L})$, $f(a) \wedge f(\neg a) = f(a \wedge \neg a) = f(\bot) = \bot$ and $f(a) \vee f(\neg a) = f(a \vee \neg a) = f(\top) = \top$, which uniquely identifies $f(\neg a)$ as $\neg f(a)$.

Now for the converse, consider a homomorphism $f : \mathsf{LA}(\mathbb{L}) \to \mathsf{LA}(\mathbb{K})$. Since all elements of $\mathsf{Idl}(\mathsf{LA}(\mathbb{L}))$ are unions of (principal ideals of) elements of $\mathsf{LA}(\mathbb{L})$, and since frame homomorphisms preserve unions, f uniquely determines a frame homomorphism, where preservation of meets follows from the properties of f. It is obvious that this frame homomorphism meets the requirements of (iii).

Finally, if the considered logics have disjunctions, then the principal ideals (or attribute intents of the complemented contexts) are exactly the compact ideals. Consider a consequence S as above, the continuous function f induced by S^∇ as in Theorem 5.3.1, and a closed set C. Results from Stone duality assure that $f^{-1}(C) = S^{-1}(C^{\not\chi})^{\chi}$. Now co-compactness of closed sets C is equivalent to the compactness of the associated ideal $C^{\not\chi}$, and so it is clear that compact-preserving frame homomorphisms as in (iii) correspond to coherent maps as in (v), and vice versa. □

While this result gives a unified view on the morphisms of topology, institution theory, FCA, and universal algebra, various places expose slight mismatches. First of all, coherent maps do not agree with infomorphisms and homomorphisms of the Lindenbaum algebra in all cases. This is largely due to the fact that infomorphisms are based on the chosen set of attributes, which can be any meet-dense set of the represented locale, while coherent maps refer to the compact elements. In this light, the above results appear to be rather coincidental, since one can generally choose meet-dense subsets quite arbitrarily. In fact, there are algebraic lattices, with meet-dense subsets that are entirely disjoint from their set of compact elements. From the logical perspective, choosing the compact elements or a dense subset thereof is certainly to be preferred, such that the (partial) coincidence of coherent maps and infomorphisms seems to be rather typical.

A second mismatch relates to the algebraic side. While infomorphisms generally present homomorphisms of the Lindenbaum algebra, the result for $\mathfrak{I}$ in case (b) is not completely satisfactory: homomorphisms of Heyting algebras are assumed to preserve implication, but homomorphisms of bounded lattices may fail to have this property.

5.5 Future work

This chapter explained the relationships between various kinds of morphisms that suggest themselves for modelling interrelations between logics. The most general proof theoretic view using consequence relations was specialized to scale measures/topological continuous functions and infomorphisms/coherent maps/Lindenbaum algebra homomorphisms. Together with the considered classes of logics, this gives rise to a stock of categories, most of which can be motivated and explained from a logical perspective.

Nonetheless, it is not known which of the newly introduced categories feature additional desirable properties, such as completeness, co-completeness, or cartesian closedness. Since these properties are vital for applications that utilize such categorical constructions, their investigation is a reasonable goal for future work.

In this respect, the categories studied herein rather are to be considered as examples for possible categories of logics. Concrete applications are more likely to begin with particular requirements, especially with a clear restriction on the semantic structures that provide the objects for a category. Likewise, the requirement of specific categorical properties may lead to restrictions on morphisms that have not been considered here. For example, the least Boolean algebra $\mathbf{2} = \{\top, \bot\}$

does not qualify as a terminal object in a category of consequence relations. Instead, one may want to disallow those relations that relate satisfiable formulae with $\bot$, thus assuring the uniqueness of any morphism to $\mathbf{2}$. We think that the achieved representations provide a high degree of flexibility for similar restrictions. Furthermore, one can readily specialize the bijections of Theorem 5.3.1 and Theorem 5.4.2 to obtain a more comprehensive understanding of the semantics of such modifications.

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List of Symbols

Given references consist of a section and page number for the corresponding definitions. Categories are typeset in bold face, (object parts of) functors in sans-serif font.

