Foundations for Machine Learning

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Reference

- Jon Kleinberg and Éva Tardos. Algorithm Design. Pearson Education / Addison-Wesley, 2006.
- Mauro Caresta. The Meaning of the Convolution.

1D CONVOLUTION:
THE CONTINUOUS CASE

The meaning of the Convolution

- The convolution of two functions $x(t)$ and $h(t)$ is defined as:

$$f(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(u)h(t-u)du$$

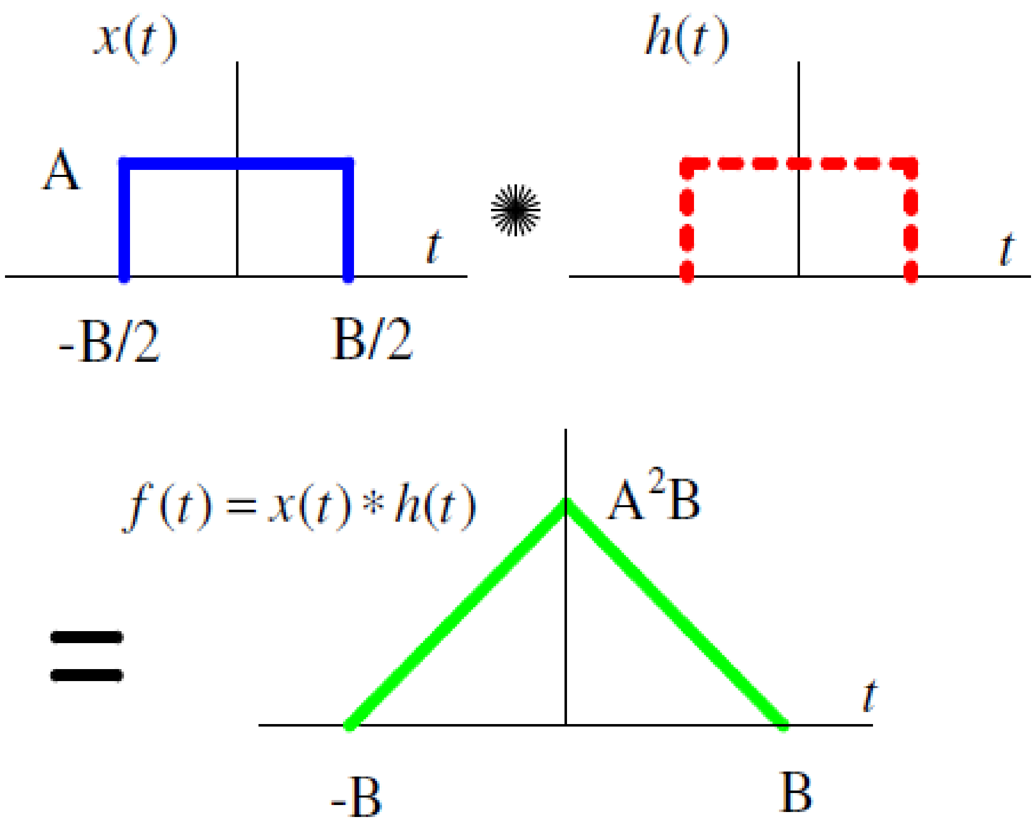
- To understand the meaning of the convolution we break it down into the following steps:

