TECHNISCHE UNIVERSITÄT DRESDEN

DATABASE THEORY

Lecture 7: Tree-Like Conjunctive Queries

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Knowledge-Based Systems

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More recent versions of this slide deck might be available.
For the most current version of this course, see
`https://iccl.inf.tu-dresden.de/web/Database_Theory/en`

Review

Conjunctive queries (CQs) are simpler than FO-queries:

- NP combined and query complexity (instead of PSpace)
- data complexity remains in AC^0

CQs become even simpler if they are tree-shaped:

- GYO algorithm defines acyclic hypergraphs
- acyclic hypergraphs have join trees
- join trees can be evaluated in P with Yannakakis' Algorithm

This time:

- Find more general conditions that make CQs tractable
  $\leadsto$ "tree-like" queries that are not really trees
- Play some games

Is Yannakakis' Algorithm Optimal?

We saw that tree queries can be evaluated in polynomial time,
but we know that there are much simpler complexity classes:

$$\mathrm{NC}^0 \subset \mathrm{AC}^0 \subset \mathrm{NC}^1 \subseteq \mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{AC}^1 \subseteq \ldots \subseteq \mathrm{NC} \subseteq \mathrm{P}$$

Indeed, tighter bounds have been shown:

Theorem 7.1 (Gottlob, Leone, Scarcello: J. ACM 2001): Answering tree BCQs is complete for LOGCFL.

LOGCFL: the class of problems LogSpace-reducible to the word problem of a context-free language:

$$\mathrm{NC}^0 \subset \mathrm{AC}^0 \subset \mathrm{NC}^1 \subseteq \mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{LOGCFL} \subseteq \mathrm{AC}^1 \subseteq \ldots \subseteq \mathrm{NC} \subseteq \mathrm{P}$$

$\leadsto$ highly parallelisable

Generalising Tree Queries

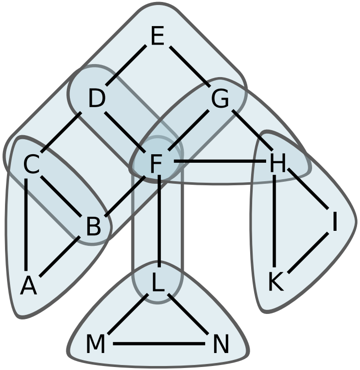
In practice, many queries are tree queries,
but even more queries are "almost" tree queries, but not quite . . .

How can we formalise this idea?

Several attempts to define "tree-like" queries:

- Treewidth: a way to measure tree-likeness of graphs
- Query width: towards tree-like query graphs
- Hypertree width: adoption of treewidth to hypergraphs

. . . from quite a long way away:



Tree Decompositions

Idea: if we can group the edges of a graph into bigger pieces, these pieces might form a tree structure

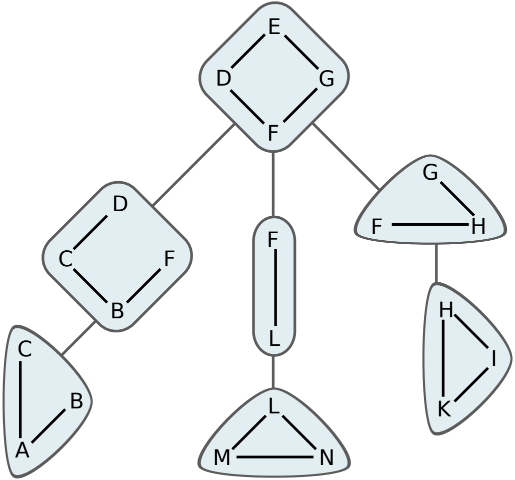
Definition 7.2: Consider a graph $G = \langle V, E \rangle$. A tree decomposition of G is a tree structure T where each node of T is a subset of V, such that:

- The union of all nodes of T is V.
- For each edge $(v_1 \to v_2) \in E$, there is a node N in T such that $v_1, v_2 \in N$.
- For every vertex $v \in V$, the set of nodes of T that contain v form a subtree of T; equivalently: if two nodes contain v, then all nodes on the path between them also contain v (connectedness condition).

Nodes of a tree decomposition are often called bags

(not related to the common use of “bag” as a synonym for “multiset”)

Tree Decompositions: Example



Treewidth

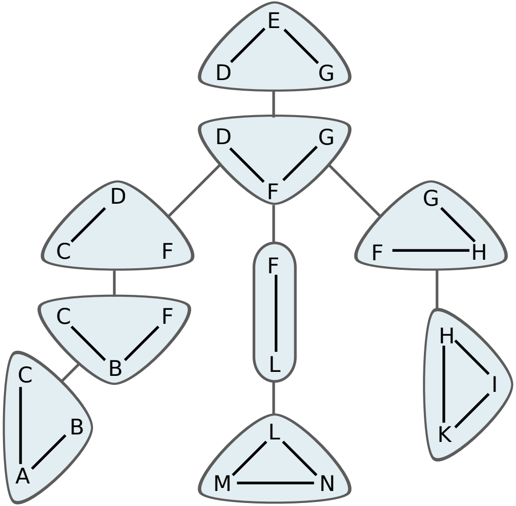
The treewidth of a graph defines how "tree-like" it is:

Definition 7.3: The width of a tree decomposition is the size of its largest bag minus one.
The treewidth of a graph G, denoted $\mathrm{tw}(G)$, is the smallest width of any of its tree decompositions.