Symbol	*Meaning*	*See*
$\bot$, $\top$	least, greatest element of a given poset	2.1, *14*
$\downarrow X$, $\uparrow X$	Lower, upper closure of X	2.1, *15*
$\bigwedge X$, $\bigvee X$	Meet, join of set X	2.1, *14*
2^X	Power set lattice of X	2.1, *15*
Alg	Algebraic lattices and Scott continuous functions	3, *35*
$\mathsf{Alg}(\mathbb{K})$	Algebraic lattice of approximable concepts of $\mathbb{K}$	3.3, *43*
$\mathsf{B}_o(\mathbb{K})$, $\mathsf{B}_a(\mathbb{K})$	Object and attribute concept-lattice of a context $\mathbb{K}$	2.3, *23*
$\mathbf{C}^{\mathsf{op}}$	Opposite of category **C**	2.5, *31*
CCxt	Conjunctive logical contexts and consequence relations	5.2, *92*
Cxt	Formal contexts and approximable mappings	3.3, *43*
CP	Deductive systems of CP logic and approximable mappings	3.4.1, *51*
DCxt	Positive logical contexts and consequence relations	5.2, *92*
$F \vdash G$	Proof theoretic consequence between formulae	3.4.1, *49*
$F \approx G$	Proof theoretic equivalence of F and G	3.4.1, *49*
$\mathsf{Fin}(X)$	Poset of finite subsets of X, $\mathsf{Fin}(X) = \mathsf{K}(2^X)$	2.1, *15*
$\mathsf{Flt}(P)$	Set of filters of P	3.4.3, *57*
$\not I$, $\not J$	Complements of the relations I, J	2.3, *24*
ICxt	Intuitionistic logical contexts and consequence relations	5.2, *92*
id_A	Identity morphism of identity functor on A	2.5, *30*
$\mathsf{Idl}(P)$	Ideal completion (set of ideals) of P	3.1, *36*
$\mathbb{K}^{\mathsf{c}}$	Complementary context to $\mathbb{K}$	2.3, *24*
$\mathbb{K}^{\mathsf{d}}$	Dual context to $\mathbb{K}$	2.3, *24*
$[\mathbb{K} \rightsquigarrow \mathbb{L}]$	Function space in **Cxt**	3.3, *47*
$\mathbb{K} + \mathbb{L}$	Direct sum of the contexts $\mathbb{K}$ and $\mathbb{L}$, product in **Cxt**	3.3, *45*
$\mathbb{K} \bowtie \mathbb{L}$	Alternative product of $\mathbb{K}$ and $\mathbb{L}$ in **Cxt**	3.3, *45*

List of Symbols

$\mathbb{K} \times \mathbb{L}$	Direct product of contexts $\mathbb{K}$ and $\mathbb{L}$	4.1, *66*
$\mathsf{K}(P)$	Compact elements of a dcpo P	2.4.1, *25*
$\mathsf{LA}(\mathbb{K})$	Lindenbaum algebra of the logical context $\mathbb{K}$	5.1, *88*
$\mathsf{LA}(\mathscr{S}(A), \vdash)$	Lindenbaum algebra of a deductive system of CP logic	3.4.1, *50*
NCxt	Classical logical contexts and consequence relations	5.2, *92*
$\Omega(\mathbb{K})$	Associated topological space of the logical context $\mathbb{K}$	5.1, *88*
$\omega(X)$	Open set lattice of topological space X	2.4.2, *27*
P^{op}	Dual order of P	2.1, *14*
$\mathsf{pt}(\omega)$	Points of the locale ω	3.4.3, *57*
R^{∇}	Intent/extent of R wrt. some direct product of contexts	4.1, *66*
$R^{(-1)}(X)$	(Pre)image of X under R	2.3, *22*
$\mathbf{Sem}_{\wedge}$	Meet-semilattices with greatest element and approximable mappings	3.4.1, *51*
$\mathbf{Sem}_{\vee}$	Join-semilattices with least element and approximable mappings	3.2, *40*
$\mathsf{Sem}(\mathbb{K})$	Join-semilattice of finitely generated intents of $\mathbb{K}$	3.3, *42*
$\boldsymbol{\Sigma}_{\mathbf{Alg}}$	Scott topologies on algebraic lattices and continuous functions	3.4.2, *55*
$\Sigma(P)$	Topological space of the Scott topology on P	2.4.2, *28*
$\sigma_{\mathbf{Alg}}$	Locales isomorphic to $\sigma(A)$ for some algebaic lattice A and frame homomorphisms	3.4.2, *58*
$\sigma(P)$	Lattice of Scott open sets of P	2.4.2, *28*
SIS	Scott information systems (with trivial consistency predicate) and approximable mappings	3.4.1, *53*
$X \uplus Y$	disjoint union of X and Y	3.3, *44*
X^I, X^J	Intent or extent of X	2.3, *22*
$x \wedge y$, $x \vee y$	Meet, join of elements x and y	2.1, *14*
$x \ll y$	Way-below relation: x is way-below y	2.4.1, *25*

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