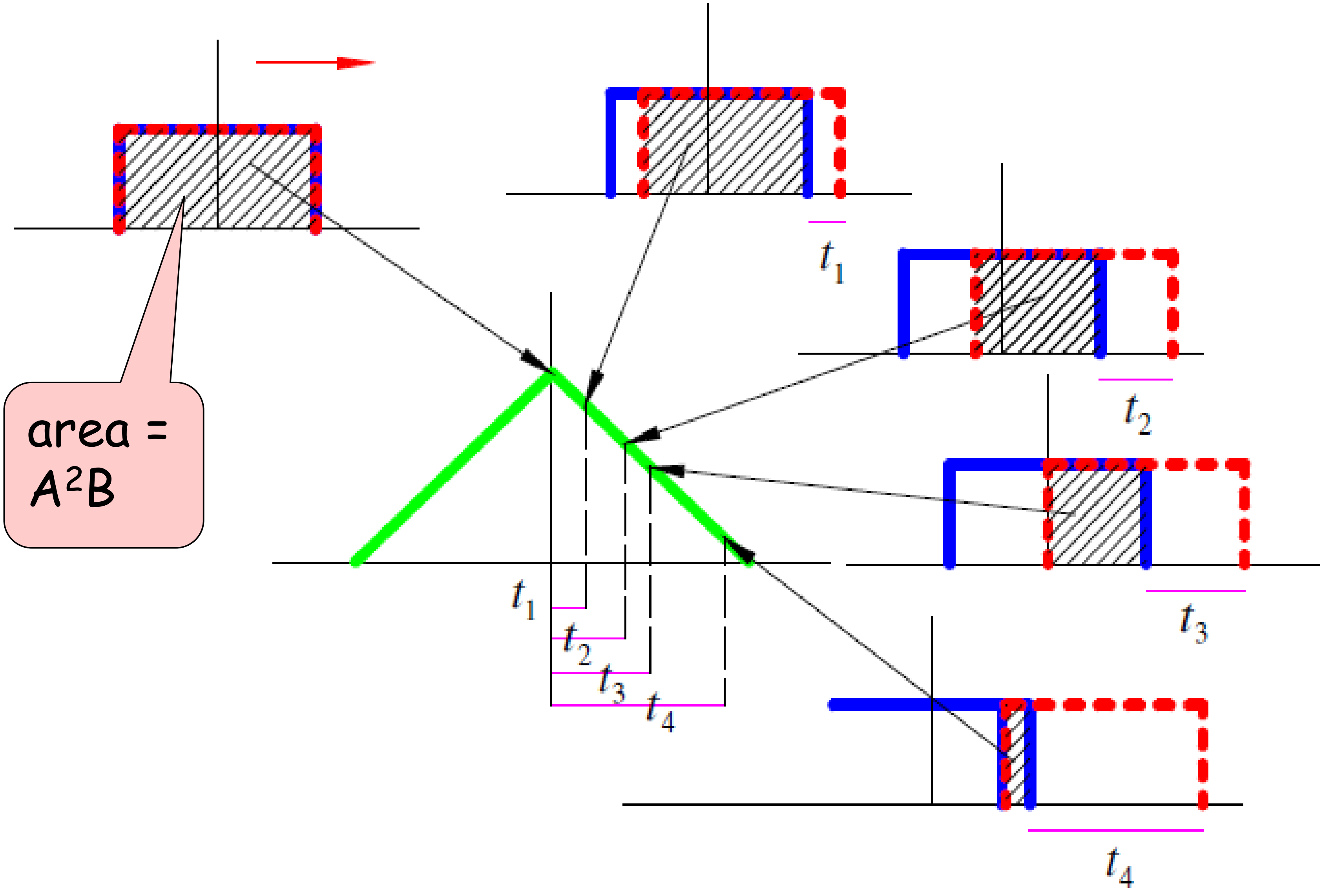
1) x and h are given as function of a dummy variable u.
2) Get the mirror of the function h: h(u) -> h(-u).
3) Add an offset t which allows $h(t - u)$ to slide along the u axis in the right direction, as t increases.
4) The value at a fixed t_1 is given by the area of the curve resulted by the product of the two functions x(u) and $h(t_1 - u)$ *i.e.*