Simple observations:

- If G is a tree, then we can decompose it into bags that contain only one edge
  $\rightsquigarrow$ trees have treewidth 1
- Every graph has at least one tree decomposition where all vertices are in one bag
  $\rightsquigarrow$ maximal treewidth = number of vertices -1

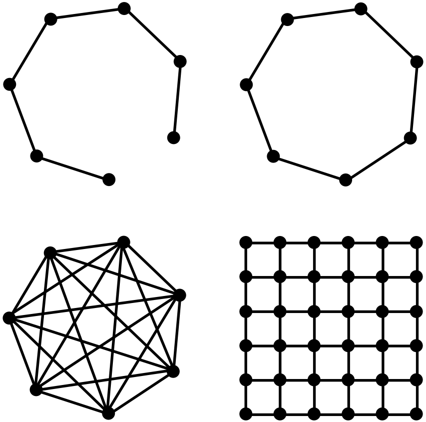
Treewidth: Example



$\rightsquigarrow$ tree decomposition of width 2 = treewidth of the example graph

More Examples

What is the treewidth of the following graphs?

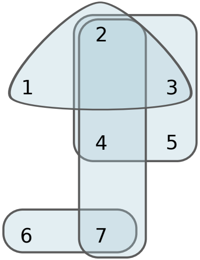


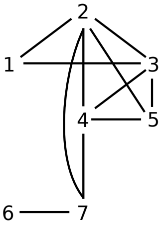
Treewidth and Conjunctive Queries

Treewidth is based on graphs, not hypergraphs

⇝ treewidth of CQ = treewidth of primal graph of query hypergraph

Query graph and corresponding primal graph:





⇝ Treewidth 3

Observation: acyclic hypergraphs can have unbounded treewidth!

Exploiting Treewidth in CQ Answering

Queries of low treewidth can be answered efficiently:

Theorem 7.4 (Dechter/Chekuri+Rajamaran '97/Kolaitis+Vardi '98/Gottlob & al. '98): Answering BCQs of treewidth k is possible in time $O(n^k \log n)$, and thus in polynomial time if k is fixed.
The problem is also complete for LOGCFL.

Checking for low treewidths can also be done efficiently:

Theorem 7.5 (Bodlaender '96): Given a graph G and a fixed number k, one can check in linear time if $\text{tw}(G) \leq k$, and the corresponding tree decomposition can also be found in linear time.

Warning: neither CQ answering nor tree decomposition might be practically feasible if k is big

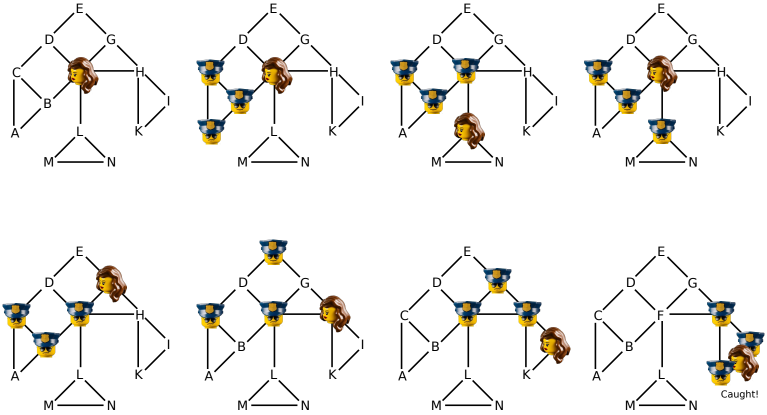
Treewidth via Games

Seymour and Thomas [1993] gave
an alternative characterisation of treewidth:

The Cops-and-Robber Game

- The game is played on a graph G
- There are k cops and one robber that may be positioned at vertices
- In the first turn, the robber places herself at an arbitrary vertex of the graph; the cops are all in a "helicopter" (i.e., not yet placed on any vertex)
- In each turn:
  - one of the cops can decide to "fly" to an arbitrary vertex in the graph
  - if the moving cop is already in the game, he is lifted from his vertex
  - before "landing" (i.e. positioning the cop at his new vertex), the target vertex is announced to the robber (the robber sees the helicopter approaching)
  - the robber can run along the edges of the graph, as far as she likes, as long as she does not use any vertex currently occupied by a cop
  - the moving cop arrives at his destination vertex
- The cops' goal is to catch the robber; the robber's goal is never to be caught

Cops and Robbers: Example



Cops & Robbers and Treewidth

Theorem 7.6 (Seymour and Thomas): A graph G is of treewidth $\leq k - 1$ if and only if k cops have a winning strategy in the cops & robber game on G.

Intuition: the cops together can block even the widest branch and still move in on the robber

Treewidth via Logic

Kolaitis and Vardi [1998] gave a logical characterisation of treewidth

Bounded treewidth CQs correspond to certain FO-queries:

- We allow FO-queries with $\exists$ and $\wedge$ as only operators
- But operators can be nested in arbitrary ways (unlike in CQs)
- Theorem: A query can be expressed with a CQ of treewidth k if and only if it can be expressed in this logic using a query with at most $k + 1$ distinct variables

Intuition: variables can be reused by binding them in more than one $\exists$

$\leadsto$ Apply a kind of “inverted prenex-normal-form transformation”

$\leadsto$ Variables that occur in the same atom or in a “tightly connected” atom must use different names

$\leadsto$ minimum number of variables $\Leftrightarrow$ treewidth (+1)

Summary and Outlook

Treewidth has Pros and Cons:

Advantages:

- Bounded treewidth is easy to check
- Bounded treewidth CQs are easy to answer

Disadvantages:

- Even families of acyclic graphs may have unbounded treewidth
- Loss of information when using primal graph
  (cliques might be single hyperedges – linear! –
  or complex query patterns – exponential!)

Open questions:

- Are there better ways to capture “tree-like” queries?