$$f(t_1) = \int_{-\infty}^{\infty} x(u)h(t_1-u)du$$

Example

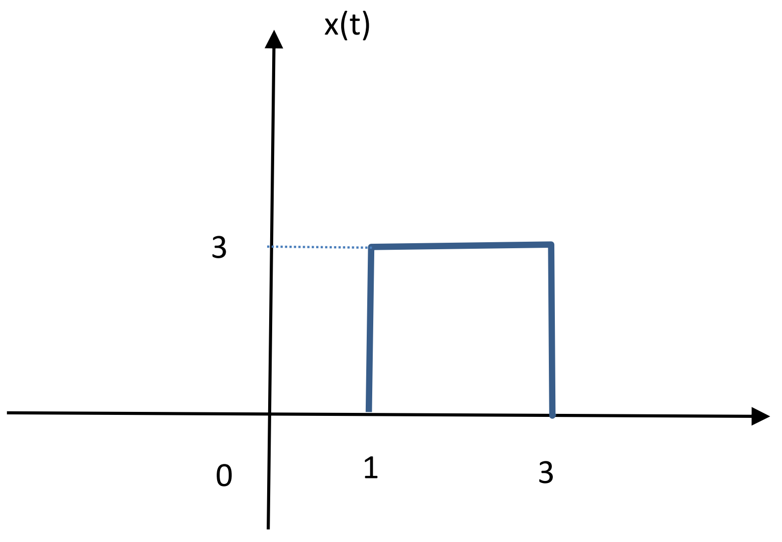
- The following example is the convolution of two identical window-shape functions. Note that in this case $h(-u) = h(u)$.

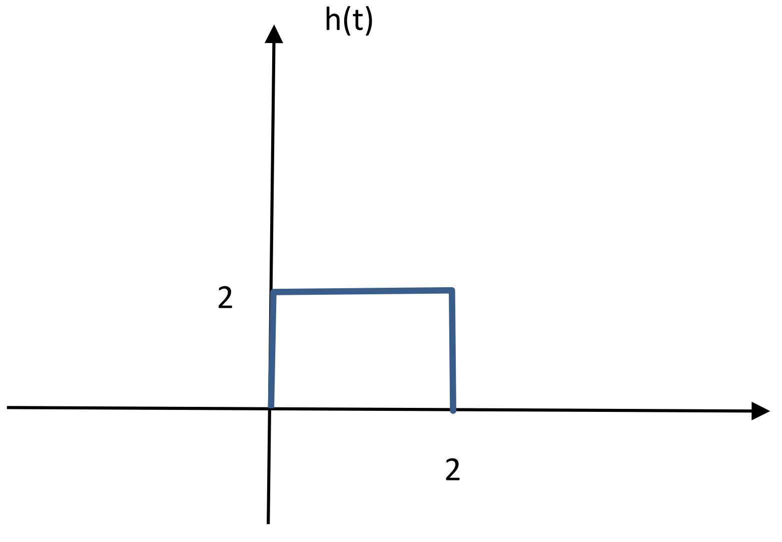



area =
A²B
t_1
t_2
t_3
t_4
t_1
t_2
t_3
t_4

Exercise:

Find the convolution of the following two functions: $x(t)$ and $h(t)$.





Properties of Convolution

- Commutative: x * h = h * x

$$\int_{-\infty}^{+\infty} x(u)h(t-u)du = \int_{-\infty}^{+\infty} h(w)x(t-w)dw$$

Proof: By the substitution $w = t-u$.

- Associative: f * (g * h) = (f * g) * h

(With the assumption that all convolution integral exist.)

- Distributive: f *(g + h) = f * g + f * h

CONVOLUTION:
THE DISCRETE CASE

The Convolution of Two Vectors

- Given two vectors $a = (a_0, a_1, \ldots, a_{n-1})$ and $b = (b_0, b_1, \ldots, b_{n-1})$ of length n, the convolution of a and b is a vector $c = a*b$ with $2n - 1$ components, where component k is defined as

$$c_k = \sum_{\substack{(i,j):i+j=k \\ i,j<n}} a_i b_j = \sum_{i=0}^{k} a_i b_{k-i}$$

- In other words,

$$a * b = (a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, \ldots, a_{n-1} b_{n-1})$$

- Another way to think about the convolution is to picture an n x n table whose (i, j) entry is $a_i b_j$, like this:

$$
\begin{array}{ccccc}
a_0b_0 & a_0b_1 & \ldots & a_0b_{n-2} & a_0b_{n-1} \\
a_1b_0 & a_1b_1 & \ldots & a_1b_{n-2} & a_1b_{n-1} \\
a_2b_0 & a_2b_1 & \ldots & a_2b_{n-2} & a_2b_{n-1} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
a_{n-1}b_0 & a_{n-1}b_1 & \ldots & a_{n-1}b_{n-2} & a_{n-1}b_{n-1}
\end{array}
$$

and then compute the components of the convolution vector by summing along the diagonals, as shown.

Example

- $a = (1, 2, 2)$
- $b = (2, 5, 4)$
- $c = a * b =$

$$
\begin{aligned}
& (a_0 b_0, \\
& a_0 b_1 + a_1 b_0, \\
& a_0 b_2 + a_1 b_1 + a_2 b_0, \\
& a_1 b_2 + a_2 b_1, \\
& a_2 b_2) \\
= \ & (2, 9, 18, 18, 8)
\end{aligned}
$$

$a_0 b_0$	$a_0 b_1$	$a_0 b_2$
$a_1 b_0$	$a_1 b_1$	$a_1 b_2$
$a_2 b_0$	$a_2 b_1$	$a_2 b_2$

Exercise

- Compute the convolution of a = (2,1,2,3) and b = (4,3,2,-1).

General Case

- The convolution can be easily generalized to vectors of different lengths, $a = (a_0, a_1, \ldots, a_{m-1})$ and $b = (b_0, b_1, \ldots, b_{n-1})$.
- In this more general case, $c = a * b$ is defined to be a vector with $m + n - 1$ components, where component k is equal

$$c_k = \sum_{\substack{(i,j):i+j=k \\ i<m,\, j<n}} a_i b_j = \sum_{i=0}^{k} a_i b_{k-i}$$

- Like in the continuous case, the discrete convolution of two vectors $a = (a_0, a_1, \dots, a_{m-1})$ and $b = (b_0, b_1, \dots, b_{n-1})$ can be interpreted as follows:

1. Write the vector b in reverse: $b' = (b_{n-1}, b_{n-2}, \dots, b_0)$.
2. Slide b′ into successive positions relative to vector a for each successive value of the convolution, by summing products of the corresponding values of the two vectors.

$$\left.\begin{array}{l} \qquad\qquad a_0\ a_1\ a_2\ a_3 \cdots a_{m-1} \\ b_{n-1} \cdots b_3\ b_2\ b_1\ b_0 \end{array}\right\} c_0 = a_0 b_0$$

$$\left.\begin{array}{l} \qquad\qquad a_0\ a_1\ a_2\ a_3 \cdots a_{m-1} \\ \quad b_{n-1} \cdots b_3\ b_2\ b_1\ b_0 \end{array}\right\} c_1 = (a_0 b_1 + a_1 b_0)$$

$$\left.\begin{array}{l} \qquad\qquad a_0\ a_1\ a_2\ a_3 \cdots a_{m-1} \\ \qquad b_{n-1} \cdots b_3\ b_2\ b_1\ b_0 \end{array}\right\} c_2 = (a_0 b_2 + a_1 b_1 + a_2 b_0)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\left.\begin{array}{l} a_0\ a_1\ a_2\ a_3 \cdots a_{m-1} \\ \qquad\qquad\qquad\qquad b_{n-1} \cdots b_3\ b_2\ b_1\ b_0 \end{array}\right\} c_{m+n-2} = a_{m-1} b_{n-1}$$

convolution as matrix multiplication

- **Toeplitz matrix** or **diagonal-constant matrix** is a matrix in which each descending diagonal from left to right is constant.
- $Example: \begin{bmatrix} a & b & c & d & e \\ e & a & b & c & d \\ f & e & a & b & c \\ g & f & e & a & b \end{bmatrix}$
- The convolution operation can be expressed as a matrix multiplication, where one of the inputs is converted into a Toeplitz matrix.
- For example, the convolution of a = (a0, a1, a2, a3) and b = (b0, b1, b2) can be formulated as:

$$c = a * b = \begin{bmatrix} a0 & 0 & 0 \\ a1 & a0 & 0 \\ a2 & a1 & a0 \\ a3 & a2 & a1 \\ 0 & a3 & a2 \\ 0 & 0 & a3 \end{bmatrix} \begin{bmatrix} b0 \\ b1 \\ b2 \end{bmatrix}$$

- In general,

$$
c = a * b = \overbrace{\begin{bmatrix}
a_1 & 0 & \cdots & 0 & 0 \\
a_2 & a_1 & \cdots & \vdots & \vdots \\
a_3 & a_2 & \cdots & 0 & 0 \\
\vdots & a_3 & \cdots & a_1 & 0 \\
a_{m-1} & \vdots & \cdots & a_2 & a_1 \\
a_m & a_{m-1} & \vdots & \vdots & a_2 \\
0 & a_m & \cdots & a_{m-2} & \vdots \\
0 & 0 & \cdots & a_{m-1} & a_{m-2} \\
\vdots & \vdots & \vdots & a_m & a_{m-1} \\
0 & 0 & 0 & \cdots & a_m
\end{bmatrix}}^{n \text{ columns}}
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n
\end{bmatrix}
$$

Exercise

- Compute the convolution of $a = (2,2,3,3,4)$ and $b = (1,1,2)$ using:
  1. the sliding method
  2. matrix multiplication
  3. polynomial multiplication

Applications

- Image/Signal Smoothing.
- String Matching.
- Convolutional Neural Networks.
- Etc.

Implementation in Python 3 (part 1)

```
# -*- coding: utf-8 -*-
"""
Created on Sat Jul  7 20:17:15 2018

Version 1:
Compute the convolution of two sequences a and b
given as lists of numbers, using the convolution
formula directly

@author: L Y Stefanus
"""
```

Implementation in Python 3 (part 1)

```
def convo1(a,b):
    na = len(a)
    nb = len(b)
    nc = na + nb - 1
    c = [0]*nc
    a = a + [0]*(nc-na)
    b = b + [0]*(nc-nb)

    for k in range(nc):
        for i in range(k+1):
            c[k] = c[k] + a[i]*b[k-i]
    return c
```

Implementation in Python 3 (part 2)

```
# -*- coding: utf-8 -*-
"""
Created on Sat Jul  7 21:19:13 2018

# version 2:
Compute the convolution of two sequences a and b
given as lists of numbers, using the
multiplication of Toeplitz matrix of a
and matrix of b

@author: L Y Stefanus
"""
```

Implementation in Python 3 (part 2)

```
import numpy as np
from scipy import linalg

def convo2(a,b):
    h = np.array(a)
    padding = np.zeros(len(b) - 1, h.dtype)
    first_col = np.r_[h, padding]
    first_row = np.r_[h[0], padding]
    T = linalg.toeplitz(first_col, first_row)
    H = np.mat(T)
    g = np.mat(b).T
    hasil = H*g
    return hasil.T.tolist()[0]
```

Implementation in Python 3

```
>>> a = [1,2,2]
>>> b = [2,5,4]
>>> c = convo1(a,b)
>>> print(c)
[2, 9, 18, 18, 8]
>>>
>>> a = [2,2,3,3,4]
>>> b = [1,1,2]
>>> c = convo2(a,b)
>>> print(c)
[2, 4, 9, 10, 13, 10, 8]
>>>
```